The Schrödinger-equation presentation of any oscillatory classical linear system that is homogeneous and conservative

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Abstract

The time-dependent Schrödinger equation with time-independent Hamiltonian operator is a linear homogeneous system that is conservative and purely oscillatory. We investigate whether a classical system that is itself linear, homogeneous, conservative and purely oscillatory is assured to have a one-to-one linear mapping into some Schrödinger-format equation. Schrödinger equations are first order in time and have an even number of real-valued variables because they are complex-valued. Any first-order in time classical system as well has an even number of real-valued variables. Its Hermitian aspect gives a Schrödinger equation a more restricted presentation than that of an arbitrary homogeneous linear oscillatory conservative classical system, but general one-to-one linear mappings have enough parameters to bridge this presentation gap. As examples of invertible linear mappings of such classical systems into Schrödinger equations, we derive the mapping of the real-valued classical Klein-Gordon equation into the nonzero-mass free particle's relativistic scalar Schrödinger equation, that of the source-free Maxwell electric and magnetic field equations into the free photon's transverse-vector Schrödinger equation, and that of the classical simple harmonic oscillator equation into the Schrödinger equation for a basic single-state system. Once such a classical system has been mapped into a Schrödinger equation, it is automatically in canonical Hamiltonian form, and second quantization of that Schrödinger equation is usually the most transparent way to quantize the original classical system.

Introduction

Just as Lagrangian classical systems that are conservative can generally be presented in canonical Hamiltonian form, it turns out that oscillatory classical linear systems that are homogeneous and conservative can generally be presented, via linear isomorphism, in *Schrödinger-equation form*, which *itself* is *automatically* in canonical Hamiltonian form. Therefore the quantization of such a classical system is, ipso facto, *the second quantization of a Schrödinger equation*, an undertaking that is exceptionally physically transparent and technically straightforward. The consequent tight relationship of the *classical wave phenomena* of many-body or continuum versions of such classical systems to the *corresponding quanta* of their Schrödinger-equation presentations and second quantizations is obviously the quintessence of classical-quantum *complementarity*. We shall in particular point out that the real-valued classical scalar-field Klein-Gordon equation with mass parameter m is linearly isomorphic to the Schrödinger equation which is characterized by a scalar wave function and the Hamiltonian operator $(|c\widehat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$ [1]. That Hamiltonian operator is in precise accord with the natural correspondence-principle mandate for a relativistic free particle of mass m. Also the source-free Maxwell equations for the electric and magnetic fields are linearly isomorphic to the Schrödinger equation which is characterized by a transverse-vector wave function and the Hamiltonian operator $|c\widehat{\mathbf{p}}|$. That Hamiltonian operator is precisely appropriate to the massless free photon [2]. Furthermore, the classical wave equation for the electromagnetic radiation-gauge vector potential is linearly isomorphic to this *very same* Schrödinger equation [1]. We shall as well point out that the classical equation of motion for the simple harmonic oscillator of natural angular frequency ω is linearly isomorphic to the Schrödinger equation for a basic single-state quantum system which has the lone energy eigenvalue $\hbar\omega$.

Oscillatory classical linear systems which are homogeneous and conservative are described by equations of motion that have the form,

$$\dot{d} = Wd, \tag{1a}$$

where d is a real-valued vector of any one-to-one linear transformation of *all* of this classical system's coordinates and velocities, and W is a real-valued time-independent (matrix) operator for which the eigenvalues of W^2 are real, nonpositive and do not all vanish (i.e., the system is *oscillatory*). Such systems *specifically include* those described by the *second-order in time* equation,

$$\ddot{f} + Kf = 0, \tag{1b}$$

where f is a real-valued vector and K is a real-valued time-independent (matrix) operator whose eigenvalues are real, nonnegative and do not all vanish. This is so because d can consist of both f and an auxiliary real-valued vector g which is equal to $\dot{f}$ and satisfies $\dot{g} = -Kf$. In that instance,

$$d = (f, g), \qquad W = \begin{pmatrix} 0 & \mathrm{I} \\ -K & 0 \end{pmatrix}. \tag{1c}$$

Likewise, if the more general second-order in time equation,

$$\ddot{f} + C\dot{f} + Kf = 0, \tag{1d}$$

where C is also a real-valued time-independent (matrix) operator, is *oscillatory*, it is as well included via,

$$d = (f, g), \qquad W = \begin{pmatrix} 0 & \mathrm{I} \\ -K & -C \end{pmatrix}. \tag{1e}$$

Here it needs to be *checked* that the eigenvalues of W^2 are real, nonpositive and do not all vanish, i.e., that the system *is* in fact oscillatory.

Because d is a real-valued vector of any one-to-one linear transformation of *all* of this classical system's coordinates *and* velocities, it is an even-dimensional vector which, in fact, can *always* be written as,

$$d = (f, g), \tag{2a}$$

where f and g *each* have *half* as many dimensions as d—this is true *even* in the continuum limit, where one counts *field degrees of freedom* rather than finite dimensions. Therefore we can also *always* express W as the *block two-by-two matrix*,

$$W = \begin{pmatrix} W_{ff} & W_{fg} \\ W_{gf} & W_{gg} \end{pmatrix}, \tag{2b}$$

and correspondingly write the equation $\dot{d} = Wd$ in the *block-expanded form*,

$$\dot{f} = W_{ff}f + W_{fg}g, \qquad \dot{g} = W_{gf}f + W_{gg}g. \tag{2c}$$

The Schrödinger equation, which is the equation to which we would like to demonstrate the linear equivalence of $\dot{d} = Wd$ (presented in block-expanded form in Eq. (2c)), is normally, however, incompatibly presented in the form,

$$i\hbar\dot{\psi} = \widehat{H}\psi, \tag{3a}$$

where ψ is a *complex-valued* vector and $\widehat{H}$ is a *Hermitian* time-independent (matrix) operator. Before we proceed further, we clearly must *first* recast the Schrödinger equation into the form,

$$\dot{\chi}_\psi = \Omega\chi_\psi, \tag{3b}$$

where χ_ψ is a *real-valued* vector and Ω is a *real-valued* time-independent (matrix) operator, a form which can be *directly compared* with the form $\dot{d} = Wd$ of Eq. (1a) for the *classical physics*.

