A Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$

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Abstract

A Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$ is rigorously presented extending our results in prior work. The $Cl(5, C) = Cl(4, C) \oplus Cl(4, C)$ algebraic structure of the Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills unification program advanced in this work is that the group structure given by the *direct* products $U(2, 2) \times U(4) \times U(4) = [SU(2, 2)]_{spacetime} \times [U(1) \times U(4) \times U(4)]_{internal}$ is ultimately tied down to four-dimensions and does *not* violate the Coleman-Mandula theorem because the spacetime symmetries (conformal group $SU(2, 2)$ in the absence of a mass gap, Poincare group when there is mass gap) do *not* mix with the internal symmetries. Similar considerations apply to the supersymmetric case when the symmetry group structure is given by the *direct* product of the superconformal group (in the absence of a mass gap) with an internal symmetry group so that the Haag-Lopuszanski-Sohnius theorem is not violated. A generalization of the de Sitter and Anti de Sitter gravitational theories based on the gauging of the $Cl(4, 1, R), Cl(3, 2, R)$ algebras follows. We conclude with a few remarks about the complex extensions of the Metric Affine theories of Gravity (MAG) based on $GL(4, C) \times_s C^4$, the realizations of twistors and the $\mathcal{N} = 1$ superconformal $su(2, 2|1)$ algebra purely in terms of Clifford algebras and their plausible role in Witten's formulation of perturbative $\mathcal{N} = 4$ super Yang-Mills theory in terms of twistor-string variables.

Keywords: C-space Gravity, Clifford Algebras, Grand Unification.

1 Introduction

Clifford, Division, Exceptional and Jordan algebras are deeply related and essential tools in many aspects in Physics [7], [8], [9], [20]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [18] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hypervolumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds. Octonionic gravity has been studied by [26], [25].

Grand-Unification models in $4D$ based on the exceptional E_8 Lie algebra have been known for sometime [1], [4]. The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [2]. Supersymmetric non-linear sigma models of Exceptional Kahler coset spaces are known to contain three generations of quarks and leptons as (quasi) Nambu-Goldstone superfields [3]. The low-energy phenomenology of superstring-inspired E_6 models has been reviewed by [6].

A Chern-Simons E_8 Gauge theory of Gravity, based on the octic E_8 invariant construction by [12], was proposed [10] as a unified field theory (at the Planck scale) of a Lanczos-Lovelock Gravitational theory with a E_8 Generalized Yang-Mills field theory which is defined in the $15D$ boundary of a $16D$ bulk space. The role of the Clifford algebra $Cl(16)$ associated with a $16D$ bulk was essential [10]. In particular, it was discussed how an E_8 Yang-Mills in 8D, after a sequence of symmetry breaking processes based on the $non-compact$ forms of exceptional groups as follows $E_{8(-24)} \rightarrow E_{7(-5)} \times SU(2) \rightarrow E_{6(-14)} \times SU(3) \rightarrow SO(8,2) \times U(1)$, leads to a Conformal gravitational theory in $8D$ based on gauging the non-compact conformal group $SO(8,2)$ in $8D$. Upon performing a Kaluza-Klein-Batakis [13] compactification on CP^2, involving a nontrivial $torsion$ which bypasses the no-go theorems that one cannot obtain $SU(3) \times SU(2) \times U(1)$ from a Kaluza-Klein mechanism in $8D$, leads to a Conformal Gravity-Yang-Mills unified theory based on the Standard Model group $SU(3) \times SU(2) \times U(1)$ in 4D.

A candidate action for an Exceptional E_8 gauge theory of gravity in $8D$ was constructed [11]. It was obtained by recasting the E_8 group as the semi-direct product of $GL(8,R)$ with a deformed Weyl-Heisenberg group associated with canonical-conjugate pairs of vectorial and antisymmetric tensorial generators of rank two and three. Other actions were proposed, like the quartic E_8 group-invariant action in 8D associated with the Chern-Simons E_8 gauge theory defined on the 7-dim boundary of a $8D$ bulk. The E_8 gauge theory of gravity can be embedded into a more general extended gravitational theory in Clifford spaces associated with the Clifford $Cl(16)$ algebra due to the fact that $E_8 \subset Cl(8) \otimes Cl(8) = Cl(16)$.

Quantum gravity models in $4D$ based on gauging the (covering of the) $GL(4,R)$ group were shown to be renormalizable by [16] however, due to the presence of fourth-derivatives terms in the metric which appeared in the quantum effective action, upon including gauge fixing terms and ghost terms, the prospects of unitarity were spoiled. The key question remains if this novel gravitational model based on gauging the E_8 group in $8D$ may still be renormalizable without spoiling unitarity at the quantum level.

Most recently it was proposed in [35] how a Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills Grand Unification model in $four$ dimensions can be attained from a Clifford Gauge Field Theory formulated in C-spaces (Clifford spaces). More precisely, the ordinary $Cl(4)$-algebra valued one-forms $(\mathcal{A}_\mu^A\ \Gamma_A)\ dx^\mu$ of a $4D$ spacetime are extended to *polyvector*-valued $(\mathcal{A}_M^A\ \Gamma_A)\ dX^M$ differential forms defined over the Clifford-space (C-space) associated with the $Cl(4)$ algebra. X^M is a *polyvector* valued coordinate corresponding to the C-space of dimensionality $2^4 = 16$. Other approaches to unification based on Clifford algebras and Noncommutative Geometry can be found in [22], [21], [23], [32], [29].

The main aim of this work is to show rigorously how a Clifford $Cl(5, C)$ Unified Gauge Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$ can be attained *without* having to recur to *polyvector* valued differential forms in the (2^4) 16-dim C-space. The upshot of the $Cl(5, C) = Cl(4, C) \oplus Cl(4, C)$ algebraic structure of the Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills unification program in $4D$ advanced in this work is that the group structure given by the *direct* products

$$U(2,2) \times U(4) \times U(4) \ = \ [SU(2,2)]_{spacetime} \times [U(1) \times U(4) \times U(4)]_{internal} \quad (1.1)$$

is ultimately tied down to four-dimensions and does *not* violate the Coleman-Mandula theorem because the spacetime symmetries (conformal group $SU(2, 2)$ in the absence of a mass gap, Poincare group when there is mass gap) do *not* mix with the internal symmetries. Similar considerations apply to the supersymmetric case when the symmetry group structure is given by the *direct* product of the superconformal group (in the absence of a mass gap) with an internal symmetry group so that the Haag-Lopuszanski-Sohnius theorem is not violated. Furthermore, the complex Clifford algebra $Cl(5, C)$ is associated with the tangent space of a complexified $5D$ spacetime which corresponds to 10 real dimensions and which is the arena of the anomaly free quantum superstring [30].

In section **2** we present our construction of a $Cl(5, C)$ Unified Gauge Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills. In section **3** we extend our prior results [36] pertaining a generalization of the de Sitter and Anti de Sitter gravitational theories based on the gauging of the $Cl(4, 1, R), Cl(3, 2, R)$ algebras. We end with a few concluding remarks about the complex extension of the Metric Affine theories of Gravity (MAG) [16] based in gauging the semidirect product of $GL(4, C) \times_s C^4$; the realizations of twistors [38] and the superconformal $su(2, 2|1)$ algebra [34] purely in terms of Clifford algebras and their plausible role in Witten's formulation [39] of the scattering amplitudes of perturbative $\mathcal{N} = 4$ super Yang-Mills theory in terms of twistor-string variables.

2 $Cl(5, C)$ Unified Gauge Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills

2.1 Clifford-algebra-valued Gauge Field Theories and Conformal (super) Gravity, (super) Yang Mills

Let $\eta_{ab} = (-, +, +, +)$, $\epsilon_{0123} = -\epsilon^{0123} = 1$, the real Clifford $Cl(3, 1, R)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \ \ \Gamma_5 = -i\,\Gamma_0\,\Gamma_1\,\Gamma_2\,\Gamma_3, \ \ (\Gamma_5)^2 = 1; \ \ \{\Gamma_5, \Gamma_a\} = 0; \quad (2.1)$$

$$\Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5; \quad \Gamma_{ab} = \frac{1}{2}\left(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a\right). \quad (2.2a)$$

$$\Gamma_{abc} = \epsilon_{abcd}\,\Gamma_5\,\Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5. \quad (2.2b)$$

$$\Gamma_a\,\Gamma_b = \Gamma_{ab} + \eta_{ab}, \quad \Gamma_{ab}\,\Gamma_5 = \frac{1}{2}\epsilon_{abcd}\,\Gamma^{cd}, \quad (2.2c)$$

$$\Gamma_{ab}\,\Gamma_c = \eta_{bc}\,\Gamma_a - \eta_{ac}\,\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2.2d)$$

$$\Gamma_c\,\Gamma_{ab} = \eta_{ac}\,\Gamma_b - \eta_{bc}\,\Gamma_a + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2.2e)$$

$$\Gamma_a\,\Gamma_b\,\Gamma_c = \eta_{ab}\,\Gamma_c + \eta_{bc}\,\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2.2f)$$

$$\Gamma^{ab}\,\Gamma_{cd} = \epsilon^{ab}{}_{cd}\,\Gamma_5 \ - 4\delta^{[a}_{[c}\,\Gamma^{b]}{}_{d]} \ - 2\delta^{ab}_{cd}. \quad (2.2g)$$

$$\delta^{ab}_{cd} = \frac{1}{2}\left(\delta^a_c\,\delta^b_d \ - \ \delta^a_d\,\delta^b_c\right). \quad (2.2.h)$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(3, 1)$ algebra exists where the generators

$$\mathbf{1}; \ \Gamma_a = \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4 = -i\Gamma_0; \ \Gamma_5; \quad a = 1, 2, 3, 4 \quad (2.3)$$

are Hermitian; while the generators $\Gamma_a\Gamma_5; \Gamma_{ab}$ for $a, b = 1, 2, 3, 4$ are anti-Hermitian. Using eqs-(2.1-2.3) allows to write the $Cl(3, 1)$ algebra-valued one-form as

$$\mathbf{A} = \left(a_\mu\,\mathbf{1} + \ b_\mu\,\Gamma_5 \ + \ e^a_\mu\,\Gamma_a \ + \ f^a_\mu\,\Gamma_a\,\Gamma_5 \ + \ \frac{1}{4}\omega^{ab}_\mu\,\Gamma_{ab} \right) dx^\mu. \quad (2.4)$$

