Gravitation on a Spherically Symmetric Metric Manifold

Stephen J. Crothers

Queensland, Australia

E-mail: thenarmis@yahoo.com

The usual interpretations of solutions for Einstein's gravitational field satisfying the spherically symmetric condition contain anomalies that are not mathematically permissible. It is shown herein that the usual solutions must be modified to account for the intrinsic geometry associated with the relevant line elements.

1 Introduction

The standard interpretation of spherically symmetric line elements for Einstein's gravitational field has not taken into account the fundamental geometrical features of spherical symmetry about an arbitrary point in a metric manifold. This has led to numerous misconceptions as to distance and radius that have spawned erroneous theoretical notions.

The nature of spherical symmetry about an arbitrary point in a three dimensional metric manifold is explained herein and applied to Einstein's gravitational field.

It is plainly evident, *res ipsa locquitur*, that the standard claims for black holes and Big Bang cosmology are not consistent with elementary differential geometry and are consequently inconsistent with General Relativity.

2 Spherical symmetry of three-dimensional metrics

Denote ordinary Efcleethean* 3-space by $\mathbf{E}^3$. Let $\mathbf{M}^3$ be a 3-dimensional metric manifold. Let there be a one-to-one correspondence between all points of $\mathbf{E}^3$ and $\mathbf{M}^3$. Let the point $O \in \mathbf{E}^3$ and the corresponding point in $\mathbf{M}^3$ be O'. Then a point transformation T of $\mathbf{E}^3$ into itself gives rise to a corresponding point transformation of $\mathbf{M}^3$ into itself.

A rigid motion in a metric manifold is a motion that leaves the metric $d\ell'^2$ unchanged. Thus, a rigid motion changes geodesics into geodesics. The metric manifold $\mathbf{M}^3$ possesses spherical symmetry around any one of its points O' if each of the ∞^3 rigid rotations in $\mathbf{E}^3$ around the corresponding arbitrary point O determines a rigid motion in $\mathbf{M}^3$.

The coefficients of $d\ell'^2$ of $\mathbf{M}^3$ constitute a metric tensor and are naturally assumed to be regular in the region around every point in $\mathbf{M}^3$, except possibly at an arbitrary point, the centre of spherical symmetry $O' \in \mathbf{M}^3$.

Let a ray i emanate from an arbitrary point $O \in \mathbf{E}^3$. There is then a corresponding geodesic $i' \in \mathbf{M}^3$ issuing from the corresponding point $O' \in \mathbf{M}^3$. Let P be any point on i other than O. There corresponds a point P' on $i' \in \mathbf{M}^3$ different to O'. Let g' be a geodesic in $\mathbf{M}^3$ that is tangential to i' at P'.

Taking i as the axis of ∞^1 rotations in $\mathbf{E}^3$, there corresponds ∞^1 rigid motions in $\mathbf{M}^3$ that leaves only all the points on i' unchanged. If g' is distinct from i', then the ∞^1 rigid rotations in $\mathbf{E}^3$ about i would cause g' to occupy an infinity of positions in $\mathbf{M}^3$ wherein g' has for each position the property of being tangential to i' at P' in the same direction, which is impossible. Hence, g' coincides with i'.

Thus, given a spherically symmetric surface Σ in $\mathbf{E}^3$ with centre of symmetry at some arbitrary point $O \in \mathbf{E}^3$, there corresponds a spherically symmetric geodesic surface Σ' in $\mathbf{M}^3$ with centre of symmetry at the corresponding point $O' \in \mathbf{M}^3$.

Let Q be a point in $\Sigma \in \mathbf{E}^3$ and Q' the corresponding point in $\Sigma' \in \mathbf{M}^3$. Let $d\sigma$ be a generic line element in Σ issuing from Q. The corresponding generic line element $d\sigma' \in \Sigma'$ issues from the point Q'. Let Σ be described in the usual spherical-polar coordinates r, θ, φ. Then