The Schrödinger equation as a real-valued canonical system

The complex-valued Schrödinger wave function ψ has the dimensions of probability density amplitude. From it we can define two real-valued fields, which each have the dimensions of action density amplitude that is compatible with these fields being mutually canonically conjugate, as follows,

$$\phi_\psi \stackrel{\text{def}}{=} (\hbar/2)^{\frac{1}{2}}(\psi + \psi^*), \qquad \pi_\psi \stackrel{\text{def}}{=} -i(\hbar/2)^{\frac{1}{2}}(\psi - \psi^*), \tag{4a}$$

which implies that,

$$\psi = (\phi_\psi + i\pi_\psi)/(2\hbar)^{\frac{1}{2}}, \qquad \psi^* = (\phi_\psi - i\pi_\psi)/(2\hbar)^{\frac{1}{2}}. \tag{4b}$$

The Hermitian (matrix) operator $\widehat{H}$ likewise has a real and and imaginary part, but since it is Hermitian, we have that,

$$\widehat{H}^* = \widehat{H}^T. \tag{5a}$$

For this reason, we can express the real and imaginary parts, H_R and H_I, of $\widehat{H}$ in terms of itself and its transpose $\widehat{H}^T$ as follows,

$$H_R \stackrel{\text{def}}{=} (\widehat{H} + \widehat{H}^T)/2, \qquad H_I \stackrel{\text{def}}{=} -i(\widehat{H} - \widehat{H}^T)/2, \tag{5b}$$

so that,

$$\widehat{H} = H_R + iH_I, \qquad \widehat{H}^T = \widehat{H}^* = H_R - iH_I. \tag{5c}$$

From Eqs. (5b) and (5a) it is clear that H_R is a *symmetric* real (matrix) operator, and that H_I is an *antisymmetric* real (matrix) operator.

If we now put the first equations that occur in both Eq. (4b) and in Eq. (5c) into the Schrödinger equation of Eq. (3a), and then equate the real and imaginary parts that result on the left-hand side to those which result on the right-hand side, we obtain the two equations,

$$\dot{\phi}_\psi = \Omega_I\phi_\psi + \Omega_R\pi_\psi, \qquad \dot{\pi}_\psi = -\Omega_R\phi_\psi + \Omega_I\pi_\psi, \tag{6a}$$

where $\Omega_R \stackrel{\text{def}}{=} H_R/\hbar$ and $\Omega_I \stackrel{\text{def}}{=} H_I/\hbar$. Therefore the complex-valued Schrödinger equation of Eq. (3a) is equivalent to the real-valued equation $\dot{\chi}_\psi = \Omega\chi_\psi$ of Eq. (3b) upon making the identifications,

$$\chi_\psi = (\phi_\psi, \pi_\psi), \qquad \Omega = \begin{pmatrix} \Omega_I & \Omega_R \\ -\Omega_R & \Omega_I \end{pmatrix}. \tag{6b}$$

We note that χ_ψ has the dimensions of action density amplitude and that Ω has the dimensions of frequency. In terms of inner products that involve the two real vectors ϕ_ψ and π_ψ and the real operators Ω_R and Ω_I, we can also write down a classical Hamiltonian functional which yields the equations of motion of Eq. (6a) as its two canonical Hamilton's equations,

$$H[\phi_\psi, \pi_\psi] = \tfrac{1}{2}\left[(\phi_\psi, \Omega_R\phi_\psi) + (\pi_\psi, \Omega_R\pi_\psi) + 2(\pi_\psi, \Omega_I\phi_\psi)\right]. \tag{6c}$$

We immediately see from Eq. (6c) that the first canonical Hamilton's equation,

$$\dot{\phi}_\psi = \delta H[\phi_\psi, \pi_\psi]/\delta\pi_\psi,$$

produces the first equation of motion of Eq. (6a), bearing in mind that Ω_R is a *symmetric* real operator. We also see from Eq. (6c) that the second canonical Hamilton's equation,

$$\dot{\pi}_\psi = -\delta H[\phi_\psi, \pi_\psi]/\delta\phi_\psi,$$

produces the second equation of motion of Eq. (6a), bearing in mind that Ω_R is a *symmetric* real operator and that Ω_I is an *antisymmetric* real operator. Thus the Schrödinger-equation system is *automatically* a classical Hamiltonian one *as well*, and therefore Schrödinger-equation systems are *always* amenable to *immediate second quantization*. This is, of course, done by the usual method of *promoting* the real-valued canonical vectors ϕ_ψ and π_ψ to become the *Hermitian operators* $\widehat{\phi}_\psi$ and $\widehat{\pi}_\psi$ that are subject to the usual canonical commutation rules,

$$[(\widehat{\phi}_\psi)_\alpha, (\widehat{\pi}_\psi)_\beta] = i\hbar\delta_{\alpha\beta}, \qquad [(\widehat{\phi}_\psi)_\alpha, (\widehat{\phi}_\psi)_\beta] = 0, \qquad [(\widehat{\pi}_\psi)_\alpha, (\widehat{\pi}_\psi)_\beta] = 0. \tag{7}$$

Therewith the Hamiltonian functional of Eq. (6c) is *also* promoted to become a Hermitian operator, which is of course the Hamiltonian operator of the second-quantized system. In the Heisenberg picture that is defined by this second-quantized Hamiltonian operator, the equations of motion of Eq. (6a) *continue to hold as operator relations*.

