On Sergeyev's Grossone : How to Compute Effectively with Infinitesimal and Infinitely Large Numbers

Elemér E Rosinger

Department of Mathematics
and Applied Mathematics
University of Pretoria
Pretoria
0002 South Africa
eerosinger@hotmail.com

Dedicated to Marie-Louise Nykamp

Abstract

The recently proposed and partly developed "Grossone Theory" of Y D Sergeyev is analyzed and partly clarified.

1. Preliminaries

Recently, a remarkable avenue has been proposed and to some extent developed in [10] for an effective computation with infinitesimal and infinitely large numbers, a computation which is implementable on usual digital computers. The presentation in [10], recommended by the author himself for a first approach of the respective theory, is aimed ostensibly for a wider readership, and as such it may benefit from a four fold upgrade along the following lines : First, the lengthy motivation on pages 1-6 and in various parts of the subsequent ones could gain a lot from a more brief and crisp formulation. Second, the mathematical part itself can be introduced in a more systematic way which makes it more explicit that the whole issue is but about a certain specific and highly particular extension of the usual set of natural numbers $\mathbb{N} = \{ 1, 2, 3, \ldots \}$, and thus implicitly, of the integers in $\mathbb{Z}$ as well. Third, it should be made clear up front that the whole issue is but about a rather small subset of various ultrapower fields, among them the well known one introduced more than four decades ago by the Nonstandard Analysis of A Robinson. And such fields can be constructed by most simple undergraduate algebra means, see [11], and also [3-9]. Fourth, and by far most importantly, the presentation in [10] is based on several postulates and axioms added to the usual set $\mathbb{N}$ of natural numbers, set assumed quite likely to be defined by the classical Peano axioms. And then, it should be made clear, very clear indeed, to what extent the respective enlarged system of axioms is consistent, or at least, can be expected to be so.

Obviously, the presentation in [10] was aimed at as large a readership as possible. However, it is not at all clear whether in our times of information over-exposure and short attention span the kind of presentation in [10] may indeed manage to maximize its impact to a sufficient extent...

The above is in no way to be construed as a negative comment on the potential merit of the mentioned avenue introduced and started to be developed in [10]. Indeed, one of the obvious and major merits of that avenue is precisely in the fact that - focusing on a special and particular subset of various well known ultrapower fields, see section 6 below - it offers a *practically effective and efficient* way to introduce in a wide range of everyday numerical computations the rather easy and natural possibility of dealing at long last with infinitesimal and infinitely large numbers as well ...

And why compute with infinitesimal and infinitely large numbers, at all ?

Well, given the mentioned information overload and short attention span which so much dominate and characterize our days, one may

hopefully be excused when not presenting yet another longer attempt at trying to convince the skeptics, non-believers, opponents, and the many like, of the idea of computing with infinitesimal and infinitely large numbers ...
However, such attempts do nevertheless exist, and a few of them can be found in [3-9], for instance ...

Controversies, one the other hand, are of course quite another issue. And they still can attract a wider attention even in our excessively busy and hectic times ...

Here, therefore, it may be amusing to mention their existence related to [10]. And as an illustration, one of them can be found in some detail in [2], where a "Blog in Fragments" has a section "On Rigmarole and Pseudoscience", and in it, an item entitled "A Penn'orth of Grossone" presents several strikingly negative comments, among them : "Unfortunately, it turned out impossible to stop the flood of Sergeyev's publications in the variety of the international journals having little if any in common with foundations of analysis. Miraculously, there are no Sergeyev's publications on his grossone in Russian in the Russian mathematical database Math-Net.Ru."

Since the issue of the status of the "Grossone Theory" still seems to be pending to a certain extent, one should start nevertheless by pointing out several remarkable observations in [10] which are worth considering most seriously by mathematicians.