$$d\sigma^2 = r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{1}$$

$$r = |\overline{OQ}|.$$

Clearly, if r, θ, φ are known, Q is determined and hence also Q' in Σ'. Therefore, θ and φ can be considered to be curvilinear coordinates for Q' in Σ' and the line element $d\sigma' \in \Sigma'$ will also be represented by a quadratic form similar to (1). To determine $d\sigma'$, consider two elementary arcs of equal length, $d\sigma_1$ and $d\sigma_2$ in Σ, drawn from the point Q in different directions. Then the homologous arcs in Σ' will be $d\sigma'_1$ and $d\sigma'_2$, drawn in different directions from the corresponding point Q'. Now $d\sigma_1$ and $d\sigma_2$ can be obtained from one another by a rotation about the axis $\overline{OQ}$ in $\mathbf{E}^3$, and so $d\sigma'_1$ and $d\sigma'_2$ can be obtained from one another by a rigid motion in $\mathbf{M}^3$, and are therefore also of equal length, since the metric is unchanged by such a motion. It therefore follows that the ratio $\frac{d\sigma'}{d\sigma}$ is the same for the two different directions irrespective of $d\theta$ and $d\varphi$, and so the foregoing ratio is a function of position, i.e. of r, θ, φ. But Q is an arbitrary point in Σ, and so $\frac{d\sigma'}{d\sigma}$ must have the same ratio for any corresponding points Q and Q'. Therefore, $\frac{d\sigma'}{d\sigma}$ is a function of r alone, thus

$$\frac{d\sigma'}{d\sigma} = H(r),$$

and so

$$d\sigma'^2 = H^2(r)\, d\sigma^2 = H^2(r)\, r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2}$$

*For the geometry due to Efcleethees, usually and abominably rendered as Euclid.

where $H(r)$ is *a priori* unknown. For convenience set $R_c = R_c(r) = H(r)\,r$, so that (2) becomes

$$d\sigma'^2 = R_c^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3}$$

where R_c is a quantity associated with $\mathbf{M}^3$. Comparing (3) with (1) it is apparent that R_c is to be rightly interpreted in terms of the Gaussian curvature K at the point Q', i.e. in terms of the relation $K = \frac{1}{R_c^2}$ since the Gaussian curvature of (1) is $K = \frac{1}{r^2}$. This is an intrinsic property of all line elements of the form (3) [1, 2]. Accordingly, R_c can be regarded as a radius of curvature. Therefore, in (1) the radius of curvature is $R_c = r$. Moreover, owing to spherical symmetry, all points in the corresponding surfaces Σ and Σ' have constant Gaussian curvature relevant to their respective manifolds and centres of symmetry, so that all points in the respective surfaces are umbilic.

Let the element of radial distance from $O \in \mathbf{E}^3$ be dr. Clearly, the radial lines issuing from O cut the surface Σ orthogonally. Combining this with (1) by the theorem of Pythagoras gives the line element in $\mathbf{E}^3$

$$d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4}$$

Let the corresponding radial geodesic from the point $O' \in \mathbf{M}^3$ be dg. Clearly the radial geodesics issuing from O' cut the geodesic surface Σ' orthogonally. Combining this with (3) by the theorem of Pythagoras gives the line element in $\mathbf{M}^3$ as,

$$d\ell'^2 = dg^2 + R_c^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{5}$$

where dg is, by spherical symmetry, also a function only of R_c. Set $dg = \sqrt{B(R_c)}dR_c$, so that (5) becomes

$$d\ell'^2 = B(R_c)dR_c^2 + R_c^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{6}$$

where $B(R_c)$ is an *a priori* unknown function.

Setting $dR_p = \sqrt{B(R_c)}dR_c$ carries (6) into

$$d\ell'^2 = dR_p^2 + R_c^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{7}$$

Expression (6) is the most general for a metric manifold $\mathbf{M}^3$ having spherical symmetry about some arbitrary point $O' \in \mathbf{M}^3$ [1, 3].

Considering (4), the distance $R_p = |\overline{OQ}|$ from the point at the centre of spherical symmetry O to a point $Q \in \Sigma$, is given by

$$R_p = \int_0^r dr = r = R_c\,.$$

Call R_p the proper radius. Consequently, in the case of $\mathbf{E}^3$, R_p and R_c are identical, and so the Gaussian curvature at any point in $\mathbf{E}^3$ can be associated with R_p, the radial distance between the centre of spherical symmetry at the point $O \in \mathbf{E}^3$ and the point $Q \in \Sigma$. Thus, in this case, we have $K = \frac{1}{R_c^2} = \frac{1}{R_p^2} = \frac{1}{r^2}$. However, this is not a general relation, since according to (6) and (7), in the case of $\mathbf{M}^3$, the radial geodesic distance from the centre of spherical symmetry at the point $O' \in \mathbf{M}^3$ is not given by the radius of curvature, but by

$$R_p = \int_0^{R_p} dR_p = \int_{R_c(0)}^{R_c(r)} \sqrt{B(R_c(r))}\, dR_c(r) = \int_0^r \sqrt{B(R_c(r))}\, \frac{dR_c(r)}{dr}\, dr\,,$$