The Clifford-valued gauge field A_μ transforms according to $A'_\mu = U^{-1}\,A_\mu\,U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1}\,F\,U$. Decomposing the field strength in terms of the Clifford algebra generators gives

$$F_{\mu\nu} = F^1_{\mu\nu}\,\mathbf{1} \ + \ F^5_{\mu\nu}\,\Gamma_5 \ + \ F^a_{\mu\nu}\,\Gamma_a \ + \ F^{a5}_{\mu\nu}\,\Gamma_a\,\Gamma_5 \ + \ \frac{1}{4}F^{ab}_{\mu\nu}\,\Gamma_{ab}. \quad (2.5)$$

where $F = \frac{1}{2}\ F_{\mu\nu}\ dx^\mu \wedge dx^\nu$. The field-strength components are given by

$$F^1_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{2.6a}$$

$$F^5_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu + 2 e^a_\mu f_{\nu a} - 2 e^a_\nu f_{\mu a} \tag{2.6b}$$

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2 f^a_\mu b_\nu - 2 f^a_\nu b_\mu \tag{2.6c}$$

$$F^{a5}_{\mu\nu} = \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2 e^a_\mu b_\nu - 2 e^a_\nu b_\mu \tag{2.6d}$$

$$F^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \longleftrightarrow \nu. \tag{2.6e}$$

At this stage we may provide the relation among the $Cl(3,1)$ algebra generators and the the conformal algebra $so(4,2) \sim su(2,2)$ in $4D$. The operators of the Conformal algebra can be written in terms of the Clifford algebra generators as [18]

$$P_a = \frac{1}{2}\Gamma_a\ (1\ -\ \Gamma_5);\quad K_a = \frac{1}{2}\Gamma_a\ (1\ +\ \Gamma_5);\quad D = -\ \frac{1}{2}\ \Gamma_5,\ \ L_{ab} = \frac{1}{2}\ \Gamma_{ab}. \tag{2.7}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra generators (2.7), the explicit evaluation of the commutators yields

$$[P_a,\ D] = P_a;\quad [K_a,\ D] = -K_a;\quad [P_a,\ K_b] = -2g_{ab}\ D\ +\ 2\ L_{ab}$$

$$[P_a,\ P_b] = 0;\quad [K_a, K_b] = 0; .... \tag{2.8}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations. We should notice that the K_a, P_a generators in (2.7) are both comprised of Hermitian Γ_a *and* anti-Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively. The dilation D operator is Hermitian, while the Lorentz generator L_{ab} is anti-Hermitian. The fact that Hermitian and anti-Hermitian generators are required is consistent with the fact that $U(2,2)$ is a pseudo-unitary group as we shall see bellow.

Having established this one can infer that the real-valued tetrad V^a_μ field (associated with translations) and its real-valued partner $\tilde V^a_\mu$ (associated with conformal boosts) can be defined in terms of the real-valued gauge fields e^a_μ, f^a_μ as follows

$$e^a_\mu\ \Gamma_a\ +\ f^a_\mu\ \Gamma_a\Gamma_5 = V^a_\mu\ P_a\ +\ \widetilde V^a_\mu\ K_a \tag{2.9}$$

From eq-(2.7) one learns that eq-(2.9) leads to

$$e^a_\mu\ -\ f^a_\mu = V^a_\mu;\quad e^a_\mu\ +\ f^a_\mu = \tilde V^a_\mu \Rightarrow$$

$$e^a_\mu = \frac{1}{2}\ (V^a_\mu + \tilde V^a_\mu),\quad f^a_\mu = \frac{1}{2}(\tilde V^a_\mu - V^a_\mu). \tag{2.10}$$

The components of the torsion and conformal-boost curvature of conformal gravity are given respectively by the linear combinations of eqs-(2.6c, 2.6d)

$$F^a_{\mu\nu} \ - \ F^{a5}_{\mu\nu} \ = \ \widetilde{F}^a_{\mu\nu}[P]; \quad F^a_{\mu\nu} \ + \ F^{a5}_{\mu\nu} \ = \ \widetilde{F}^a_{\mu\nu}[K] \ \Rightarrow$$

$$F^a_{\mu\nu} \ \Gamma_a \ + \ F^{a5}_{\mu\nu} \ \Gamma_a \ \Gamma_5 \ = \ \widetilde{F}^a_{\mu\nu}[P] \ P_a \ + \ \widetilde{F}^a_{\mu\nu}[K] \ K_a. \qquad (2.11a)$$

Inserting the expressions for e^a_μ, f^a_μ in terms of the vielbein V^a_μ and $\tilde{V}^a_\mu$ given by (2.10), yields the standard expressions for the Torsion and conformal-boost curvature, respectively

$$\widetilde{F}^a_{\mu\nu}[P] \ = \ \partial_{[\mu} \ V^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ V_{\nu]b} \ - \ V^a_{[\mu} \ b_{\nu]}, \qquad (2.11b)$$

$$\widetilde{F}^a_{\mu\nu}[K] \ = \ \partial_{[\mu} \ \tilde{V}^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ \tilde{V}_{\nu]b} \ + 2 \ \tilde{V}^a_{[\mu} \ b_{\nu]}, \qquad (2.11b)$$

The Lorentz curvature in eq-(2.6e) can be recast in the standard form as

$$F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} \ = \ \partial_{[\mu} \ \omega^{ab}_{\nu]} \ + \ \omega^{ac}_{[\mu} \ \omega^b_{\nu]c} \ + \ 2(\ V^a_{[\mu} \ \tilde{V}^b_{\nu]} \ + \ \tilde{V}^a_{[\mu} \ V^b_{\nu]} \ ). \qquad (2.11c)$$

The components of the curvature corresponding to the Weyl dilation generator given by $F^5_{\mu\nu}$ in eq-(2.6b) can be rewritten as

$$F^5_{\mu\nu} \ = \ \partial_{[\mu} \ b_{\nu]} \ + \ \frac{1}{2} \ (\ V^a_{[\mu} \ \tilde{V}_{\nu]a} \ - \ \tilde{V}^a_{[\mu} \ V_{\nu]a} \ ). \qquad (2.11d)$$

and the Maxwell curvature is given by $F^1_{\mu\nu}$ in eq-(2.6a). A re-scaling of the vielbein V^a_μ/l and $\tilde{V}^a_\mu/l$ by a length scale parameter l is necessary in order to endow the curvatures and torsion in eqs-(2.11) with the proper dimensions of $length^{-2}, length^{-1}$, respectively.

To sum up, the real-valued tetrad gauge field V^a_μ (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\widetilde{V}^a_\mu$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e^a_\mu \ \mp \ f^a_\mu$ associated with the $\Gamma_a, \ \Gamma_a \ \Gamma_5$ generators of the Clifford algebra $Cl(3,1)$ of the tangent space of spacetime $\mathcal{M}^4$ after performing a Wick rotation $-i \ \Gamma_0 = \Gamma_4$.