It is obvious, of course, that the equations of motion of Eq. (6a), in conjunction with the expression for ϕ_ψ and π_ψ in terms of ψ and ψ^* that is given by Eq. (4a) and the relation of Ω_R and Ω_I to $\widehat{H}$ and $\widehat{H}^T$ that is given by Eq. (5b), implies the Schrödinger equation of Eq. (3a). In the same manner it can be seen that the Hamiltonian functional $H[\phi_\psi, \pi_\psi]$ of Eq. (6c) is, upon being expressed as a functional of ψ and ψ^* in place of ϕ_ψ and π_ψ, equal to $H[\psi, \psi^*]$, where,

$$H[\psi, \psi^*] = (\psi^*, \widehat{H}\psi), \tag{8a}$$

and that the two canonical Hamilton's equations, $\dot{\phi}_\psi = \delta H[\phi_\psi, \pi_\psi]/\delta\pi_\psi$ and $\dot{\pi}_\psi = -\delta H[\phi_\psi, \pi_\psi]/\delta\phi_\psi$, are equivalent to the single complex-valued functional derivative equation,

$$i\hbar\dot{\psi} = \delta H[\psi, \psi^*]/\delta\psi^*, \tag{8b}$$

which, in conjunction with Eq. (8a) above, directly produces the Schrödinger equation of Eq. (3a). Furthermore, the canonical commutation rules of Eq. (7) that effect the second quantization can likewise be expressed in language that pertains exclusively to ψ and ψ^* and their respective non-Hermitian operator quantizations $\widehat{\psi}$ and $\widehat{\psi}^\dagger$,

$$[(\widehat{\psi})_\alpha, (\widehat{\psi}^\dagger)_\beta] = \delta_{\alpha\beta}, \qquad [(\widehat{\psi})_\alpha, (\widehat{\psi})_\beta] = 0, \qquad [(\widehat{\psi}^\dagger)_\alpha, (\widehat{\psi}^\dagger)_\beta] = 0. \tag{8c}$$

The Hermitian Hamiltonian operator of the second quantized regime is, aside from minor operator-ordering details, $H[\widehat{\phi}_\psi, \widehat{\pi}_\psi]$, which is of course equal to (again aside from minor operator-ordering details) $H[\widehat{\psi}, \widehat{\psi}^\dagger]$, i.e., it is the second quantization of the Hamiltonian functional $H[\psi, \psi^*]$ that is explicitly given by Eq. (8a). In the Heisenberg picture that is defined by this second-quantized Hamiltonian operator, the Schrödinger equation of Eq. (3a) *continues to hold as an operator relation for* $\widehat{\psi}$ *and* $\partial\widehat{\psi}/\partial t$. The commutation relations of Eq. (8c) are interpreted as identifying $(\widehat{\psi}^\dagger)_\alpha$ as the creation operator for the quantum state that is characterized by the index symbol α, and as identifying $(\widehat{\psi})_\alpha$ as the annihilation operator for this state. Therefore the second-quantized Hilbert space, called Fock space, is a relatively immense one whose *individual basis states* consist of *arbitrary sets* of basis states that can be selected from a basis system for the Hilbert space which is associated to the first-quantized Schrödinger equation of Eq. (3a). These basis-state sets are selected *with repetition* in the case of the commutation rules of Eq. (8c), but are selected *without repetition* when these rules are replaced by the anticommutation rules that are appropriate to systems which are subject to the Pauli exclusion principle.

While the canonical and second quantization properties of Schrödinger-equation systems are definitely of great interest, our *primary* concern here is with the issue of one-to-one linear time-independent mapping of an oscillatory linear classical system described by the homogeneous conservative equation $\dot{d} = Wd$ *into* such a Schrödinger-equation system, which Eq. (6b) tells us is described by $\dot{\chi}_\psi = \Omega\chi_\psi$, where Ω is a time-independent (matrix) operator which has dimensions of frequency and the block representation,

$$\Omega = \begin{pmatrix} \Omega_I & \Omega_R \\ -\Omega_R & \Omega_I \end{pmatrix}, \tag{9a}$$

where Ω_R is a symmetric real (matrix) operator and Ω_I is an antisymmetric real (matrix) operator. Now a *completely general* one-to-one linear time-independent mapping S of the classical system described by $\dot{d} = Wd$ produces $\chi_\psi = Sd$, or $d = S^{-1}\chi_\psi$. The equation of motion of χ_ψ is therefore $\dot{\chi}_\psi = SWS^{-1}\chi_\psi$, and thus the Ω which emerges from this mapping is SWS^{-1}. The *most general possible form* of such an Ω would be,

$$\Omega = \begin{pmatrix} \Omega_{\phi_\psi\phi_\psi} & \Omega_{\phi_\psi\pi_\psi} \\ \Omega_{\pi_\psi\phi_\psi} & \Omega_{\pi_\psi\pi_\psi} \end{pmatrix}, \tag{9b}$$

and we read off from Eq. (9a) that this describes a Schrödinger-equation system when it satisfies the *four* operator conditions,

$$\Omega_{\phi_\psi\phi_\psi} = \Omega_{\pi_\psi\pi_\psi}, \qquad \Omega_{\phi_\psi\pi_\psi} = -\Omega_{\pi_\psi\phi_\psi}, \qquad \Omega^T_{\phi_\psi\phi_\psi} = -\Omega_{\phi_\psi\phi_\psi}, \qquad \Omega^T_{\phi_\psi\pi_\psi} = \Omega_{\phi_\psi\pi_\psi}.$$

Now since $d = (f, g)$ and $\chi_\psi = (\phi_\psi, \pi_\psi)$, the mapping S of d into χ_ψ obviously consists of *four* block matrix operators,

$$S = \begin{pmatrix} S_{\phi_\psi f} & S_{\phi_\psi g} \\ S_{\pi_\psi f} & S_{\pi_\psi g} \end{pmatrix}, \tag{9c}$$

which should indeed be *general enough* to be able to *fulfill* those *four* operator conditions above that are required for the mapped matrix $SWS^{-1} = \Omega$ of Eq. (9b) to describe a Schrödinger-equation system.