where $R_c(0)$ is *a priori* unknown owing to the fact that $R_c(r)$ is *a priori* unknown. One cannot simply assume that because $0 \leqslant r < \infty$ in (4) that it must follow that in (6) and (7) $0 \leqslant R_c(r) < \infty$. In other words, one cannot simply assume that $R_c(0) = 0$. Furthermore, it is evident from (6) and (7) that R_p determines the radial geodesic distance from the centre of spherical symmetry at the arbitrary point O' in $\mathbf{M}^3$ (and correspondingly so from O in $\mathbf{E}^3$) to another point in $\mathbf{M}^3$. Clearly, R_c does not in general render the radial geodesic length from the centre of spherical symmetry to some other point in a metric manifold. Only in the particular case of $\mathbf{E}^3$ does R_c render both the Gaussian curvature and the radial distance from the centre of spherical symmetry, owing to the fact that R_p and R_c are identical in that special case.

It should also be noted that in writing expressions (4) and (5) it is implicit that $O \in \mathbf{E}^3$ is defined as being located at the origin of the coordinate system of (4), i.e. O is located where $r = 0$, and by correspondence O' is defined as being located at the origin of the coordinate system of (5), i.e. using (7), $O' \in \mathbf{M}^3$ is located where $R_p = 0$. Furthermore, since it is well known that a geometry is completely determined by the form of the line element describing it [4], expressions (4) and (6) share the very same fundamental geometry because they are line elements of the same form.

Expression (6) plays an important rôle in Einstein's gravitational field.

3 The standard solution

The standard solution in the case of the static vacuum field (i.e. no deformation of the space) of a single gravitating body, satisfying Einstein's field equations $R_{\mu\nu} = 0$, is (using $G = c = 1$),

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{8}$$

where m is allegedly the mass causing the field, and upon which it is routinely claimed that $2m < r < \infty$ is an exterior region and $0 < r < 2m$ is an interior region. Notwithstanding the inequalities it is routinely allowed that $r = 2m$ and $r = 0$ by which it is also routinely claimed that $r = 2m$ marks a "removable" or "coordinate" singularity and that $r = 0$ marks a "true" or "physical" singularity [5].

The standard treatment of the foregoing line-element proceeds from simple inspection of (8) and thereby upon the following assumptions:

(a) that there is only one radial quantity defined on (8);

(b) that r can approach zero, even though the line-element (8) is singular at $r = 2m$;

(c) that r is the radial quantity in (8) ($r = 2m$ is even routinely called the "Schwarzschild radius" [5]).

With these unstated assumptions, but assumptions nonetheless, it is usual procedure to develop and treat of black holes. However, all three assumptions are demonstrably false at an elementary level.

4 That assumption (a) is false

Consider standard Minkowski space (using $c = G = 1$) described by

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2, \qquad (9)$$
$$0 \leqslant r < \infty\,,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. Comparing (9) with (4) it is easily seen that the spatial components of (9) constitute a line element of $\mathbf{E}^3$, with the point at the centre of spherical symmetry at $r_0 = 0$, coincident with the origin of the coordinate system.

In relation to (9) the *calculated* proper radius R_p of the sphere in $\mathbf{E}^3$ is,

$$R_p = \int_0^r dr = r\,, \qquad (10)$$

and the radius of curvature R_c is

$$R_c = r = R_p\,. \qquad (11)$$

Calculate the surface area of the sphere:

$$A = \int_0^{2\pi}\!\!\int_0^{\pi} r^2 \sin\theta\, d\theta\, d\varphi = 4\pi r^2 = 4\pi R_p^2 = 4\pi R_c^2\,. \qquad (12)$$

Calculate the volume of the sphere:

$$V = \int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{r} r^2 \sin\theta\, dr\, d\theta\, d\varphi = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R_p^3 = \frac{4}{3}\pi R_c^3\,. \qquad (13)$$

Then for (9), according to (10) and (11),

$$R_p = r = R_c\,. \qquad (14)$$

Thus, for Minkowski space, R_p and R_c are identical. This is because Minkowski space is pseudo-Efcleethean.

Now comparing (8) with (6) and (7) is is easily seen that the spatial components of (8) constitute a spherically symmetric metric manifold $\mathbf{M}^3$ described by

$$d\ell'^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

and which is therefore in one-to-one correspondence with $\mathbf{E}^3$. Then for (8),

$$R_c = r\,,$$
$$R_p = \int \sqrt{\frac{r}{r-2m}}\, dr \neq r = R_c\,.$$

Hence, $R_P \neq R_c$ in (8) in general. This is because (8) is non-Efcleethean (it is pseudo-Riemannian). Thus, assumption (a) is false.