First, and stated more or less in the very terms of [10], comes the fact that the way much of mathematics is done recalls that of science in general. Namely, the triad made up from 'researcher', 'object of investigation' and 'tools used to observe the object' is so often present in mathematics as well.
And in this regard, we already happen to have in mathematics a variety of 'numeral systems' used to express 'numbers'. And clearly, such 'numeral systems' can be seen as some among the tools of observation used by mathematicians, used not only in the application of mathematics, but also within mathematics itself, that is, when doing mathematical research.

In this regard, no matter how much and for how long we have been accustomed to the usual set $\mathbb{N}$ of natural numbers, nowadays, and ever since Gödel's Incompleteness Theorem, we should be aware that what we mean by $\mathbb{N}$ is most definitely *not* one single 'numeral system', or for that matter, set of 'numbers'.
And then, the basic idea in [10] is in fact :

(1.1) to distinguish between 'numeral systems' and 'numbers', with the former supposed to represent some, or possibly, all of the latter,

(1.2) to focus with priority on the construction of a 'numeral system' which allows the representation of infinitesimal and infinitely large 'numbers', as well as the usual algebraic operations with them, as they are customary in a field,

(1.3) to have the usual natural numbers n in $\mathbb{N}$ both part of the 'numeral system', and of the 'numbers' represented by that system, simply by having the 'numeral' n represent the 'number' n, or equivalently, having the 'number' n be represented by the 'numeral' n,

(1.4) to allow only finitely constructible 'numerals', with the construction being recursive from the simpler 'numerals' to the more complex ones,

(1.5) to be unconcerned about 'numbers' which cannot be represented by a given 'numeral system',

(1.6) to be open to the construction of new 'numeral systems' in order to be able to represent more 'numbers', and/or to represent them with better precision.

Remarkably, all of the above can be attained by the surprisingly simple means of introducing one single infinite 'numeral' called *grossone*, as seen in section 2, next.

Second, the emerging "Grossone Theory" touches upon Set Theory as well, and does so in novel ways. Indeed, the way Cantor defines the size of sets, namely, by their cardinal numbers, allows the possibility

that "a part is not smaller than the whole". Namely, it is in fact a characteristic property of infinite sets to have strict subsets of the same cardinal, thus of the same size in the sense of Cantor. Indeed, take any infinite set X and in it any element $x \in X$. Then clearly $Y = X \setminus \{x\}$ is still an infinite set, and has the same cardinal with X, while it is obviously a strict subset of X.
Well, Postulate 3 in [10] states, on the other hand, that the part is less than the whole. In other words, the way [10] tries to measure the size of sets is clearly non-Cantorian.
Unfortunately however, that alternative way of measuring is not made sufficiently clear by a suitable general enough definition, and all one is presented with instead is a formulation in usual colloquial language, plus some examples.
The conclusion, nevertheless, is valid that, indeed, there can be many other ways to measure the size of sets. And the way Cantor did with the help of cardinals is but one of them.

Amusingly, such fundamentally important issues are completely missed by the critics in [2] who seem to become lost in arguments trying to support by all the means available to them a seemingly prejudiced negative attitude ...

Hopefully, in due time, the "Grossone Theory" in [10] may be perfected, and among others, possibly along the line of the above suggestions as well ...

Needless to say, we cannot expect a 'numeral system' to be able to represent all the 'numbers' which can possibly be conceived of, and do so by finite means. After all, even in usual Analysis we have to face uncountably many 'numbers', and most of the irrational ones we cannot represent by finite means ...

As for Nonstandard Analysis, or more generally, the various ultrapower fields, [11,3-9], they contain uncountably many different infinitesimals, and also, different infinitely large numbers. Indeed, given two such infinitesimals, one of them can be infinitely smaller, or for that matter, infinitely larger than the other one, and a similar situation happens with the infinitely large numbers. Details in this regard

can be found in [8], for instance.
And in fact, as is known, [1], such ultrapower fields can have arbitrarily large cardinals.