In order to obtain the generators of the compact $U(4) \ = \ SU(4) \times U(1)$ unitary group, in terms of the $Cl(3,1)$ generators, a $different$ basis involving a full set of Hermitian generators must be chosen of the form

$$M_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ -i \ \Gamma_5); \quad N_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ +i \ \Gamma_5); \quad \mathcal{D} \ = \ \frac{1}{2} \ \Gamma_5, \ \ \mathcal{L}_{ab} \ = \ - \ \frac{i}{2} \ \Gamma_{ab}. \qquad (2.12)$$

One may choose, instead, a full set of anti-Hermitian generators by multiplying every generator $M_a, N_a, \mathcal{D}, \mathcal{L}_{ab}$ by $\mathbf{i}$ in (2.12), if one wishes. The choice (2.12) leads to a $different$ algebra $so(6) \sim su(4)$ and whose commutators $differ$ from those in (2.8)

$$[M_a, \ \mathcal{D}] \ = \ i \ N_a; \quad [N_a, \ \mathcal{D}] \ = \ - \ i \ M_a; \quad [M_a, \ N_b] \ = \ - \ 2i \ g_{ab} \ \mathcal{D}$$

$$[M_a,\ M_b]\ = [N_a,\ N_b]\ =\ \frac{1}{2}\ \Gamma_{ab}\ =\ i\ \mathcal{L}_{ab};\ .... \tag{2.13}$$

The Hermitian generators $M_a, N_a, \mathcal{D}, \mathcal{L}_{ab}$ associated to the $so(6) \sim su(4)$ algebra are given by the one-to-one correspondence

$$M_a\ =\ \frac{1}{2}\Gamma_a\ (1\ - i\ \Gamma_5)\ \longleftrightarrow\ -\ \Sigma_{a5};\quad N_a\ =\ \frac{1}{2}\Gamma_a\ (1\ + i\ \Gamma_5)\ \longleftrightarrow\ \Sigma_{a6}$$

$$\mathcal{D}\ =\ \frac{1}{2}\ \Gamma_5\ \longleftrightarrow\ \Sigma_{56};\quad \mathcal{L}_{ab}\ =\ -\ \frac{i}{2}\ \Gamma_{ab}\ \longleftrightarrow\ \Sigma_{ab} \tag{2.14}$$

The $so(6)$ Lie algebra in $6D$ associated to the Hermitian generators Σ_{AB} $(A, B = 1, 2, ..., 6)$ is defined by the commutators

$$[\Sigma_{AB},\ \Sigma_{CD}]\ =\ i\ (g_{BC}\ \Sigma_{AD}\ -\ g_{AC}\ \Sigma_{BD}\ -\ g_{BD}\ \Sigma_{AC}\ +\ g_{AD}\ \Sigma_{BC}\ ) \tag{2.15}$$

where g_{AB} is a diagonal $6D$ metric with signature $(-,-,-,-,-,-)$. One can verify that the realization (2.12) and correspondence (2.14) is consistent with the $so(6) \sim su(4)$ commutation relations (2.15). The extra $U(1)$ Abelian generator in $U(4) = U(1) \times SU(4)$ is associated with the unit $\mathbf{1}$ generator.

Since $su(4) \sim so(6)$ (isomorphic algebras) and the unitary algebra $u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6)$, the Hermitian $u(1) \oplus so(6)$ valued field $\mathbf{A}_\mu$ may be expanded in a $Cl(3, 1, R)$ basis of Hermitian generators as

$$\mathbf{A}_\mu\ =\ a_\mu\ \mathbf{1} +\ b_\mu\ \Gamma_5\ +\ e^a_\mu\ \Gamma_a\ + \mathbf{i}\ f^a_\mu\ \Gamma_a\ \Gamma_5\ + \mathbf{i}\ \frac{1}{4}\omega^{ab}_\mu\ \Gamma_{ab}\ =$$

$$a_\mu\ \mathbf{1} +\ A^{56}_\mu\ \Sigma_{56}\ +\ A^{a5}_\mu\ \Sigma_{a5}\ +\ A^{a6}_\mu\ \Sigma_{a6}\ +\ \frac{1}{4}A^{ab}_\mu\ \Sigma_{ab} \tag{2.16}$$

One should notice the key presence of **i** factors in the last two (Hermitian) terms of the first line of eq-(2.16), compared to the last two terms of (2.4) devoid of **i** factors. All the terms in eq-(2.4) are devoid of **i** factors such that the last two terms of (2.4) are comprised of anti-Hermitian generators while the first three terms involve Hermitian generators. The dictionary between the real-valued fields in the first and second lines of (2.16) is given by

$$a_\mu = a_\mu,\ b_\mu = A^{56}_\mu,\ A^{a5}_\mu\ =\ e^a_\mu - f^a_\nu,\ A^{a6}_\mu\ =\ e^a_\mu + f^a_\nu,\ A^{ab}_\mu = \omega^{ab}_\mu \tag{2.17}$$

the dictionary (2.17) is inferred from the relation

$$e^a_\mu\ \Gamma_a\ + \mathbf{i}\ f^a_\mu\ \Gamma_a\ \Gamma_5\ =\ A^{a5}_\mu\ \Sigma_{a5}\ +\ A^{a6}_\mu\ \Sigma_{a6} \tag{2.18}$$

and from eq-(2.12) (all terms in (2.18) are comprised of Hermitian generators as they should). The evaluation of the $u(1) \oplus so(6)$ valued field strengths $F_{\mu\nu}, F^{MN}_{\mu\nu}$, $M, N = 1, 2, 3, ..., 6$ proceeds in a similar fashion as in the conformal Gravity-Maxwell case based on the pseudo-unitary algebra $u(2, 2) = u(1) \oplus su(2, 2) \sim u(1) \oplus so(4, 2)$.

Gauge invariant actions involving Yang-Mills terms of the form $\int Tr(F \wedge^* F)$ and theta terms of the form $\int Tr(F \wedge F)$ are straightforwardly constructed. For example, a $SO(4,2)$ gauge-invariant action for conformal gravity is [33]

$$S = \int d^4x \; \epsilon_{abcd} \; \epsilon^{\mu\nu\rho\sigma} \; R^{ab}_{\mu\nu} \; R^{cd}_{\rho\sigma} \tag{2.19}$$

where the components of the Lorentz curvature 2-form $R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu$ are given by eq-(2.11c) after re-scaling the vielbein V^a_μ / l and $\tilde{V}^a_\mu / l$ by a length scale parameter l in order to endow the curvature with the proper dimensions of $length^{-2}$. The conformal boost symmetry can be fixed by choosing the gauge $b_\mu = 0$ because under infinitesimal conformal boosts transformations the field b_μ transforms as $\delta b_\mu = -2 \; \xi^a \; e_{a\mu} = -2 \; \xi_\mu$; i.e the parameter ξ_μ has the same number of degrees of feedom as b_μ. After further fixing the dilational gauge symmetry, setting the torsion to zero which constrains the spin connection $\omega^{ab}_\mu (V^a_\mu)$ to be of the Levi-Civita form given by a function of the vielbein V^a_μ, and eliminating the $\tilde{V}^a_\mu$ field algebraically via its (non-propagating) equations of motion [5] leads to the de Sitter group $SO(4,1)$ invariant Macdowell-Mansouri-Chamseddine-West action [14], [15] (suppressing spacetime indices for convenience)

$$S = \int d^4x \; (\; R^{ab}(\omega) + \frac{1}{l^2} \; V^a \wedge V^b \;) \wedge (\; R^{cd}(\omega) + \frac{1}{l^2} \; V^c \wedge V^d \;) \; \epsilon_{abcd}. \tag{2.20}$$

the action (2.20) is comprised of the topological invariant Gauss-Bonnet term $R^{ab}(\omega) \wedge R^{cd}(\omega) \epsilon_{abcd}$; the standard Einstein-Hilbert gravitational action term $\frac{1}{l^2} R^{ab}(\omega) \wedge V^c \wedge V^d \epsilon_{abcd}$, and the cosmological constant term $\frac{1}{l^4} V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd}$. l is the de Sitter throat size; i.e. l^2 is proportional to the square of the Planck scale (the Newtonian coupling constant).

The familiar Einstein-Hilbert gravitational action can also be obtained from a coupling of gravity to a scalar field like it occurs in a Brans-Dicke-Jordan theory of gravity

$$S = \frac{1}{2} \int d^4x \; \sqrt{g} \; \phi \; \left(\frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} \; g^{\mu\nu} \; D^c_\mu \phi) \; + \; b^\mu \; (D^c_\mu \phi) \; + \; \frac{1}{6} \; R \; \phi \right). \tag{2.21a}$$

where the conformally covariant derivative acting on a scalar field ϕ of Weyl weight one is

$$D^c_\mu \phi = \partial_\mu - b_\mu \; \phi \tag{2.21b}$$

Fixing the conformal boosts symmetry by setting $b_\mu = 0$ and the dilational symmetry by setting $\phi = constant$ leads to the Einstein-Hilbert action for ordinary gravity.