Hamiltonian operators that apply to practical cases almost always turn out to be purely real and symmetric, aside from rather trivial spin one-half exceptions. In any event, there *invariably* exist unitary transformations which *purge* a Hamiltonian of any nonvanishing antisymmetric imaginary part: the unitary transformation that actually *diagonalizes* the Hamiltonian is obviously one of those that does this job. In practice, then, we shall be looking for a one-to-one linear mapping S of our classical system vector d such that 1) $Sd = \chi_\psi$ has the dimensions of action density amplitude and 2) $SWS^{-1} = \Omega$, where Ω has the simple form,

$$\Omega = \begin{pmatrix} 0 & \Omega_R \\ -\Omega_R & 0 \end{pmatrix}, \tag{10a}$$

Ω_R being a nonzero real symmetric (matrix) operator with the dimensions of frequency. In other words, S maps the general equations of motion of Eq. (2c), namely,

$$\dot{f} = W_{ff}f + W_{fg}g, \qquad \dot{g} = W_{gf}f + W_{gg}g,$$

into,

$$\dot{\phi}_\psi = \Omega_R \pi_\psi, \quad \dot{\pi}_\psi = -\Omega_R \phi_\psi, \tag{10b}$$

where ϕ_ψ and π_ψ have the dimensions of action density amplitude and Ω_R is a real symmetric (matrix) operator with dimensions of frequency. Comparing our simple form of Ω in Eq. (10a) with its most general possible form that is given by Eq. (9b), we *again* see that we must impose *four* operator conditions, namely,

$$\Omega_{\phi_\psi\phi_\psi} = 0, \qquad \Omega_{\pi_\psi\pi_\psi} = 0, \qquad \Omega_{\phi_\psi\pi_\psi} = -\Omega_{\pi_\psi\phi_\psi}, \qquad \Omega^T_{\phi_\psi\pi_\psi} = \Omega_{\phi_\psi\pi_\psi}.$$

Now since Eq. (9c) shows that the general one-to-one transform S is comprised of four operators, it should clearly be possible to fulfill these four operator requirements on $\Omega = SWS^{-1}$.

Finally, we note from Eq. (10a) that,

$$\Omega^2 = \begin{pmatrix} -\Omega_R^2 & 0 \\ 0 & -\Omega_R^2 \end{pmatrix}. \tag{10c}$$

Because Ω_R is a nonzero real symmetric (matrix) operator, this implies that Ω^2 has *nonpositive* real eigenvalues which do not all vanish. That is *precisely* the restriction we have imposed on W^2 to ensure that $\dot{d} = Wd$ is an *oscillatory system*—eigenvalues of the square of a (matrix) operator are, of course, *invariant* under one-to-one linear mappings of the type $W \to SWS^{-1} = \Omega$.

Just as there is no cut and dried recipe for constructing the unitary transformation which diagonalizes an arbitrary Hermitian operator, neither can we here provide such a cut and dried recipe for constructing the one-to-one linear time-independent mapping S which converts arbitrary classical equations of motion of the form $\dot{d} = Wd$, where W^2 has only nonpositive real eigenvalues that do not all vanish, to the Schrödinger-equation presentation form of Eq. (10b), where ϕ_ψ and π_ψ have dimensions of action density amplitude and Ω_R is a nonzero real symmetric (matrix) operator with dimensions of frequency. But merely knowing that such a mapping *exists* is sufficient to motivate the unearthing of its explicit form for selected classical systems which are of widespread physical interest. We thus now proceed to the actual realizations of such mappings for the real-valued scalar-field classical Klein-Gordon equation and the source-free Maxwell equations [1, 2], and as well for the classical simple harmonic oscillator.

As a matter of fact, the well-known theorem that any homogeneous linear oscillatory conservative classical system can be linearly decomposed into *normal modes*, which are *completely independent* classical simple harmonic oscillators, permits one to in principle parlay the Schrödinger-equation presentation of the classical simple harmonic oscillator that will be presented in detail further on into a Schrödinger-equation presentation of *any* such homogeneous linear oscillatory conservative classical system—indeed into a Schrödinger-equation presentation whose Hamiltonian operator is *already* in diagonal form.

The relativistic quantum free particle from the classical Klein-Gordon equation

The classical Klein-Gordon equation for the real-valued scalar field ϕ differs from the classical wave equation by a simple mass term [3, 1],

$$\ddot{\phi}/c^2 + (-\nabla^2 + \mu^2)\phi = 0, \tag{11a}$$

where $\mu = ((mc)/\hbar)$. We convert this second-order in time equation to two equations that are first-order in time in the standard way,

$$\dot{\phi} = \xi, \qquad \dot{\xi} = -c^2(-\nabla^2 + \mu^2)\phi. \tag{11b}$$

To carry out the Schrödinger-equation presentation of such an equation system, we know that we need to pin down a real symmetric operator Ω_R with the dimensions of frequency. This should not be difficult in the least in this particular case, as the second of our two equations very prominently manifests the real-valued nonnegative symmetric operator $c^2(-\nabla^2 + \mu^2)$, which has dimensions of frequency squared. It is therefore immediately clear that the square-root of this operator will necessarily figure very prominently in the consequent Schrödinger equation.

The second touchstone of Schrödinger-equation presentation is that its canonical fields ϕ_ψ and π_ψ must have dimensions of action density amplitude. Now the *conventional choice of dimensions* for the classical Klein-Gordon field ϕ is the *same* as that of the electromagnetic vector potential $\mathbf{A}$ [3, 1]. With this choice, the field ξ/c will have the same dimensions as the electric field, i.e., that of energy density amplitude. To obtain the desired dimensions of action density amplitude, we must multiply ξ/c by an object which has the dimensions of the square-root of time. Since the nonnegative symmetric operator $c^2(-\nabla^2 + \mu^2)$ has the dimensions of frequency squared, we shall take it to negative one quarter power, and multiply that into ξ/c to obtain a proposed π_ψ,

$$\pi_\psi = (c^3)^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}}\,\xi. \tag{11c}$$

Now ξ is the time derivative of ϕ, so in order to construct the *second* proposed canonical field ϕ_ψ from ϕ *itself*, we require a further factor of frequency, which is readily provided by the square root of the operator $c^2(-\nabla^2 + \mu^2)$. These considerations lead us to,

$$\phi_\psi = (c)^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}}\,\phi. \tag{11d}$$

Now applying the equations of motion given by Eq. (11b) to calculate the time derivatives of ϕ_ψ and π_ψ defined by Eqs. (11d) and (11c), we obtain,

$$\dot{\phi}_\psi = c(-\nabla^2 + \mu^2)^{\frac{1}{2}}\,\pi_\psi, \qquad \dot{\pi}_\psi = -c(-\nabla^2 + \mu^2)^{\frac{1}{2}}\,\phi_\psi. \tag{11e}$$