In conclusion, to the extent that some of the infinitely small and infinitely large numbers can be included in the representation available in a 'numeral system', a system that may even be performed on digital computers, we are, and for the first time, given a truly precious gift ...

2. A Few Basic Details on Grossone

Here we shall present a few first steps in making the "Grossone Theory" more clear. Needless to say, the completion of the respective venture in clarification may fall in its entirety upon the author of [10] himself.

The presentation proper in [10] starts with the following three postulates :

P1. "We postulate existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only a finite number of operations."

P2. "We shall not tell what are the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects."

P3. "We adopt the principle : 'The part is less than the whole' to all numbers (finite, infinite, infinitesimal) and to all sets and processes (finite and infinite)."

In certain parts of [10] it is claimed that one or another of the above three postulates is used in an essential, and in fact, sine-qua-non manner. That claim itself may need a further consideration.

As for the above first postulate P1, the term "operation" is not at

all made clear enough. After all, we humans can within Calculus, for instance, compute limits, derivatives and integrals. And they contain by their definitions infinitely many arithmetic operations.
Regarding [10], that first postulate P1 is actually an overstatement, since what it mostly means is that the 'numeral system' is composed of elements which can be constructed by finite means from usual natural numbers n in the infinite set $\mathbb{N}$, with these numbers seen as 'numerals' according to (1.3) above, as well as from a special infinite 'numeral' introduced next.

The development proper in [10] starts with section 3, where the "numeral $< 1 >$ called *grossone*" is introduced as "the infinite unit of measure" which is declared to be, or rather, to represent "the number of elements of the set $\mathbb{N}$".

Then the so called "Infinite Unit Axiom", or in short, IUA is introduced as consisting of three parts, namely :

Infinity : Any finite natural number n is less than the grossone, that is, $n << 1 >$.

Identity : The following relations hold :

$$0. < 1 > \;=\; < 1 > .0 \;=\; 0, \quad < 1 > - < 1 > \;=\; 0, \quad \frac{<1>}{<1>} \;=\; 1$$

$$< 1 >^0 \;=\; 1, \quad 1^{<1>} \;=\; 1, \quad 0^{<1>} \;=\; 0$$

Divisibility : For any finite natural number n, the infinite sets

$$\mathbb{N}_{k,n} \;=\; \{\, k, k+n, k+2n, k+3n, \ldots \}, \qquad 1 \leq k \leq n$$

have the same number of elements given by the numeral $\frac{<1>}{n}$.

Here as a motivation of the above, one can note that

$$\bigcup_{1 \leq k \leq n} \mathbb{N}_{k,n} \;=\; \mathbb{N}$$

is a partition of $\mathbb{N}$, and the sets $\mathbb{N}_{k,n}$, with $1 \leq k \leq n$, are translates of one another.

This is about all in [10] regarding the introduction of the basic novelties.

As one can see, these novelties include :

(2.1) the numeral $< 1 >$, called grossone,

(2.2) the assumption that $\mathbb{N}$, when extended with grossone, can be included in a numeral system, say, $\mathbb{S}$,

(2.3) certain infinite sets can - in a non-Cantorian manner - have their sizes measured, or rather represented by elements in $\mathbb{S}$.

(2.4) the numeral system $\mathbb{S}$ is *not* supposed to be the same with the set of numbers it can represent, and is not expected to represent all possible numbers

(2.5) the concern is *not* with the set of numbers, but with the numeral system,

(2.6) various numeral systems may describe various sets of numbers,

(2.7) the concern is to introduce a numeral system which, for the first time in the literature, can represent in addition to the usual natural numbers, also infinitesimal and infinitely large numbers, and can operate with them rigorously, including on digital computers, according to the customary operations in a field.

Obviously, the numeral $\frac{<1>}{n}$ above is considered to be infinite, and therefore, its inverse $\frac{n}{<1>}$ is considered to be infinitesimal. Furthermore, $\frac{<1>}{n}$ is, in fact, considered to be an infinite integer.