5 That assumption (b) is false

On (8),

$$R_p = R_p(r) = \int \sqrt{\frac{r}{r-2m}}\, dr = \\ = \sqrt{r\,(r-2m)} + 2m \ln\left|\sqrt{r} + \sqrt{r-2m}\right| + K, \qquad (15)$$

where K is a constant of integration.

For some r_0, $R_p(r_0) = 0$, where r_0 is the corresponding point at the centre of spherical symmetry in $\mathbf{E}^3$ to be determined from (15). According to (15), $R_p(r_0) = 0$ when $r = r_0 = 2m$ and $K = -m \ln 2m$. Hence,

$$R_p(r) = \sqrt{r\,(r-2m)} + 2m \ln\left(\frac{\sqrt{r} + \sqrt{r-2m}}{\sqrt{2m}}\right). \qquad (16)$$

Therefore, $2m < r < \infty \Rightarrow 0 < R_p < \infty$, where $R_c = r$. The inequality is required to maintain Lorentz signature, since the line-element is undefined at $r_0 = 2m$, which is the only possible singularity on the line element. Thus, assumption (b) is false.

It follows that the centre of spherical symmetry of $\mathbf{E^3}$, in relation to (8), is located not at the point $r_0 = 0$ in $\mathbf{E^3}$ as usually assumed according to (9), but at the point $r_0 = 2m$, which corresponds to the point $R_p(r_0 = 2m) = 0$ in the metric manifold $\mathbf{M^3}$ that is described by the spatial part of (8). In other words, the point at the centre of spherical symmetry in $\mathbf{E^3}$ in relation to (8) is located at any point Q in the spherical surface Σ for which the radial distance from the centre of the coordinate system at $r = 0$ is $r = 2m$, owing to the one-to-one correspondence between all points of $\mathbf{E^3}$ and $\mathbf{M^3}$. It follows that (8) is not a generalisation of (9), as usually claimed. The manifold $\mathbf{E^3}$ of Minkowski space corresponding to the metric manifold $\mathbf{M^3}$ of (8) is not described by (9), because the point at the centre of spherical symmetry of (9), $r_0 = 0$, does not coincide with that required by (15) and (16), namely $r_0 = 2m$.

In consequence of the foregoing it is plain that the expression (8) is not general in relation to (9) and the line element (8) is not general in relation to the form (6). This is due to the incorrect way in which (8) is usually derived from (9), as pointed out in [6, 7, 8]. The standard derivation of (8) from (9) unwittingly shifts the point at the centre of sphericaly symmetry for the $\mathbf{E^3}$ of Minkowski space from $r_0 = 0$

to $r_0 = 2m$, with the consequence that the resulting line element (8) is misinterpreted in relation to $r = 0$ in the $\mathbf{E^3}$ of Minkowski space as described by (9). This unrecognised shift actually associates the point $r_0 = 2m \in \mathbf{E^3}$ with the point $R_p(2m) = 0$ in the $\mathbf{M^3}$ of the gravitational field. The usual analysis then incorrectly associates $R_p = 0$ with $r_0 = 0$ instead of with the correct $r_0 = 2m$, thereby conjuring up a so-called "interior", as typically alleged in [5], that actually has no relevance to the problem — a completely meaningless manifold that has nothing to do with the gravitational field and so is disjoint from the latter, as also noted in [6, 9, 10, 11]. The point at the centre of spherical symmetry for Einstein's gravitational field is $R_p = 0$ and is also the origin of the coordinate system for the gravitational field. Thus the notion of an "interior" manifold under some other coordinate patch (such as the Kruskal-Szekeres coordinates) is patently false. This is clarified in the next section.

6 That assumption (c) is false

Generalise (9) so that the centre of a sphere can be located anywhere in Minkowski space, relative to the origin of the coordinate system at $r = 0$, thus

$$\begin{aligned} ds^2 &= dt^2 - \left(d\,|r - r_0|\right)^2 - |r - r_0|^2\, d\Omega^2 = \\ &= dt^2 - \frac{(r - r_0)^2}{|r - r_0|^2} dr^2 - |r - r_0|^2\, d\Omega^2 = \\ &= dt^2 - dr^2 - |r - r_0|^2\, d\Omega^2, \end{aligned} \tag{17}$$