Comparing this result to Eq. (10b), we positively identify the real symmetric operator Ω_R as $c(-\nabla^2 + \mu^2)^{\frac{1}{2}}$ in this classical Klein-Gordon field case. We know that $H_R = \hbar\Omega_R$, and of course the antisymmetric Ω_I and corresponding H_I are entirely absent in this case. Therefore the first-quantized Hamiltonian operator which corresponds to the classical Klein-Gordon field is $\hbar c(-\nabla^2 + \mu^2)^{\frac{1}{2}}$. Taking account of the facts that $\mu = ((mc)/\hbar)$ and that, in configuration representation, $\widehat{\mathbf{p}} = -i\hbar\nabla$, this Hamiltonian operator is equal to $(|c\widehat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$, which is *identical* to the first quantized Hamiltonian operator that is *mandated by the correspondence principle* for a free relativistic particle of mass m. We as well, of course, have available the *precise details* of the one-to-one linear mapping from the classical Klein-Gordon field ϕ and its time derivative $\xi = \dot{\phi}$ into the Schrödinger-equation wave function ψ,

$$\psi = (\phi_\psi + i\pi_\psi)/(2\hbar)^{\frac{1}{2}} = (2\hbar c)^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}}\,\phi + i(2\hbar c^3)^{-\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}}\,\dot{\phi}. \tag{11f}$$

It can be *explicitly verified from this result* that if ϕ simply satisfies the second-order in time Klein-Gordon equation of Eq. (11a), then this ψ *definitely satisfies* the first-order in time Schrödinger equation with the correspondence-principle first quantized Hamiltonian operator $\hbar c(-\nabla^2 + \mu^2)^{\frac{1}{2}}$. This ψ has as well, of course, been painstakingly crafted to have the proper dimensions of probability density amplitude that is appropriate to a Schrödinger-equation wave function. *Second quantization of this* ψ along the lines described in the previous section is, of course, completely straightforward. It is quite stunning that there exists a one-to-one linear map of the *classical* Klein-Gordon fields ϕ and $\dot{\phi}$ which *links them in such detail* to that theory's *latent quantum characteristics*. What we thus have in front of our eyes in the one-to-one linear map of Eq. (11f) is a *tour de force* of classical-quantum *complementarity*.

The one-to-one linear map of Eq. (11f) can, of course be explicitly *inverted*,

$$\phi = ((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{-\frac{1}{4}}(\psi + \psi^*), \qquad \dot{\phi} = -i((\hbar c^3)/2)^{\frac{1}{2}}(-\nabla^2 + \mu^2)^{\frac{1}{4}}(\psi - \psi^*). \tag{11g}$$

As we have previously mentioned, the most straightforward and physically transparent route to the *quantization* of the classical Klein-Gordon field ϕ, which is explicitly given by Eq. (11g) above, is via the *second quantization* of the first-quantized Schrödinger-equation wave function ψ. This, of course, entails *promotion* of that wave function ψ to become *the non-Hermitian operator* $\widehat{\psi}$ which obeys the canonical commutation rules,

$$[\widehat{\psi}(\mathbf{r}), \widehat{\psi}^\dagger(\mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \qquad [\widehat{\psi}(\mathbf{r}), \widehat{\psi}(\mathbf{r}')] = 0, \qquad [\widehat{\psi}^\dagger(\mathbf{r}), \widehat{\psi}^\dagger(\mathbf{r}')] = 0. \tag{11h}$$

The interpretation of Eq. (11h) is of course that $\widehat{\psi}^\dagger(\mathbf{r})$ is the operator which creates a free Klein-Gordon scalar quantum of mass m at the point $\mathbf{r}$, and that $\widehat{\psi}(\mathbf{r})$ is the operator which annihilates such a quantum at the point $\mathbf{r}$. The Hamiltonian functional $H[\psi,\psi^*]$ of Eq. (8a), which for this classical Klein-Gordon case is explicitly,

$$H[\psi,\psi^*] = (\psi^*, \hbar c(-\nabla^2+\mu^2)^{\frac{1}{2}}\,\psi), \tag{11i}$$

becomes, in the form $\widehat{H}[\widehat{\psi},\widehat{\psi}^\dagger]$, the Hamiltonian operator of the second-quantized system. In the Heisenberg picture which this Hamiltonian operator defines, the time-dependent Schrödinger equation that ψ satisfies *continues to hold* for the annihilation operator $\widehat{\psi}$ as *an operator relation*. Also, upon being transcribed into second-quantized form, where $\widehat{\psi}$ and $\widehat{\psi}^\dagger$ respectively *replace* ψ and ψ^*, Eq. (11g) explicitly yields the *quantized version* $\widehat{\phi}$ of the *classical Klein-Gordon field* ϕ as a *Hermitian operator*, and it as well does the same for the quantized version of the *time derivative* of the classical Klein-Gordon field. It is interesting to note from the quantized transcription of Eq. (11g) that $\widehat{\phi}$, the Hermitian quantized version of the classical Klein-Gordon field, can *both* create *and* annihilate free Klein-Gordon scalar quanta.

Free-photon quantum mechanics from the source-free Maxwell equations

In the source-free case, the Coulomb and Gauss laws tell us that both the electric and magnetic fields are purely transverse, i.e., $\nabla\cdot\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$. The results of Faraday's law and the Maxwell law in the source-free case are,

$$\dot{\mathbf{B}} = -c\nabla\times\mathbf{E}, \qquad \dot{\mathbf{E}} = c\nabla\times\mathbf{B}. \tag{12a}$$

Both $\mathbf{B}$ and $\mathbf{E}$ have dimensions of energy density amplitude. We need to multiply them both by the operator $(-c^2\nabla^2)^{-\frac{1}{4}}$, which has the dimensions of the square root of time to convert them to the dimensions of action density amplitude,

$$\mathbf{\Phi_B} = (-c^2\nabla^2)^{-\frac{1}{4}}\,\mathbf{B}, \qquad \mathbf{\Pi_E} = -(-c^2\nabla^2)^{-\frac{1}{4}}\,\mathbf{E}. \tag{12b}$$