This construction of Conformal Gravity and Yang-Mills based on a Clifford-algebra valued gauge field theory can also be extended to the superconformal Yang-Mills and conformal Supergravity case. The $\mathcal{N} = 1$ superconformal algebra $su(2,2|1)$ involving the additional fermionic generators Q_α, S_α and the chiral generator A, admits a Clifford algebra realization as well [34]. The realization of the 15 bosonic generators is given by (2.7) after one embeds the 4×4 matrices into a 5×5 matrix where one adds zero elements in the 5-th column and in the 5-th row. Whereas the 8 fermionic Q_α, S_α generators are represented by the 5×5 matrices with zeros everywhere *except* in the four entries along the 5-th column and along the 5-th row as follows

$$(Q_\alpha)^{5\beta} = -\frac{1}{2}(1-\Gamma_5)_{\alpha\beta}, \quad (Q_\alpha)^{55} = 0, \quad (Q_\alpha)^{\beta 5} = [\frac{1}{2}(1+\Gamma_5)C]_{\alpha\beta}$$

$$(S_\alpha)^{5\beta} = \frac{1}{2}(1+\Gamma_5)_{\alpha\beta}, \quad (S_\alpha)^{55} = 0, \quad (S_\alpha)^{\beta 5} = -[\frac{1}{2}(1-\Gamma_5)C]_{\alpha\beta} \quad (2.22a)$$

The indices $\alpha,\beta = 1,2,3,4$. $C = C_{\alpha\beta}$ is the charge conjugation matrix $C = -C^{-1} = -C^T$ satisfying $C\Gamma_\mu C^{-1} = -(\Gamma_\mu)^T$. In the represenation chosen in (2.22a) $C = \Gamma_0$. The chiral generator A is represented by $-\frac{i}{4}$ times a diagonal 5×5 matrix whose entries are $(1,1,1,1,4)$. The nonzero (anti) commutators of the $\mathcal{N}=1$ superconformal algebra $su(2,2|1)$ are [34]

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\Gamma^\mu P_\mu)_{\alpha\beta}, \quad \{S_\alpha, \bar{S}_\beta\} = -2(\Gamma^\mu K_\mu)_{\alpha\beta}$$

$$\{Q_\alpha, \bar{S}_\beta\} = -\frac{1}{2}C_{\alpha\beta}\, D + \frac{1}{2}\,(\Gamma^{ab}C)_{\alpha\beta}\, L_{ab} + (i\Gamma_5 C)_{\alpha\beta}\, A$$

$$[S_\alpha, L_{ab}] = \frac{1}{2}\,(\Gamma_{ab})_{\alpha\beta}\, S_\beta, \quad [Q_\alpha, L_{ab}] = \frac{1}{2}\,(\Gamma_{ab})_{\alpha\beta}\, Q_\beta$$

$$[S_\alpha, A] = i\,\frac{3}{4}\,(\Gamma_5)_{\alpha\beta}\, S_\beta, \quad [Q_\alpha, A] = -i\,\frac{3}{4}\,(\Gamma_5)_{\alpha\beta}\, Q_\beta$$

$$[S_\alpha, D] = -\frac{1}{2}\, S_\alpha, \quad [Q_\alpha, D] = \frac{1}{2}\, Q_\alpha$$

$$[S_\alpha, P_a] = -\frac{1}{2}\,(\Gamma_a)_{\alpha\beta}\, Q_\beta, \quad [Q_\alpha, P_a] = -\frac{1}{2}\,(\Gamma_a)_{\alpha\beta}\, S_\beta..... \quad (2.22b)$$

The remaining commutators involving the bosonic generators are given by (2.8).

2.2 $U(p,q)$ from $U(p+q)$ via the Weyl unitary trick

In general, the unitary *compact* group $U(p+q;C)$ is related to the *noncompact* unitary group $U(p,q;C)$ by the Weyl unitary trick [17] mapping the anti-Hermitian generators of the compact group $U(p+q;C)$ to the anti-Hermitian *and* Hermitian generators of the noncompact group $U(p,q;C)$ as follows : The $(p+q)\times(p+q)$ $U(p+q;C)$ complex matrix generator is comprised of the diagonal blocks of $p\times p$ and $q\times q$ complex anti-Hermitian matrices $M_{11}^\dagger = -M_{11}$; $M_{22}^\dagger = -M_{22}$, respectively. The off-diagonal blocks are comprised of the $q\times p$ complex matrix M_{12} and the $p\times q$ complex matrix $-M_{12}^\dagger$, i.e. the off-diagonal blocks are the anti-Hermitian complex conjugates of each other. In this fashion the $(p+q)\times(p+q)$ $U(p+q;C)$ complex matrix generator $\mathbf{M}$ is anti-Hemitian $\mathbf{M}^\dagger = -\mathbf{M}$ such that upon an exponentiation $U(t) = e^{t\mathbf{M}}$ it generates a unitary group element obeying the condition $U^\dagger(t) = U^{-1}(t)$ for $t = real$. This is what occurs in the $U(4)$ case.

In order to retrieve the noncompact group $U(2,2;C)$ case, the Weyl unitary trick requires leaving M_{11}, M_{22} intact but performing a Wick rotation of the off-diagonal block matrices $i\ M_{12}$ and $-i\ M_{12}^\dagger$. In this fashion, M_{11}, M_{22} still retain their anti-Hermitian character, while the off-diagonal blocks are now $Hermitian$ complex conjugates of each-other. This is precisely what occurs in the realization of the Conformal group generators in terms of the $Cl(3,1,R)$ algebra generators. For example, P_a, K_a both contain Hermitian Γ_a and anti-Hermitian $\Gamma_a\Gamma_5$ generators. Despite the name "unitary" group $U(2,2;C)$, the exponentiation of the P_a and K_a generators does not furnish a truly unitary matrix obeying $U^\dagger = U^{-1}$. For this reason the groups $U(p,q;C)$ are more properly called *pseudo*-unitary. The complex extension of $U(p+q,C)$ is $GL(p+q;C)$. Since the algebras $u(p+q;C), u(p,q;C)$ differ only by the Weyl unitary trick, they both have identical complex extensions $gl(p+q;C)$ [17]. $gl(N,C)$ has $2N^2$ generators whereas $u(N,C)$ has N^2.

The covering of the general linear group $GL(N,R)$ admits $infinite$-dimensional spinorial representations but *not* finite-dimensional ones. For a thorough discussion of the physics of infinite-component fields and the perturbative renormalization property of metric affine theories of gravity based on (the covering of) $GL(4,R)$ we refer to [16]. The group $U(2,2)$ consists of the 4×4 complex matrices which preserve the *sesquilinear* symmetric metric $g_{\alpha\beta}$ associated to the following quadratic form in C^4

$$< u, u > \ = \ \bar{u}^\alpha\ g_{\alpha\beta}\ u^\beta \ = \ \bar{u}^1 u^1 \ + \ \bar{u}^2 u^2 \ - \ \bar{u}^3 u^3 \ - \ \bar{u}^4 u^4. \qquad (2.23a)$$

obeying the *sesquilinear* conditions

$$< \lambda\ v, u > \ = \ \bar{\lambda} < v, u >; \quad < v, \lambda\ u > \ = \ \lambda < v, u > . \qquad (2.23b)$$

where λ is a complex parameter and the bar operation denotes complex conjugation. The metric $g_{\alpha\beta}$ can be chosen to be given precisely by the chirality $(\Gamma_5)_{\alpha\beta}$ 4×4 matrix representation whose entries are $\mathbf{1}_{2\times2}, \ - \ \mathbf{1}_{2\times2}$ along the main diagonal blocks, respectively, and 0 along the off-diagonal blocks. The Lie algebra $su(2,2) \sim so(4,2)$ corresponds to the conformal group in $4D$. The special unitary group $SU(p+q;C)$ in addition to being sesquilinear metric-preserving is also volume-preserving.

The group $U(4)$ consists of the 4×4 complex matrices which preserve the *sesquilinear* symmetric metric $g_{\alpha\beta}$ associated to the following quadratic form in C^4

$$< u, u > \ = \ \bar{u}^\alpha\ g_{\alpha\beta}\ u^\beta \ = \ \bar{u}^1 u^1 \ + \ \bar{u}^2 u^2 \ + \ \bar{u}^3 u^3 \ + \ \bar{u}^4 u^4. \qquad (2.24)$$

The metric $g_{\alpha\beta}$ is now chosen to be given by the unit $\mathbf{1}_{\alpha\beta}$ diagonal 4×4 matrix. The $U(4) = U(1)\times SU(4)$ metric-preserving group transformations are generated by the 15 Hermitian generators Σ_{AB} and the unit $\mathbf{1}$ generator.

In the most general case one has the following isomorphisms of Lie algebras [17]

$$so(5,1) \sim su^*(4) \sim sl(2,H); \quad so^*(6) \sim su(3,1); \quad so(3,2) \sim sp(4,R)$$

$$so(4,2) \sim su(2,2); \quad so(3,3) \sim sl(4,R); \quad so(6) \sim su(4), \; etc..... \qquad (2.25)$$

where the asterisks like $su^*(4), so^*(6)$ denote the algebras associated with the *noncompact* versions of the compact groups $SU(4), SO(6)$. $sl(2,H)$ is the special linear Mobius algebra over the field of quaternions H. The $SU(4)$ group is a two-fold covering of $SO(6)$ but their algebras are isomorphic.