$$0 \leqslant |r - r_0| < \infty\,,$$

which is well-defined for all real r. The value of r_0 is arbitrary. The spatial components of (17) describe a sphere of radius $D = |r - r_0|$ centred at some *point* r_0 on a common radial line through r and the origin of coordinates at $r = 0$ (i.e. centred at the *point* of orthogonal intersection of the common radial line with the spherical surface $r = r_0$). Thus, the arbitrary point r_0 is the centre of spherical symmetry in $\mathbf{E^3}$ for (17) in relation to the problem of Einstein's gravitational field, the spatial components of which is a spherically symmetric metric manifold $\mathbf{M^3}$ with line element of the form (6) and corresponding centre of spherical symmetry at the point $R_p(r_0) = 0\; \forall\, r_0$. If $r_0 = 0$, (9) is recovered from (17). One does not need to make $r_0 = 0$ so that the centre of spherical symmetry in $\mathbf{E^3}$, associated with the metric manifold $\mathbf{M^3}$ of Einstein's gravitational field, coincides with the origin of the coordinate system itself, at $r = 0$. Any point in $\mathbf{E^3}$, relative to the coordinate system attached to the arbitrary point at which $r = 0$, can be regarded as a point at the centre of spherical symmetry in relation to Einstein's gravitational field. Although it is perhaps desirable to make the point $r_0 = 0$ the centre of spherical symmetry of $\mathbf{E^3}$ correspond to the point $R_p = 0$ at the centre of symmetry of $\mathbf{M^3}$ of the gravitational field, to simplify matters somewhat, this has not been done in the usual analysis of Einstein's gravitational field, despite appearances, and in consequence thereof false conclusions have been drawn owing to this fact going unrecognised in the main.

Now on (17),

$$\begin{aligned} R_c &= |r - r_0|\,, \\ R_p &= \int_0^{|r-r_0|} d\,|r - r_0| = \int_{r_0}^{r} \frac{(r - r_0)}{|r - r_0|} dr = |r - r_0| \equiv R_c, \end{aligned} \tag{18}$$

and so $R_p \equiv R_c$ on (17), since (17) is pseudo-Efcleethean. Setting $D = |r - r_0|$ for convenience, generalise (17) thus,

$$ds^2 = A\big(C(D)\big)dt^2 - B\big(C(D)\big)d\sqrt{C(D)}^{\,2} - C(D)d\Omega^2, \tag{19}$$

where $A\big(C(D)\big)$, $B\big(C(D)\big)$, $C(D) > 0$. Then for $R_{\mu\nu} = 0$, metric (19) has the solution,

$$\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{\sqrt{C(D)}}\right) dt^2 - \\ &- \frac{1}{1 - \frac{\alpha}{\sqrt{C(D)}}} d\sqrt{C(D)}^{\,2} - C(D)\, d\Omega^2, \end{aligned} \tag{20}$$

where α is a function of the mass generating the gravitational field [3, 6, 7, 9]. Then for (20),

$$\begin{aligned} R_c &= R_c(D) = \sqrt{C(D)}, \\ R_p &= R_p(D) = \int \sqrt{\frac{\sqrt{C(D)}}{\sqrt{C(D)} - \alpha}}\, d\sqrt{C(D)} = \\ &= \int \sqrt{\frac{R_c(D)}{R_c(D) - \alpha}}\, dR_c(D) = \sqrt{R_c(D)\big(R_c(D) - \alpha\big)} + \\ &+ \alpha \ln\left(\frac{\sqrt{R_c(D)} + \sqrt{R_c(D) - \alpha}}{\sqrt{\alpha}}\right), \end{aligned} \tag{21}$$

where $R_c(D) \equiv R_c\left(|r - r_0|\right) = R_c(r)$. Clearly r is a parameter, located in Minkowski space according to (17).

Now $r = r_0 \Rightarrow D = 0$, and so by (21), $R_c(D = 0) = \alpha$ and $R_p(D = 0) = 0$. One must ascertain the admissible form of $R_c(D)$ subject to the conditions $R_c(D = 0) = \alpha$ and $R_p(D = 0) = 0$ and $dR_c(D)/dD > 0$ [6, 7], along with the requirements that $R_c(D)$ must produce (8) from (20) at will, must yield Schwarzschild's [12] original solution at will (which is *not* the line element (8) with r down to zero), must produce Brillouin's [13] solution at will, must produce Droste's [14] solution at will, and must yield an infinite number of equivalent metrics [3]. The only admissible form satisfying these conditions is [7],

$$R_c = R_c(D) = \left(D^n + \alpha^n\right)^{\frac{1}{n}} \equiv \left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}} = R_c(r), \tag{22}$$

$$D > 0, \quad r \in \Re, \quad n \in \Re^+, \quad r \neq r_0,$$

where r_0 and n are entirely arbitrary constants.

Choosing $r_0 = 0$, $r > 0$, $n = 3$,

$$R_c(r) = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}, \tag{23}$$

and putting (23) into (20) gives Schwarzschild's original solution, defined on $0 < r < \infty$.