They satisfy the equations of motion,

$$\dot{\mathbf{\Phi}}_\mathbf{B} = c\nabla\times\mathbf{\Pi_E}, \qquad \dot{\mathbf{\Pi}}_\mathbf{E} = -c\nabla\times\mathbf{\Phi_B}, \tag{12c}$$

which yield $\Omega_I=0$ and the Schrödinger equation with Hermitian Hamiltonian operator $\widehat{H}=\hbar\Omega_R=\hbar c\,\mathbf{curl}$. This Hamiltonian operator is *odd* in parity, which, of course, *does not suit elecromagnetism*. We can, however, readily elicit the *parity-conserving* Schrödinger equation by *instead* choosing $\mathbf{\Phi}$ and $\mathbf{\Pi}$ to *both* be of *the same definite parity*, as well as having the dimensions of action density amplitude, e.g.,

$$\mathbf{\Phi} \stackrel{\text{def}}{=} (-\nabla^2)^{-\frac{1}{2}}(\nabla\times\mathbf{\Phi_B}) = c^{-\frac{1}{2}}(-\nabla^2)^{-\frac{3}{4}}(\nabla\times\mathbf{B}), \qquad \mathbf{\Pi} \stackrel{\text{def}}{=} \mathbf{\Pi_E} = -c^{-\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}}\,\mathbf{E}. \tag{12d}$$

These two transverse *polar* vector fields $\mathbf{\Phi}$ and $\mathbf{\Pi}$ satisfy the equations of motion,

$$\dot{\mathbf{\Phi}} = c(-\nabla^2)^{\frac{1}{2}}\,\mathbf{\Pi}, \qquad \dot{\mathbf{\Pi}} = -c(-\nabla^2)^{\frac{1}{2}}\,\mathbf{\Phi}, \tag{12e}$$

which yield $\Omega_I=0$ and the Schrödinger equation with the *even-parity* Hermitian Hamiltonian operator $\widehat{H}=\hbar\Omega_R=\hbar c(-\nabla^2)^{\frac{1}{2}}$. Because $\widehat{\mathbf{p}}=-i\hbar\nabla$ in configuration representation, this Hamiltonian operator equals $|c\widehat{\mathbf{p}}|$, which is, of course, appropriate to the free photon. Eq. (12d) thus yields the one-to-one linear mapping of the source-free electric and magnetic fields into the free photon's Schrödinger-equation wave function,

$$\mathbf{\Psi} = (\mathbf{\Phi}+i\mathbf{\Pi})/(2\hbar)^{\frac{1}{2}} = (2\hbar c)^{-\frac{1}{2}}\left[(-\nabla^2)^{-\frac{3}{4}}(\nabla\times\mathbf{B}) - i(-\nabla^2)^{-\frac{1}{4}}\,\mathbf{E}\right]. \tag{12f}$$

The inverse of this mapping is given by,

$$\mathbf{B} = ((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}}\left(\nabla\times(\mathbf{\Psi}+\mathbf{\Psi}^*)\right), \qquad \mathbf{E} = i((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}}(\mathbf{\Psi}-\mathbf{\Psi}^*). \tag{12g}$$

For source-free electromagnetism, an appropriate gauge for the four-vector potential A^μ is the *radiation gauge*, for which $A^0=0$ and $\nabla\cdot\mathbf{A}=0$ [4]. In radiation gauge, $\mathbf{E}=-\dot{\mathbf{A}}/c$ and $\mathbf{B}=\nabla\times\mathbf{A}$, so that we can reexpress the mapping of Eq. (12f) in terms of the radiation gauge $\mathbf{A}$ and $\dot{\mathbf{A}}$,

$$\mathbf{\Psi} = (2\hbar c)^{-\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}}\,\mathbf{A} + i(2\hbar c^3)^{-\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}}\,\dot{\mathbf{A}}, \tag{12h}$$

which is *highly analogous* to Eq. (11f) for the Schrödinger-equation presentation of the wave function that corresponds to classical Klein-Gordon field theory. Its inverse mapping is consequently *highly analogous* to the classical Klein-Gordon field theory inverse of Eq. (11g),

$$\mathbf{A} = ((\hbar c)/2)^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}}(\mathbf{\Psi} + \mathbf{\Psi}^*), \qquad \dot{\mathbf{A}} = -i((\hbar c^3)/2)^{\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}}(\mathbf{\Psi} - \mathbf{\Psi}^*). \tag{12i}$$

Indeed, if we *begin* with the radiation-gauge vector potential approach rather than the electric and magnetic field Maxwell equation approach, the steps that are are involved turn out to *rigidly parallel* those of classical Klein-Gordon field theory. This is so because the transverse radiation-gauge vector potential satisfies the *classical wave equation*, which is simply the *special case* of the Klein-Gordon equation that has $\mu = 0$. The transverse part of the vector potential $\mathbf{A}_T$ always satisfies the equation,

$$\ddot{\mathbf{A}}_T/c^2 - \nabla^2\mathbf{A}_T = \mathbf{j}_T/c, \tag{13a}$$

where $\mathbf{j}_T$ is the transverse part of the source current. When there is no transverse source current, Eq. (13a) reduces to the classical wave equation for $\mathbf{A}_T$. When there is no source whatsoever, one can use the radiation gauge [4], wherein $A^0 = 0$ and $\nabla\cdot\mathbf{A} = 0$, which imply that $\mathbf{A}_T$ is the *only* nonvanishing part of the four-vector potential A^μ, and $\mathbf{A}_T$ of course satisfies the classical wave equation.

One now approaches the classical wave equation for the radiation-gauge vector potential $\mathbf{A}$ in *rigid parallel* with the approach of Eqs. (11a) through (11g) to the classical Klein-Gordon scalar field ϕ—these two fields even have the same dimensions [3, 1]. One merely sets the Klein-Gordon mass parameter m to zero, which causes μ to also equal zero, and substitutes the transverse vector field $\mathbf{A}$ for ϕ. Thus instead of Eq. (11a) one has,

$$\ddot{\mathbf{A}}/c^2 + (-\nabla^2)\mathbf{A} = 0. \tag{13b}$$

In rigid parallel with Eq. (11b), this second-order in time equation is converted to two equations which are first-order in time,

$$\dot{\mathbf{A}} = \mathbf{\Xi}, \qquad \dot{\mathbf{\Xi}} = -c^2(-\nabla^2)\mathbf{A}. \tag{13c}$$

From this point on everything proceeds in perfect analogy with the Klein-Gordon development of Eqs. (11c) through (11g), which results in the one-to-one linear mapping of Eqs. (12h) and (12i) above, and also the first-quantized Hamiltonian $|c\widehat{\mathbf{p}}|$ for the free photon.