2.3 $U(4) \times U(4)$ Yang-Mills and Conformal Gravity, Maxwell Unification from a $Cl(5,C)$ Gauge Theory

To complete this section it is necessary to recall the following isomorphisms among real and complex Clifford algebras

$$Cl(2m+1,C) = Cl(2m,C) \oplus Cl(2m,C) \sim M(2^m,C) \oplus M(2^m,C) \;\Rightarrow$$

$$Cl(5,C) \;=\; Cl(4,C) \oplus Cl(4,C) \qquad (2.26a)$$

and

$$Cl(4,C) \sim M(4,C) \sim Cl(4,1,R) \sim Cl(2,3,R) \sim Cl(0,5,R) \qquad (2.26b)$$

$$Cl(4,C) \sim M(4,C) \sim Cl(3,1,R) \oplus \mathbf{i}\, Cl(3,1,R) \sim M(4,R) \oplus \mathbf{i}\, M(4,R) \quad (2.26c)$$

$$Cl(4,C) \sim M(4,C) \sim Cl(2,2,R) \oplus \mathbf{i}\, Cl(2,2,R) \sim M(4,R) \oplus \mathbf{i}\, M(4,R) \quad (2.26d)$$

$M(4,R), M(4,C)$ is the 4×4 matrix algebra over the reals and complex numbers, respectively. From each one of the $Cl(3,1,R)$ algebra factors in the above decomposition (2.26c) of the complex $Cl(4,C)$ algebra, one can generate a $u(2,2)$ algebra by writing the $u(2,2)$ generators explicitly in terms of the $Cl(3,1,R)$ gamma matrices as displayed above in eqs-(2.7) ; i.e. one may convert a $Cl(3,1,R)$ gauge theory into a Conformal Gravity-Maxwell theory based on $U(2,2) = SU(2,2) \times U(1)$. Therefore, a $Cl(4,C)$ gauge theory is algebraically equivalent to a *bi*-Conformal Gravity-Maxwell theory based on the complex group $U(2,2) \otimes \mathbf{C} = GL(4,C)$; i.e. the $Cl(4,C)$ gauge theory is algebraically equivalent to a *complexified* Conformal Gravity-Maxwell theory in four real dimensions based on the complex algebra $u(2,2) \oplus \mathbf{i}\, u(2,2) = gl(4,C)$. The algebra $gl(N,C)$ is the complex extension of $u(p,q)$ for all p,q such that $p+q=N$.

Furthermore, from each $Cl(3,1,R)$ commuting sub-algebra inside the $Cl(4,C)$ algebra one can also generate a $u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6)$ algebra by writing the latter generators in terms of the $Cl(3,1,R)$ gamma matrices as displayed explicitly in eqs-(2.12). Therefore, the $Cl(4,C)$ gauge theory is also algebraically equivalent to a Yang-Mills gauge theory based on the algebra

$u(4) \oplus \mathbf{i}\ u(4) = gl(4,C)$ and associated with the *two* $Cl(3,1,R)$ commuting sub-algebras inside $Cl(4,C)$. The complex group is $U(4) \otimes \mathbf{C} = GL(4,C)$ also.

From eq-(2.26d) : $Cl(4,C) \sim Cl(4,1,R)$ one learns that the complex Clifford $Cl(4,C)$ algebra is also isomorphic to a *real* Clifford algebra $Cl(4,1,R)$ (and also to $Cl(2,3,R), Cl(0,5,R)$). A Wick rotation (Weyl unitary trick) transforms $Cl(4,1,R) \to Cl(3,2,R) = Cl(3,1,R) \oplus Cl(3,1,R) \sim M(4,R) \oplus M(4,R)$ such that there are two commuting sub-algebras of $Cl(3,2,R)$ which are isomorphic to $Cl(3,1,R)$. From each one of the latter $Cl(3,1,R)$ algebras one can build an $u(4)$ (and $u(2,2)$) algebra as described earlier. A typical example of this feature in ordinary Lie algebras is the case of $so(3) \sim su(2)$ such that there are two commuting sub-algebras of $so(4)$ and isomorphic to $so(3)$ furnishing the decomposition $so(4) = su(2) \oplus su(2) \sim so(3) \oplus so(3)$. Concluding, one can generate a $U(4) \times U(4)$ Yang-Mills gauge theory from a $Cl(4,C)$ gauge theory via a $Cl(4,1,R)$ gauge theory (based on a *real* Clifford algebra) after the Wick rotation (Weyl unitary trick) procedure to the $Cl(3,2,R)$ algebra is performed.

The physical reason why one needs a $U(4) \times U(4)$ Yang-Mills theory is because the group $U(4)$ by itself is *not* large enough to accommodate the Standard Model Group $SU(3) \times SU(2) \times U(1)$ as its maximally compact subgroup [24]. The GUT groups $SU(5), SU(2) \times SU(2) \times SU(4)$ are large enough to achieve this goal. In general, the group $SU(m+n)$ has $SU(m) \times SU(n) \times U(1)$ for compact subgroups. Therefore, $SU(4) \to SU(3) \times U(1)$ or $SU(4) \to SU(2) \times SU(2) \times U(1)$ is allowed but one cannot have $SU(4) \to SU(3) \times SU(2)$. For this reason one cannot rely only on a $Cl(4,C) = Cl(3,1,R) \oplus \mathbf{i}\ Cl(3,1)$ gauge theory to build a unifying model; i.e. because one cannot have the branching $SU(4) \to SU(3) \times SU(2)$, one would not able to generate the full Standard Model group despite that the other group inside $Cl(4,C)$ given by $U(2,2) = SU(2,2) \times U(1)$ furnishes Conformal Gravity *and* Maxwell's Electro-Magnetism based on $U(1)$.

A breaking [28], [31], [5] of $U(4) \times U(4) \longrightarrow SU(2)_L \times SU(2)_R \times SU(4)$ leads to the Pati-Salam [27] GUT group which contains the Standard Model Group, which in turn, breaks down to the ordinary Maxwell Electro-Magnetic (EM) $U(1)_{EM}$ and color (QCD) group $SU(3)_c$ after the following chain of symmetry breaking patterns

$$SU(2)_L \times SU(2)_R \times SU(4) \ \to \ SU(2)_L \times U(1)_R \times U(1)_{B-L} \times SU(3)_c \ \to$$
$$SU(2)_L \times U(1)_Y \times SU(3)_c \ \to \ U(1)_{EM} \times SU(3)_c. \tag{2.27}$$

where $B-L$ denotes the Baryon minus Lepton number charge; $Y =$ hypercharge and the Maxwell EM charge is $Q = I_3 + (Y/2)$ where I_3 is the third component of the $SU(2)_L$ isospin. It is noteworthy to remark that since we had already identified the $U(1)_{EM}$ symmetry stemming from the ($U(2,2)$ group-based) Conformal Gravity-Maxwell sector, it is not necessary to follow the symmetry breaking pattern of the second line in (2.27) in order to retrieve the desired $U(1)_{EM}$ symmetry.

The fermionic matter and Higgs sector of the Standard Model within the context of Clifford gauge field theories has been analyzed in [35]. The 16 fermions of each generation can be assembled into the entries of a 4×4 matrix representation of the $Cl(3,1)$ algebra. A unified model of strong, weak and electromagnetic interactions based on the flavor-color group $SU(4)_f \times SU(4)_c$ of Pati-Salam has been described by Rajpoot and Singer [27]. Fermions were placed in left-right multiplets which transform as the representation $(\bar{4}, 4)$ of $SU(4)_f \times SU(4)_c$. Further investigation is warranted to explore the group $SU(4)_f \times SU(4)_c$ of Pati-Salam within the context of the $U(4) \times U(4)$ group symmetry associated with the $Cl(4,C)$ algebra presented here.

The $u(4)$ algebra can also be realized in terms of $so(8)$ generators, and in general, $u(N)$ algebras admit realizations in terms of $so(2N)$ generators [5]. Given the Weyl-Heisenberg "superalgebra" involving the N fermionic creation and annihilation (oscillators) operators

$$\{a_i, a_j^\dagger\} \; = \; \delta_{ij}, \quad \{a_i, a_j\} \; = \; 0, \; \{a_i^\dagger, a_j^\dagger\} \; = \; 0; \quad i, j \; = \; 1, 2, 3, ... \; N. \tag{2.28}$$

one can find a realization of the $u(N)$ algebra bilinear in the oscillators as $E_i^{\ j} \; = \; a_i^\dagger \; a_j$ and such that the commutators

$$[E_i^{\ j}, \; E_k^{\ l}] \; = \; a_i^\dagger \; a_j \; a_k^\dagger \; a_l \; - \; a_k^\dagger \; a_l \; a_i^\dagger \; a_j \; =$$

$$a_i^\dagger \; (\delta_{jk} - a_k^\dagger \; a_j) \; a_l \; - \; a_k^\dagger \; (\delta_{li} - a_i^\dagger \; a_l) \; a_j \; = \; a_i^\dagger \; (\delta_{jk}) \; a_l \; - \; a_k^\dagger \; (\delta_{li}) \; a_j \; =$$

$$\delta_k^j \; E_i^{\ l} - \; \delta_i^l \; E_k^{\ j}. \tag{2.29}$$

reproduce the commutators of the Lie algebra $u(N)$ since

$$-a_i^\dagger \; a_k^\dagger \; a_j \; a_l \; + \; a_k^\dagger \; a_i^\dagger \; a_l \; a_j \; = \; - \; a_k^\dagger \; a_i^\dagger \; a_l \; a_j \; + \; a_k^\dagger \; a_i^\dagger \; a_l \; a_j \; = \; 0. \tag{2.30}$$

due to the anti-commutation relations (2.28) yielding a double negative sign $(-)(-) = +$ in (2.30). Furthermore, one also has an explicit realization of the Clifford algebra $Cl(2N)$ Hermitian generators by defining the even-number and odd-number generators as

$$\Gamma_{2j} \; = \; \frac{1}{2} \; (a_j \; + \; a_j^\dagger); \quad \Gamma_{2j-1} \; = \; \frac{1}{2i} \; (a_j \; - \; a_j^\dagger). \tag{2.31}$$

The Hermitian generators of the $so(2N)$ algebra are defined as usual $\Sigma_{mn} = \frac{i}{2}[\Gamma_m, \Gamma_n]$ where $m, n \; = \; 1, 2, ...2N$. Therefore, the $u(4), so(8), Cl(8)$ algebras admit an explicit realization in terms of the fermionic Weyl-Heisenberg oscillators $a_i, a_j^\dagger$ for $i, j \; = \; 1, 2, 3, 4$. $u(4)$ is a subalgebra of $so(8)$ which in turn is a subalgebra of the $Cl(8)$ algebra. The Conformal algebra in $8D$ is $so(8,2)$ and also admits an explicit realization in terms of the $Cl(8)$ generators, similar to the realization of the algebra $so(4,2) \sim su(2,2)$ in terms of the $Cl(3,1,R)$ generators as displayed in eq- (2.7). The compact version of the group $SO(8,2)$ is $SO(10)$ which is a GUT group candidate. In particular, the algebras $u(5)$, $so(10)$, $Cl(10)$ admit a realization in terms of the fermionic Weyl-Heisenberg oscillators $a_i, a_j^\dagger$ for $i, j = 1, 2, 3, 4, 5$.