Choosing $r_0 = 0$, $r > 0$, $n = 1$,

$$R_c(r) = r + \alpha, \tag{24}$$

and putting (24) into (20) gives Marcel Brillouin's solution, defined on $0 < r < \infty$.

Choosing $r_0 = \alpha$, $r > \alpha$, $n = 1$,

$$R_c(r) = (r - \alpha) + \alpha = r, \tag{25}$$

and putting (25) into (20) gives line element (8), but defined on $\alpha < r < \infty$, as found by Johannes Droste in May 1916. Note that according to (25), and in general by (22), r is not a radial quantity in the gravitational field, because $R_c(r) = (r - \alpha) + \alpha = D + \alpha$ is really the radius of curvature in (8), defined for $0 < D < \infty$.

Thus, assumption (c) is false.

It is clear from this that the usual line element (8) is a restricted form of (22), and by (22), with $r_0 = \alpha = 2m$, $n = 1$ gives $R_c = |r - 2m| + 2m$, which is well defined on $-\infty < r < \infty$, i.e. on $0 \leqslant D < \infty$, so that when $r = 0$, $R_c(0) = 4m$ and $R_P(0) > 0$. In the limiting case of $r = 2m$, then $R_c(2m) = 2m$ and $R_p(2m) = 0$. The latter two relationships hold for any value of r_0.

Thus, if one insists that $r_0 = 0$ to match (9), it follows from (22) that,

$$R_c = \left(|r|^n + \alpha^n\right)^{\frac{1}{n}},$$

and if one also insists that $r > 0$, then

$$R_c = \left(r^n + \alpha^n\right)^{\frac{1}{n}}, \tag{26}$$

and for $n = 1$,

$$R_c = r + \alpha,$$

which is the simplest expression for R_c in (20) [6, 7, 13].

Expression (26) has the centre of spherical symmetry of $\mathbf{E}^3$ located at the point $r_0 = 0 \,\forall\, n \in \Re^+$, corresponding to the centre of spherical symmetry of $\mathbf{M}^3$ for Einstein's gravitational field at the point $R_p(0) = 0 \;\forall\; n \in \Re^+$. Then taking $\alpha = 2m$ it follows that $R_p(0) = 0$ and $R_c(0) = \alpha = 2m$ for all values of n.

There is no such thing as an interior solution for the line element (20) and consequently there is no such thing as an interior solution on (8), and so there can be no black holes.

7 That the manifold is inextendable

That the singularity at $R_p(r_0) \equiv 0$ is insurmountable is clear by the following ratio,

$$\lim_{r \to r_0^{\pm}} \frac{2\pi R_c(r)}{R_p(r)} = \lim_{r \to r_0^{\pm}} \frac{2\pi \left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}}}{R_p(r)} = \infty,$$

since $R_p(r_0) = 0$ and $R_c(r_0) = \alpha$ are invariant.

Hagihara [15] has shown that all radial geodesics that do not run into the boundary at $R_c(r_0) = \alpha$ (i.e. that do not run into the boundary at $R_p(r_0) = 0$) are geodesically complete.

Doughty [16] has shown that the acceleration a of a test particle approaching the centre of mass at $R_p(r_0) = 0$ is given by,

$$a = \frac{\sqrt{-g_{00}}\left(-g^{11}\right)|g_{00,1}|}{2g_{00}}.$$

By (20) and (22), this gives,

$$a = \frac{\alpha}{2R_c^{\frac{3}{2}}\sqrt{R_c(r) - \alpha}}.$$

Then clearly as $r \to r_0^{\pm}$, $a \to \infty$, independently of the value of r_0.

J. Smoller and B. Temple [10] have shown that the Oppenheimer-Volkoff equations do not permit gravitational collapse to form a black hole and that the alleged interior of the Schwarzschild spacetime (i.e. $0 \leqslant R_c(r) \leqslant \alpha$) is therefore *disconnected* from Schwarzschild spacetime and so does not form part of the solution space.

N. Stavroulakis [17, 18, 19, 20] has shown that an object cannot undergo gravitational collapse into a singularity, or to form a black hole.

Suppose $0 \leqslant \sqrt{C(D(r))} < \alpha$. Then (20) becomes

$$ds^2 = -\left(\frac{\alpha}{\sqrt{C}} - 1\right)dt^2 + \left(\frac{\alpha}{\sqrt{C}} - 1\right)^{-1} d\sqrt{C}^{\,2} - C\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

which shows that there is an interchange of time and length. To amplify this set $r = \bar{t}$ and $t = \bar{r}$. Then

$$ds^2 = \left(\frac{\alpha}{\sqrt{C}} - 1\right)^{-1} \frac{\dot{C}^2}{4C}\, d\bar{t}^2 - \left(\frac{\alpha}{\sqrt{C}} - 1\right) d\bar{r}^2 - C\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

where $C = C(\bar{t})$ and the dot denotes $d/d\bar{t}$. This is a time dependent metric and therefore bears no relation to the problem of a static gravitational field.