In addition to its zero mass parameter, a second special feature of electromagnetic theory vis-à-vis classical Klein-Gordon theory is, of course, the free photon's *always transverse* polarization (spin) states. This signature free-photon characteristic does not cause much in the way of complications, but there is one formula concerning second quantization which it *notationally* impacts, albeit *no substantive physical effect is involved*. The canonical commutation rule for second quantization of the free photon's transverse vector wave function might naively be expected to read,

$$[(\widehat{\mathbf{\Psi}}(\mathbf{r}))_i, (\widehat{\mathbf{\Psi}}^\dagger(\mathbf{r}'))_j] = \delta_{ij}\delta^{(3)}(\mathbf{r} - \mathbf{r}'), \tag{14a}$$

but this is *not* mathematically consistent with the transverse character of the second-quantized photon wave-functions, i.e., it is mathematically inconsistent with the fact that $\nabla\cdot\widehat{\mathbf{\Psi}} = 0$. The nature of the right-hand of Eq. (14a) is one of completeness, but the transverse wave function creation and annihilation operators are *incomplete* in that they do *not* pertain to vector fields which are the gradients of scalar fields, i.e., they do *not* pertain to vector fields *which fail to be transverse*. Now the ij components of the *projection operator* onto the subspace of such purely gradient vector fields is given by,

$$P_{ij} = -\partial_i(-\nabla^2)^{-1}\partial_j. \tag{14b}$$

We note that P_{ij} is Hermitian, and that its contraction with itself yields itself, which are the two essential properties of the ij components of projection operators. Of course its contraction with the components of any transverse vector field vanishes. Thus $(\delta_{ij} - P_{ij})$ are the ij components of the *projection operator onto the subspace of transverse vector fields*, and therefore,

$$[(\widehat{\mathbf{\Psi}}(\mathbf{r}))_i, (\widehat{\mathbf{\Psi}}^\dagger(\mathbf{r}'))_j] = \langle\mathbf{r}|(\delta_{ij} - P_{ij})|\mathbf{r}'\rangle = (2\pi)^{-3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left(\delta_{ij} - \mathbf{k}_i\mathbf{k}_j|\mathbf{k}|^{-2}\right)d^3\mathbf{k}. \tag{14c}$$

Notwithstanding these fancy maneuvers with projection operators, the *only* issue which is involved here is the simple fact that free-photon creation and annihilation operators (and as well free photon wave functions in the first quantized regime) are *purely transverse*, and therefore *any expression involving these operators*, e.g., the expression which describes their canonical commutation relation, must, of course, *correctly reflect this fact*. There is obviously *no physics implication* which flows from this requirement of *mere notational correctness*.

Single-state quantum mechanics from the classical harmonic oscillator equation

The simplest homogeneous linear oscillatory conservative classical system is, of course, the classical simple harmonic oscillator, whose *generic* equation of motion can be regarded as being given by the pair of first-order differential equations,

$$\dot{q} = v \quad \text{and} \quad \dot{v} = -\omega^2 q, \tag{15}$$

or by the more familiar equivalent second-order differential equation,

$$\ddot{q} + \omega^2 q = 0, \tag{16}$$

where q specifically has the dimension of the square root of action and ω is the classical simple harmonic oscillator's natural angular frequency. As one pertinent example, a point particle of mass m moving in one dimension attached to a Hooke's Law spring of spring constant k obeys the classical equation of motion $m\ddot{x} + kx = 0$, which is invertibly linearly transformed to the generic form given by Eq. (16) by identifying ω as $(k/m)^{\frac{1}{2}}$ and taking q to be $(mk)^{\frac{1}{4}}x$.

The classical Lagrangian for the generic classical simple harmonic oscillator equation of motion given by Eq. (16) is,

$$L_{\text{sho}}(q, \dot{q}) = \tfrac{1}{2}(\dot{q}^2/\omega - \omega q^2), \tag{17a}$$

where the Lagrangian $L_{\text{sho}}(q, \dot{q})$ of course specifically has the dimension of energy. This Lagrangian yields the classical canonical momentum $p = \partial L_{\text{sho}}/\partial\dot{q} = \dot{q}/\omega$ and the classical Hamiltonian,

$$H_{\text{sho}}(q, p) = [\dot{q}p - L_{\text{sho}}(q, \dot{q})]_{\dot{q}=\omega p} = \tfrac{1}{2}\omega(q^2 + p^2), \tag{17b}$$

which in turn yields the pair of classical Hamilton's equations of motion,

$$\dot{q} = \omega p \quad \text{and} \quad \dot{p} = -\omega q, \tag{18}$$

that of course are linearly equivalent to Eq. (15) via the invertible linear relation $p = v/\omega$.

The pair of real-valued classical Hamilton's equations of motion of Eq. (18) can readily be rewritten as the single complex-valued Schrödinger equation,

$$i\hbar d\psi/dt = \widehat{H}_{\text{sho}}\psi, \tag{19a}$$

where the degenerate Hermitian Hamiltonian "operator" $\widehat{H}_{\text{sho}}$, which has the dimension of energy, is given by the trivial real-valued one-by-one "matrix",

$$\widehat{H}_{\text{sho}} \stackrel{\text{def}}{=} \hbar\omega, \tag{19b}$$

and the dimensionless $\psi(t)$ is given by a straightforward complex-valued linear combination of $q(t)$ and $p(t)$, namely,

$$\psi(t) \stackrel{\text{def}}{=} [q(t) + ip(t)]/(2\hbar)^{\frac{1}{2}}, \tag{19c}$$

which has the linear inversion,

$$q(t) = (\hbar/2)^{\frac{1}{2}}[\psi(t) + \psi^*(t)], \quad p(t) = -i(\hbar/2)^{\frac{1}{2}}[\psi(t) - \psi^*(t)]. \tag{19d}$$

The particular *normalization* which has been imposed on ψ by Eq. (19c) cannot actually be directly determined from the homogeneous linear Schrödinger equation of Eq. (19a) that ψ satisfies, but has been chosen to ensure that the bilinear energy expectation value $\psi^*\widehat{H}_{\text{sho}}\psi$ for the Schrödinger-equation "quantum system" is precisely equal to that system's classical energy, which is, of course, given by $H_{\text{sho}}(q, p) = \frac{1}{2}\omega(q^2 + p^2)$.