Conclusion : The upshot of the $Cl(5,C) \; = \; Cl(4,C) \oplus Cl(4,C)$ algebraic structure of the Conformal Gravity, Maxwell, $U(4) \times U(4)$ Yang-Mills unification program advanced in this work is that the group structure given by the *direct* products

$$U(2,2)\times U(4)\times U(4) \ = \ [SU(2,2)]_{spacetime}\times[U(1)\times U(4)\times U(4)]_{internal} \quad (2.32)$$

is ultimately tied down to four-dimensions and does *not* violate the Coleman-Mandula theorem because the spacetime symmetries (conformal group $SU(2,2)$ in the absence of a mass gap, Poincare group when there is mass gap) do *not* mix with the internal symmetries. Similar considerations apply to the supersymmetric case when the symmetry group structure is given by the *direct* product of the superconformal group (in the absence of a mass gap) with an internal symmetry group so that the Haag-Lopuszanski-Sohnius theorem is not violated.

3 Generalized Gauge Theories of Gravity based on $Cl(4,1,R), Cl(3,2,R)$ Algebras

We saw in the last section that the complex Clifford algebra $Cl(4,C) \sim M(4,C) \sim Cl(4,1,R)$ is isomorphic to a *real* Clifford algebra $Cl(4,1,R)$ which contains the de Sitter algebra $so(4,1)$. In this section we will construct generalized gauge theories of de Sitter ($SO(4,1)$) and Anti de Sitter Gravity ($SO(3,2)$) based on the real Clifford $Cl(4,1,R), Cl(3,2,R)$ Algebras. The $Cl(4,1,R), Cl(3,2,R)$ algebra-valued gauge field is defined as

$$\mathbf{A} \ = \ A_\mu \, \mathbf{1} + A^m_\mu \, \Gamma_m + A^{mn}_\mu \, \Gamma_{mn} + A^{mnp}_\mu \, \Gamma_{mnp} + A^{mnpq}_\mu \, \Gamma_{mnpq} + A^{mnpqr}_\mu \, \Gamma_{mnpqr} \quad (3.1)$$

the spacetime indices are $\mu = 1,2,3,4$ as before. The gamma generators are

$$\Gamma_I \ : \ \mathbf{1}; \ \Gamma_m = \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4, \ \Gamma_5; \ \Gamma_{m_1 m_2} = \frac{1}{2}\Gamma_{m_1} \wedge \Gamma_{m_2} = \frac{1}{2}[\Gamma_{m_1}, \Gamma_{m_2}];$$

$$\Gamma_{m_1 m_2 m_3} \ = \ \frac{1}{3!} \, \Gamma_{m_1}\wedge\Gamma_{m_2}\wedge\Gamma_{m_3}; \ ...., \ \Gamma_{m_1 m_2 ...... m_5} \ = \ \frac{1}{5!} \, \Gamma_{m_1}\wedge\Gamma_{m_2}\wedge.........\wedge\Gamma_{m_5} \quad (3.2)$$

the indices $m_1, m_2, ...$ run from $1,2,3,4,5$. The above decomposition of the connection $\mathcal{A}_\mu = \mathcal{A}^I_\mu \, \Gamma_I$ contains Hermitian *and* anti-Hermitian components (generators). It is common practice to split the de Sitter/Anti de Sitter algebra gauge connection in $4D$ into a (Lorentz) rotational piece $\omega^{a_1 a_2}_\mu \Gamma_{a_1 a_2}$ where $a_1, a_2 = 1,2,3,4$; $\mu, \nu = 1,2,3,4$, and a momentum piece $\omega^{a5}_\mu \Gamma_{a5} = \frac{1}{l} V^a_\mu P_a$, where V^a is the physical vielbein field, l is the de Sitter/Anti de Sitter throat size, and P_a is the momentum generator whose indices span $a = 1,2,3,4$. One may proceed in the same fashion in the Clifford algebra $Cl(3,2), Cl(4,1), ...$ case. The poly-momentum generator corresponds to those poly-rotations with a component along the 5-th direction in the *internal* space.

Therefore, one may assign

$$\Gamma_5 \;=\; P_{\mathbf{0}};\;\; \Gamma_{a5} \;= l\; P_a,\; a = 1,2,3,4;\;\; \Gamma_{a_1 a_2 5} = l^2\; P_{a_1 a_2},\; a_1, a_2 = 1,2,3,4$$

$$\Gamma_{a_1 a_2 a_3 5} \;=\; l^3\; P_{a_1 a_2 a_3},\; a_1, a_2, a_3 = 1,2,3,4$$

$$\Gamma_{a_1 a_2 a_3 a_4 5} \;=\; l^4\; P_{a_1 a_2 a_3 a_4},\; a_1, a_2, a_3, a_4 = 1,2,3,4; \qquad (3.3)$$

In this way the 16 components of the (*noncommutative*) poly-momentum operator $P_A = P_{\mathbf{0}}, P_a, P_{a_1 a_2}, P_{a_1 a_2 a_3}, P_{a_1 a_2 a_3 a_4}$ are identified with those poly-rotations with a component along the 5-th direction in the *internal* space. A length scale l is needed to match dimensions.

$P_{\mathbf{0}}$ does not transform as a $Cl(3,2), Cl(4,1)$ algebra scalar, but as a vector. P_a does not transform as a $Cl(3,2), Cl(4,1)$ vector but as a bivector. $P_{a_1 a_2}$ does not transform as $Cl(3,2), Cl(4,1)$ bivector but as a trivector, etc.... What about under $Cl(3,1)$ transformations ? One can notice $[\Gamma_{ab}, \Gamma_5] = [\Gamma_{ab}, P_0] = 0$ when $a, b = 1,2,3,4$. Thus under rotations along the four dimensional subspace, $\Gamma_5 = P_0$ is inert, it behaves like a scalar from the four-dimensional point of view. This justifies the labeling of Γ_5 as P_0. The commutator

$$[\Gamma_{ab},\; \Gamma_{c5}] \;=\; [\Gamma_{ab},\; l\; P_c] \;=\; -\eta_{ac}\Gamma_{b5} \;+\; \eta_{bc}\Gamma_{a5} \;=\; -\eta_{ac}\; l\; P_b \;+\; \eta_{bc}\; l\; P_a \quad (3.4)$$

so that $\Gamma_{c5} = lP_c$ does behave like a vector under rotations along the four-dim subspace. Thus this justifies the labeling of Γ_{c5} as lP_c, etc...

To sum up, one has split the $Cl(3,2), Cl(4,1)$ gauge algebra generators into two sectors. One sector represented by $\mathcal{M}$ which comprises poly-rotations along the $four$-dim subspace involving the generators

$$\mathbf{1};\;\; \Gamma_{a_1};\; \Gamma_{a_1 a_2};\;\; \Gamma_{a_1 a_2 a_3};\;\; \Gamma_{a_1 a_2 a_3 a_4},\;\; a_1, a_2, a_3, a_4 = 1,2,3,4. \qquad (3.5)$$

and another sector represented by $\mathcal{P}$ involving poly-rotations with one coordinate pointing along the internal 5-th direction as displayed in (2.8).