Thus, the Schwarzschild manifold described by (20) with (22) (and hence (8)) is inextendable.

8 That the Riemann tensor scalar curvature invariant is everywhere finite

The Riemann tensor scalar curvature invariant (the Kretschmann scalar) is given by $f = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. In the general case of (20) with (22) this is

$$f = \frac{12\alpha^2}{R_c^6(r)} = \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{\frac{6}{n}}}.$$

A routine attempt to justify the standard assumptions on (8) is the *a posteriori* claim that the Kretschmann scalar must be unbounded at a singularity [5, 21]. Nobody has ever offered a proof that General Relativity necessarily requires this. That this additional *ad hoc* assumption is false is clear from the following ratio,

$$f(r_0) = \frac{12\alpha^2}{\left(|r_0 - r_0|^n + \alpha^n\right)^{\frac{6}{n}}} = \frac{12}{\alpha^4} \;\forall\, r_0 \,.$$

In addition,

$$\lim_{r\to\pm\infty} \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{\frac{6}{n}}} = 0\,,$$

and so the Kretschmann scalar is finite everywhere.

9 That the Gaussian curvature is everywhere finite

The Gaussian curvature K of (20) is,

$$K = K\big(R_c(r)\big) = \frac{1}{R_c^2(r)}\,,$$

where $R_c(r)$ is given by (22). Then,

$$K(r_0) = \frac{1}{\alpha^2} \;\forall\, r_0\,,$$

and

$$\lim_{r\to\pm\infty} K(r) = 0\,,$$

and so the Gaussian curvature is everywhere finite.

Furthermore,

$$\lim_{\alpha\to 0} \frac{1}{\alpha^2} = \infty,$$

since when $\alpha = 0$ there is no gravitational field and empty Minkowski space is recovered, wherein R_p and R_c are identical and $0 \leqslant R_p < \infty$. A centre of spherical symmetry in Minkowski space has an infinite Gaussian curvature because Minkowski space is pseudo-Efcleethean.

10 Conclusions

Using the spherical-polar coordinates, the general solution to $R_{\mu\nu} = 0$ is (20) with (22), which is well-defined on

$$-\infty < r_0 < \infty,$$

where r_0 is entirely arbitrary, and corresponds to

$$0 < R_p(r) < \infty, \quad \alpha < R_c(r) < \infty,$$

for the gravitational field. The only singularity that is possible occurs at $g_{00} = 0$. It is impossible to get $g_{11} = 0$ because there is no value of the parameter r by which this can be attained. No interior exists in relation to (20) with (22), which contain the usual metric (8) as a particular case.

The radius of curvature $R_c(r)$ does not in general determine the radial geodesic distance to the centre of spherical symmetry of Einstein's gravitational field and is only to be interpreted in relation to the Gaussian curvature by the equation $K = 1/R_c^2(r)$. The radial geodesic distance from the point at the centre of spherical symmetry to the spherical geodesic surface with Gaussian curvature $K = 1/R_c^2(r)$ is given by the proper radius, $R_p(R_c(r))$. The centre of spherical symmetry in the gravitational field is invariantly located at the point $R_p(r_0) = 0$.

Expression (20) with (22), and hence (8) describes only a centre of mass located at $R_p(r_0) = 0$ in the gravitational field, $\forall\, r_0$. As such it does not take into account the distribution of matter and energy in a gravitating body, since $\alpha(M)$ is indeterminable in this limited situation. One cannot generally just utilise a potential function in comparison with the Newtonian potential to determine α by the weak field limit because α is subject to the distribution of the matter of the source of the gravitational field. The value of α must be calculated from a line-element describing the interior of the gravitating body, satisfying $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \neq 0$. The interior line element is necessarily different to the exterior line element of an object such as a star. A full description of the gravitational field of a star therefore requires two line elements [22, 23], not one as is routinely assumed, and when this is done, there are no singularities anywhere. The standard assumption that one line element is sufficient is false. Outside a star, (20) with (22) describes the gravitational field in relation to the centre of mass of the star, but α is nonetheless determined by the interior metric, which, in the case of the usual treatment of (8), has gone entirely unrecognised, so that the value of α is instead determined by a comparison with the Newtonian potential in a weak field limit.