It is therefore readily seen that the *second* quantization of the Schrödinger equation of Eq. (19a) via the usual imposition of the wave-"vector" commutation relation $[\widehat{\psi}, \widehat{\psi}^{\dagger}] = \mathrm{I}$ in conjunction with the order-averaged second-quantized Hamiltonian operator $\frac{1}{2}[\widehat{\psi}^{\dagger}(\hbar\omega)\widehat{\psi} + \widehat{\psi}(\hbar\omega)\widehat{\psi}^{\dagger}]$ is precisely *equivalent* to imposition of the familiar Dirac commutation rule $[\widehat{q}, \widehat{p}] = i\hbar \mathrm{I}$ in conjunction with the equally familiar *quantized classical* Hamiltonian operator $H_{\mathrm{sho}}(\widehat{q}, \widehat{p}) = \frac{1}{2}\omega(\widehat{q}^{\,2} + \widehat{p}^{\,2})$, i.e., the *second* quantization of the Schrödinger equation $i\hbar d\psi/dt = \widehat{H}_{\mathrm{sho}}\psi$ of Eq. (19a) is *exactly the same* as the familiar *standard* quantization of the *classical* generic simple harmonic oscillator.

Turning our attention now to *general* homogeneous linear oscillatory conservative classical systems, it is well-known that these can be invertibly linearly decomposed into *normal modes*, which are, of course, *mutually independent* classical simple harmonic oscillators. Since each such *independent* normal mode can be presented as a Schrödinger equation of the degenerate single-state, single-eigenvalue type described by Eqs. (19), it is apparent that the *entire* homogeneous linear oscillatory conservative classical system can as well be presented as a Schrödinger equation—indeed one whose Hamiltonian operator is already in *diagonal form* as a consequence of the normal-mode decomposition of that classical system. Of course an invertible linear transformation which "undoes" the normal-mode decomposition of that homogeneous linear oscillatory conservative classical system will simultaneously act to "undo" the diagonal form of the Hamiltonian operator of its Schrödinger-equation presentation.

Conversely, it is as well apparent that once the Hamiltonian operator of any Schrödinger equation has been diagonalized, that Schrödinger equation is easily invertibly linearly transformed into independent normal modes of a homogeneous linear oscillatory conservative classical system. Furthermore, any unitary transformation which "undoes" the diagonalization of the Schrödinger equation's Hamiltonian operator will simultaneously effectively act to "undo" the normal-mode decomposed form of its corresponding homogeneous linear oscillatory conservative classical system.

Conclusion

It is a remarkable fact that any homogeneous linear oscillatory conservative *classical* system is effectively *already first-quantized*: one but needs to work out the invertible linear transformation which brings its classical equations of motion to explicit time-dependent Schrödinger-equation form. Thus Michael Faraday and James Clerk Maxwell were actually the first to effectively elucidate a quantized particle system, namely the very important and not exactly elementary one of the ultra-relativistic massless transverse-vector free photon.

Any *complex-valued* solution wave function of a time-dependent Schrödinger-equation has the familiar characteristic expansion in terms of the complete set of mutually orthogonal eigenfunctions of that equation's Hamiltonian operator. The one-to-one linear mapping of any oscillatory linear classical system that is homogeneous and conservative into a Schrödinger equation thus implies a characteristic *two-component* eigenfunction expansion of such a classical system's solutions. For the case of certain wave equations that fall into the class of Eq. (1b), precisely such a solution expansion has been described in detail by Leung, Tong and Young [5].

The natural *correspondence-principle* version of the relativistic free-particle Schrödinger equation was *iterated* by Klein, Gordon and Schrödinger *for no physically motivated reason*, but merely in an effort to *rid* it of its *calculationally unpalatable* square-root Hamiltonian operator [6, 1, 7]. If this *iterated* equation is *still* regarded as a *complex-valued quantum-mechanical entity*, a large class of *completely extraneous, highly unphysical unbounded-below negative-energy solutions* are *injected* by that iteration. These also destroy its probability interpretation, and the fact that it depends on only the *square* of a Hamiltonian cuts it adrift from the Heisenberg picture and Ehrenfest theorem. However, if this iterated equation is regarded as the description of a *classical, real-valued* field, it thereupon becomes strongly analogous to the *classical wave equation*, and has an eminently sensible *nonnegative* conserved energy [3, 1]. This *classical* Klein-Gordon equation is *as well* one of those classical equation systems which is linearly equivalent to a Schrödinger equation: it quite marvelously *chooses* to be equivalent to *precisely* the Schrödinger equation with the natural correspondence-principle *square-root Hamiltonian operator* which Klein, Gordon and Schrödinger *had tried to escape by concocting it.*

It is a pity that Klein, Gordon and Schrödinger had no idea of the theorem presented by this paper, and thus were not equipped to unearth this astonishing fact themselves. If they had but grasped the full consequences of the real-valued classical Klein-Gordon equation, they might well have abandoned their physically unmotivated flight from the *correspondence-principle mandated* relativistic free-particle square-root Hamiltonian operator $(|c\widehat{\mathbf{p}}|^2 + m^2c^4)^{\frac{1}{2}}$ [7, 1].

References

[1] S. K. Kauffmann, arXiv:1012.5120 [physics.gen-ph] (2010).

[2] S. K. Kauffmann, arXiv:1011.6578 [physics.gen-ph] (2010).

[3] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961).

[4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[5] P. T. Leung, S. S. Tong and K. Young, J. Phys. A: Math. Gen. **30**, 2139 (1997).

[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[7] S. K. Kauffmann, arXiv:1009.3584 [physics.gen-ph] (2010).