Thus their commutation relations are of the form

$$[\mathcal{P},\; \mathcal{P}] \;\sim\; \mathcal{M};\;\; [\mathcal{M},\; \mathcal{M}] \;\sim\; \mathcal{M};\;\; [\mathcal{M},\; \mathcal{P}] \;\sim\; \mathcal{P}. \qquad (3.6)$$

which are compatible with the commutators of the Anti de Sitter, de Sitter algebra $SO(3,2), SO(4,1)$ respectively. To sum up, we have decomposed the $Cl(3,2), Cl(4,1)$ gauge connection one-form in a $4D$ spacetime as

$$\mathcal{A}_\mu\, dx^\mu \;=\; \mathcal{A}^I_\mu \Gamma_I\, dx^\mu \;=\; \left(\Omega^A_\mu\; \Gamma_A \;+\; E^A_\mu\; P_A\right)\, dx^\mu;\;\; \Gamma_A \subset \mathcal{M},\;\; P_A \subset \mathcal{P} \quad (3.7)$$

The components of the generalized curvature 2-form are defined by

$$\mathcal{R}^{a_1 a_2}_{\mu\;\nu} \;=\; \partial_{[\mu}\, \Omega^{a_1 a_2}_{\nu]} + \Omega^m_\mu\; \Omega^r_\nu\; < [\gamma_m, \gamma_r]\, \gamma^{a_1 a_2} \;> + \Omega^{mn}_\mu\; \Omega^{rs}_\nu\; < [\gamma_{mn}, \gamma_{rs}]\, \gamma^{a_1 a_2} \;> +$$

$$\Omega^{mnp}_\mu\; \Omega^{rst}_\nu\; < [\gamma_{mnp}, \gamma_{rst}]\, \gamma^{a_1 a_2} \;> + \Omega^{mnpq}_\mu\; \Omega^{rstu}_\nu\; < [\gamma_{mnpq}, \gamma_{rstu}]\, \gamma^{a_1 a_2} \;> +$$

$$\Omega_\mu^{mnpqk}\ \Omega_\nu^{rstuv}\ <\ [\gamma_{mnpqk},\gamma_{rstuv}]\ \gamma^{a_1a_2}\ >. \tag{3.8}$$

where the brackets $<\ [\gamma_{mn},\gamma_r]\ \gamma^a\ >, <\ [\gamma_{mnpq},\gamma_{rst}]\ \gamma^a\ >$ indicate the *scalar* part of the product of the $Cl(4,1,R), Cl(3,2,R)$ algebra elements; i.e it extracts the $Cl(4,1,R), Cl(3,2,R)$ invariant contribution. For example,

$$<\ [\gamma_{mn},\gamma_r]\ \gamma^a\ >\ =\ <\ -\eta_{mr}\ \gamma_n\ \gamma^a\ >\ +\ <\ \eta_{nr}\gamma_m\gamma^a\ >\ =\ -\eta_{mr}\ \delta_n^a + \eta_{nr}\ \delta_m^a. \tag{3.9}$$

The standard curvature tensor is given by

$$R_{\mu\ \nu}^{a_1a_2}\ =\ \partial_{[\mu}\ \Omega_{\nu]}^{a_1a_2}\ +\ \Omega_\mu^{mn}\ \Omega_\nu^{rs}\ <\ [\gamma_{mn},\gamma_{rs}]\ \gamma^{a_1a_2}\ >. \tag{3.10}$$

which clearly differs from the modified expression in (3.8). Since the indices m,n,r,s in general run from $1,2,3,4,5$ the standard curvature two-form becomes

$$R_{\mu\nu}^{a_1a_2}\ dx^\mu\ \wedge\ dx^\nu\ =\ \mathbf{d}\Omega^{a_1a_2}\ +\ \Omega_{\ c}^{a_1}\ \wedge\ \Omega^{ca_2}\ -\eta_{55}\ \Omega^{a_15}\ \wedge\ \Omega^{a_25}\ =$$

$$\mathbf{d}\Omega^{a_1a_2}\ +\ \Omega_{\ c}^{a_1}\ \wedge\ \Omega^{ca_2}\ -\eta_{55}\ \frac{1}{l^2}V^{a_1}\ \wedge\ V^{a_2};\quad \Omega^{a5} = \frac{1}{l}V^a \tag{3.11}$$

where the vielbein one-form is $V^a = V_\mu^a dx^\mu$. In the $l\to\infty$ limit the last terms $\frac{1}{l^2}V^{a_1}\wedge V^{a_2}$ in (3.11) decouple and one recovers the standard Riemmanian curvature two-form in terms of the spin connection one form $\omega^{a_1a_2} = \omega_\mu^{a_1a_2}dx^\mu$ and the exterior derivative operator $\mathbf{d} = dx^\mu\partial_\mu$. From (3.11) one infers that a vacuum solution $R_{\mu\nu}^{a_1a_2} = 0$ in de Sitter/ Anti de Sitter gravity leads to the relation

$$R^{a_1a_2}(\omega)\ \equiv\ \mathbf{d}\omega^{a_1a_2}\ +\ \omega_{\ c}^{a_1}\ \wedge\ \omega^{ca_2}\ =\ \frac{1}{l^2}\ \eta_{55}\ V^{a_1}\ \wedge\ V^{a_2} \tag{3.12}$$

which is tantamount to having a constant Riemannian scalar curvature in $4D$ $R(\omega) = \pm(12/l^2)$ and a cosmological constant $\Lambda = \pm(3/l^2)$; the positive (negative) sign corresponds to de Sitter (anti de Sitter space) respectively ; i.e. the de Sitter/ Anti de Sitter gravitational *vacuum* solutions are solutions of the Einstein field equations *with* a non-vanishing cosmological constant.

A different approach to the cosmological constant problem can be taken as follows. The *modified* curvature tensor in (3.8) is

$$\mathcal{R}_{\mu\ \nu}^{a_1a_2}\ =\ R_{\mu\nu}^{a_1a_2}\ +\ extra\ terms\ =$$

$$\mathbf{d}\omega^{a_1a_2}\ +\ \omega_{\ c}^{a_1}\ \wedge\ \omega^{ca_2}\ -\eta_{55}\ \frac{1}{l^2}V^{a_1}\ \wedge\ V^{a_2}\ +\ extra\ terms \tag{3.13}$$

The extra terms in (3.13) involve the second and third lines of eq-(3.8). The vacuum solutions $\mathcal{R}_{\mu\nu}^{a_1a_2} = 0$ in (3.13) imply that

$$\mathbf{d}\omega^{a_1a_2}\ +\ \omega_{\ c}^{a_1}\ \wedge\ \omega^{ca_2}\ =\ \frac{1}{l^2}\ \eta_{55}\ V^{a_1}\ \wedge\ V^{a_2}\ -\ extra\ terms. \tag{3.14}$$

Consequently, as a result of the *extra* terms in the right hand side of (3.13) obtained from the extra terms in the definition of $\mathcal{R}^{a_1 a_2}_{\mu\nu}$ in (3.8), it could be possible to have a cancellation of a cosmological constant term associated to a very large vacuum energy density $\rho \sim (L_{Planck})^{-4}$; i.e. one would have an $effective$ zero value of the cosmological constant despite the fact that the length scale in eq-(3.14) might be set to $l \sim L_{Planck}$.

For instance, one could have a cancellation (after neglecting the terms of higher order rank in eq-(3.14)) to the contribution of the cosmological constant as follows

$$\Omega^m_\mu \ \Omega^n_\nu \ < \ [\gamma_m, \gamma_r] \ \gamma^{a_1 a_2} \ > \ + \ \Omega^{m5}_\mu \ \Omega^{r5}_\nu \ < \ [\gamma_{m5}, \gamma_{r5}] \ \gamma^{a_1 a_2} \ > \ = \ 0 \ \Rightarrow$$

$$\Omega^{a_1} \ \wedge \ \Omega^{a_2} \ - \eta_{55} \ \Omega^{a_1 5} \ \wedge \ \Omega^{a_2 5} = \ 0. \qquad (3.15a)$$

Since the $Cl(3,2)$ algebra corresponds to the Anti de Sitter algebra $SO(3,2)$ case one has

$$\eta_{55} = -1 \Rightarrow \ \frac{V^a}{l} \ = \ \Omega^{a5}_\mu \ = \ \pm \ i \ \Omega^a_\mu \qquad (3.15b)$$

Hence, one can attain a cancellation of a very large cosmological constant term in (3.15) if $\Omega^{a5}_\mu = \pm i \ \Omega^a_\mu$. In the de Sitter case the group is $SO(4,1)$ so $\eta_{55} = 1$ and one would have instead the condition $\Omega^{a5}_\mu = \pm \Omega^a_\mu$ leading to a cancellation of a very large value of the cosmological constant when $l = L_{Planck}$. Having an imaginary value for Ω^a_μ in the Anti de Sitter case fits into a gravitational theory involving a complex Hermitian metric $G_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ which is associated to a complex tetrad $E^a_\mu = \frac{1}{\sqrt{2}}(\tilde{e}^a_\mu + i \tilde{f}^a_\mu)$ such that $G_{\mu\nu} = (E^a_\mu)^* E^b_\nu \eta_{ab}$ and the fields are constrained to obey $\tilde{e}^a_\mu = V^a_\mu; i \tilde{f}^a_\mu = i V^a_\mu = \mp l \ \Omega^a_\mu$. For further details on complex metrics (gravity) in connection to Born's reciprocity principle of relativity [40], [41] involving a maximal speed and maximum proper force see [42] and references therein.