Black holes are not predicted by General Relativity. The Kruskal-Szekeres coordinates do not describe a coordinate patch that covers a part of the gravitational manifold that is not otherwise covered - they describe a completely different pseudo-Riemannian manifold that has nothing to do with Einstein's gravitational field [6, 9, 11]. The manifold of Kruskal-Szekeres is not contained in the fundamental one-to-one correspondence between the $\mathbf{E}^3$ of Minkowski space and the $\mathbf{M}^3$ of Einstein's gravitational field, and is therefore a spurious augmentation.

It follows in similar fashion that expansion of the Universe and the Big Bang cosmology are inconsistent with General Relativity, as is easily demonstrated [24, 25].

Submitted on February 03, 2007
Accepted on February 09, 2007

References

1. Levi-Civita T. The Absolute Differential Calculus, Dover Publications Inc., New York, 1977.
2. O'Neill B. Semi-Riemannian geometry with applications to Relativity, Academic Press, San Deigo, 1983.
3. Eddington A. S. The mathematical theory of relativity. Cambridge University Press, Cambridge, 2nd edition, 1960.
4. Tolman R. C. Relativity, thermodynamics and cosmology. Dover Publications Inc., Mineola (NY), 1987.

5. Misner C. W., Thorne K. S. and Wheeler J. A. Gravitation. W. H. Freeman and Company, New York, 1973.
6. Abrams L. S. Black holes: the legacy of Hilbert's error. *Can. J. Phys.*, v. 67, 919, 1989; arXiv: gr-qc/0102055.
7. Crothers S. J. On the geometry of the general solution for the vacuum field of the point-mass. *Progress in Physics*, 2005, v. 2, 3–14.
8. Antoci S. David Hilbert and the origin of the "Schwarzschild" solution. arXiv: physics/0310104.
9. Loinger A. On black holes and gravitational waves. La Goliardica Paves, Pavia, 2002.
10. Smoller S. and Temple B. On the Oppenheimer-Volkoff equations in General Relativity. *Arch. Rational Mech. Anal.*, v. 142, 177–191, Springer-Verlag, 1998.
11. Crothers S. J. On the Regge-Wheeler tortoise and the Kruskal-Szekeres coordinates. *Progress in Physics*, 2006, v. 3, 30–34.
12. Schwarzschild K. On the gravitational field of a mass point according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 189; arXiv: physics/9905030.
13. Brillouin M. The singular points of Einstein's Universe. *Journ. Phys. Radium*, 1923, v. 23, 43; accessed online on http://www.geocities.com/theometria/brillouin.pdf.
14. Droste J. The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field. *Ned. Acad. Wet., S. A.*, 1917, v. 19, 197; accessed online on http://www.geocities.com/theometria/Droste.pdf.
15. Hagihara Y. *Jpn. J. Astron. Geophys.*, 1931, v. 8, 97.
16. Doughty N. *Am. J. Phys.*, 1981, v. 49, 720.
17. Stavroulakis N. A statical smooth extension of Schwarzschild's metric. *Lett. Nuovo Cim.*, Ser.2, 1974, v.11, No.8, 427–430; http://www.geocities.com/theometria/Stavroulakis-3.pdf.
18. Stavroulakis N. On the principles of General Relativity and the $\Theta(4)$-invariant metrics. *Proceedings of the* 3^{rd} *Panhellenic Congress of Geometry*, Athens, 1997, 169–182; assessed online on http://www.geocities.com/theometria/Stavroulakis-2.pdf.
19. Stavroulakis N. On a paper by J. Smoller and B. Temple. *Annales Fond. Louis de Broglie*, 2002, v. 27, No. 3, 511–521; http://www.ptep-online.com/theometria/Stavroulakis-1.pdf.
20. Stavroulakis N. Non-Euclidean geometry and gravitation. *Progress in Physics*, 2006, v. 2, 68–75.
21. Kruskal M. D. Maximal extension of Schwarzschild metric. *Phys. Rev.*, 1960, v. 119, 1743.
22. Schwarzschild K. On the gravitational field of a sphere of incompressible fluid according to Einstein's theory. *Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl.*, 1916, 424; arXiv: physics/9912033.
23. Crothers S. J. On the vacuum field of a sphere of incompressible fluid. *Progress in Physics*, 2005, v. 2, 76–81.
24. Crothers S. J. On the general solution to Einstein's vacuum field for the point-mass when $\lambda \neq 0$ and its consequences for relativistic cosmology. *Progress in Physics*, 2005, v. 3, 7–18.
25. Crothers S. J. Relativistic cosmology revisited. *Progress in Physics*, 2007, v. 2, 27–30.