The $modified$ torsion is

$$\mathcal{T}^a_{\mu \ \nu} \ = \mathcal{R}^{a5}_{\mu \ \nu} \ = \partial_{[\mu} \ \Omega^{a5}_{\nu]} \ +$$

$$\Omega^m_\mu \ \Omega^r_\nu \ < \ [\gamma_m, \gamma_r] \ \gamma^{a5} \ > \ + \ \Omega^{mn}_\mu \ \Omega^{rs}_\nu \ < \ [\gamma_{mn}, \gamma_{rs}] \ \gamma^{a5} \ > \ +$$

$$\Omega^{mnp}_\mu \ \Omega^{rst}_\nu \ < \ [\gamma_{mnp}, \gamma_{rst}] \ \gamma^{a5} \ > \ + \ \Omega^{mnpq}_\mu \ \Omega^{rstu}_\nu \ < \ [\gamma_{mnpq}, \gamma_{rstu}] \ \gamma^{a5} \ > \ +$$

$$\Omega^{mnpqk}_\mu \ \Omega^{rstuv}_\nu \ < \ [\gamma_{mnpqk}, \gamma_{rstuv}] \ \gamma^{a5} \ > . \qquad (3.16)$$

Form (3.16) one can see that the $Cl(3,2), Cl(4,1)$-algebraic expression for the torsion $\mathcal{T}^a_{\mu \ \nu}$ contains many *more* terms than the standard expression for the torsion in Riemann-Cartan spacetimes

$$T^a_{\mu\nu} \ dx^\mu \ \wedge \ dx^\nu \ = \ R^{a5}_{\mu\nu} \ dx^\mu \ \wedge \ dx^\nu \ = \ l \ (\mathbf{d} \ \Omega^{a5} \ + \ \Omega^a{}_b \wedge \Omega^{b5} \ ) \ =$$

$$\mathbf{d} \ V^a \ + \ \Omega^a{}_b \wedge V^b. \qquad (3.17)$$

The vielbein one-form is $V^a = V^a_\mu dx^\mu = l\ \Omega^{a5}_\mu dx^\mu$ and the spin connection one-form is $\Omega^{ab} = \Omega^{ab}_\mu dx^\mu$ (it is customary to denote the spin connection by ω^{ab}_μ instead).

The analog of the Abelian $U(1)$ field strength sector is $\mathcal{F}^{\mathbf{0}}_{\mu\nu} = \partial_{[\mu}\ \Omega^0_{\nu]}$. The other relevant components of the $Cl(3,2)$-valued gauge field strengths/curvatures $F^A_{\mu\nu}$ ($\mathcal{R}^A_{\mu\nu}$) are

$$\mathcal{R}^a_{\mu\nu} = \partial_{[\mu}\ \Omega^a_{\nu]} + \Omega^{mn}_\mu\ \Omega^r_\nu < [\gamma_{mn}, \gamma_r]\ \gamma^a > + \Omega^{mnpq}_\mu\ \Omega^{rst}_\nu < [\gamma_{mnpq}, \gamma_{rst}]\ \gamma^a > . \tag{3.18}$$

A quadratic $Cl(3,2), Cl(4,1)$ gauge invariant action in a $4D$ spacetime involving the modified curvature $\mathcal{R}^A_{\mu\nu}$ and torsion terms $\mathcal{T}^A_{\mu\nu}$ is given by

$$\int d^4x\ \sqrt{|g|}\ [\ (\mathcal{R}^{\mathbf{0}}_{\mu\nu})^2 + (\mathcal{R}^a_{\mu\nu})^2 + (\mathcal{R}^{a_1a_2}_{\mu\nu})^2 + \ ....\ (\mathcal{R}^{a_1a_2a_3a_4}_{\mu\nu})^2\ +$$

$$(\mathcal{R}^{\mathbf{5}}_{\mu\nu})^2 + (\mathcal{R}^{a5}_{\mu\nu})^2 + (\mathcal{R}^{a_1a_5}_{\mu\nu})^2 + \ ....\ (\mathcal{R}^{a_1a_2a_35}_{\mu\nu})^2 + (\mathcal{R}^{a_1a_2a_3a_45}_{\mu\nu})^2\ ] \tag{3.19}$$

The modifications to the ordinary scalar Riemmanian curvature $R(\omega)$ is given in terms of the inverse vielbein V^μ_a by the expression $\mathcal{R}^{a_1a_2}_{\mu\nu} V^{[\mu}_{[a_1} V^{\nu]}_{a_2]}$ which is comprised of $R(\omega)$, plus the cosmological constant term , plus the extra terms stemming from the additional connection pieces in (3.8)

$$\Omega^{a_1} \wedge \Omega^{a_2},\ \ \Omega^{a_1}{}_{b_1b_2} \wedge \Omega^{b_1b_2a_2},\ ....,\ \Omega^{a_1}{}_{b_1b_2b_3b_4} \wedge \Omega^{b_1b_2b_3b_4a_2} \tag{3.20}$$

One can introduce an $SO(3,2), SO(4,1)$-valued scalar multiplet $\phi^1, \phi^2, ..., \phi^5$ and construct an $SO(3,2), SO(4,1)$ invariant action of the form

$$S = \int_M d^4x\ \left(\ \phi^5\ \mathcal{R}^{ab}_{\mu\nu}\ \mathcal{R}^{cd}_{\rho\sigma}\ +\ \phi^a\ \mathcal{R}^{bc}_{\mu\nu}\ \mathcal{R}^{d5}_{\rho\sigma}\ +\ ....\ \right)\ \epsilon_{abcd5}\ \epsilon^{\mu\nu\rho\sigma}. \tag{3.21}$$

As described above the *modified* curvature two-form $\mathcal{R}^{ab}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ is given by the standard expression $R^{ab}_{\mu\nu}(\omega)\ dx^\mu \wedge dx^\nu + \frac{1}{l^2} V^a_\mu dx^\mu \wedge V^b_\nu dx^\nu$ *plus* the addition of many *extra* terms as shown in (3.8, 3.20). Also the modified torsion $\mathcal{R}^{a5}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ in (3.16) is given by the standard torsion expression *plus* extra terms. Therefore, by a simple inspection, the action (3.21) contains many *more* terms than the Macdowell-Mansouri-Chamseddine-West gravitational action given by eq-(2.20).

An invariant action linear in the curvature is

$$S = \frac{1}{2\kappa^2} \int d^4x\ \sqrt{|g|}\ \mathcal{R}^{a_1a_2}_{\mu\nu}\ V^{[\mu}_{[a_1}\ V^{\nu]}_{a_2]};\quad g_{\mu\nu} = V^a_\mu V^b_\nu\ \eta_{ab},\quad |g| = |det\ g_{\mu\nu}|. \tag{3.22}$$

where $\kappa^2 = 8\pi G_N$, G_N is the Newtonian gravitational constant, V^μ_a is the inverse vielbein and the components of the curvature two-form are antisymmetric under the exchange of indices by construction $\mathcal{R}^{a_1a_2}_{\mu\nu} = -\mathcal{R}^{a_1a_2}_{\nu\mu}$, $\mathcal{R}^{a_1a_2}_{\mu\nu} = -\mathcal{R}^{a_2a_1}_{\mu\nu}$. The action (3.22) contains clear *modifications* to the Einstein-Hilbert action due to the extra terms stemming from the corrections to the curvature as shown by eq-(3.8, 3.20).

The generalized gravitational theory based on the $Cl(4,1,R) \sim Cl(4,C)$ and $Cl(3,2,R)$ algebras, must not be confused with a Metric Affine Gravitational (MAG) theory based on the complex affine group $GA(4,C) = GL(4,C) \times_s C^4$ given by the semi-direct product of $GL(4,C)$ with the translations group in C^4 and involving $32 + 8 = 40$ generators. The real MAG based on $GA(4,R) = GL(4,R) \times_s R^4$ is a very intricate non-Riemannian theory of gravity with propagating non-metricity and torsion [16]. The most general Renormalizable Lagrangian of MAG contains a very large number of terms. We refer to [16] for an extensive list of references. The rich particle classification and dynamics in $GL(2,C)$ Gravity was analyzed by [37]. In addition to orbits associated with standard massive and massless particles, a number of novel orbits can be identified based on the quadratic and quartic Casimirs invariants of $GL(2,C)$. Noncommutative generalizations of $GL(2,C)$ gravity based on star products and the Seiberg-Witten map should be straightforward [19].

The $Cl(5,C)$ algebra-valued gauge field theory defined over a $4D$ real spacetime raises the possibility of embedding this gauge theory into one defined over the full fledged Clifford-space (C-space) associated with the tangent space of a $complexified$ $5D$ spacetime. Namey, having the ordinary one-forms $(\mathcal{A}^I_\mu \, \Gamma_I) \, dz^\mu$ of a complexified $5D$ spacetime extended to polyvector-valued $(\mathcal{A}^I_M \; \Gamma_I) \; dZ^M$ differential forms defined over the complex Clifford-space (C-space) associated with the complexified $Cl(5,C)$ algebra. Z^M is a polyvector valued coordinate corresponding to the complex Clifford-space. Since a complexified $5D$ spacetime has 10 real-dimensions, this is a very suggestive task due to the fact that 10-dimensions are the critical dimensions of an anomaly-free quantum superstring theory [30]. Since twistors admit a natural reformulation in terms of Clifford algebras [38], and in section **2** we displayed the realization of the superconformal $su(2,2|1)$ algebra generators explicitly in terms of Clifford algebra generators [34], it is very natural to attempt to reformulate Witten's twistor-string picture [39] of $\mathcal{N} = 4$ super Yang-Mills theory from the perspective of Clifford algebras, mainly because C-space is the natural background where point particles, strings, membranes, ... , p-branes propagate [18] .

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