Here, a surprising novelty is introduced in [10] by declaring that, now, with the above *new* numeral system, we can consider that, in fact, we have

$$(2.8) \qquad \mathbb{N} = \{1, 2, 3, \ldots, <1> -3, <1> -2, <1> -1, <1>\}$$

while on the other hand, and much unlike the above, the usual numeral system gives of course

$$(2.9) \qquad \mathbb{N} = \{1, 2, 3, \ldots\}$$

All of this is but a simple example of the fact that different numeral systems can give different ways of representing numbers. And the numbers represented can be the same or different ones. And some of such systems are more sophisticated than other ones, and thus may allow a greater precision in the representation of numbers, and/or can allow the representation of more numbers.
In this regard, the above new numeral system is clearly more sophisticated and with greater precision than the usual one since it can represent numbers, for instance, infinite and infinitesimal ones, which the usual one cannot.
For instance, the usual representation only allows $\infty + 1 = \infty + 2 = \infty$, while in the new representation we have

$$<1> \; \neq \; <1> +1 \; \neq \; <1> +2$$

Next, [10] introduces the extension of the new set $\mathbb{N}$ of natural numbers in (2.1), namely

$$(2.10) \qquad \widehat{\mathbb{N}} = \{1, 2, 3, \ldots, <1> -3, <1> -2, <1> -1, <1>, \\ <1> +1, <1> +2, <1> +3, \ldots, \\ <1>^2 -3, <1>^2 -2, <1>^2 -1, <1>^2, \\ <1>^2 +1, \ldots\}$$

By the way, and quite amusingly, (2.8) and (2.10) are not seen in [10] as conflicting with postulate P1. This fact alone shows that the mentioned postulate may have to be reconsidered and formulated in more precise form, a form which is considerably weaker than as it presently stands in [10]. Indeed, in view of (2.8) and (2.10), for instance, the term "to operate" in that postulate seems not to include the term "to conceive" ...
Here, of course, one can recall that the concept of infinity is notori-

ously difficult not only in mathematics, but more widely, in philosophy as well. Therefore, no doubt, a special care need be taken when formulating postulates like P1 above.

3. New Positional Numeral System

Next, in [10], a new positional numeral system is introduced in which the grossone $< 1 >$ plays an essential role. And this new system is a rather straightforward and natural extension of the usual one, with the only exception that each numeral in it is but a finite construction, thus keeping with the postulate P1. Namely, the numerals are of the general form

$$(3.1) \qquad C \; = \; c_{p_m} < 1 >^{p_m} + \ldots + c_{p_1} < 1 >^{p_1} + c_{p_0} < 1 >^{p_0} + \\ + c_{p_{-1}} < 1 >^{p_{-1}} + \ldots + c_{p_{-k}} < 1 >^{p_{-k}}$$

or written more briefly

$$(3.2) \qquad C \; = \; c_{p_m} < 1 >^{p_m} \ldots c_{p_1} < 1 >^{p_1} c_{p_0} < 1 >^{p_0} c_{p_{-1}} < 1 >^{p_{-1}} \\ \ldots c_{p_{-k}} < 1 >^{p_{-k}}$$

where m, k are natural numbers, or zero, further, the *grosspowers*

$$(3.3) \qquad p_m > \ldots p_1 > p_0 \; = \; 0 > p_{-1} > \ldots > p_{-k}$$

are arbitrary numerals already constructed according to (3.1), and as such can be finite, infinitesimal or infinite, while finally, the *grossdigits* $c_i \neq 0$ are arbitrary usual rational numbers in $\mathbb{Q}$, expressed in finite terms in a positional system.

The surprising fact is the amount of infinitesimal and infinitely large numbers which such a positional numeral system can represent. Some examples in this regard are presented in [10]. Here, let us only mention the following immediate consequence of (3.1). Clearly, $< 1 >$ is of form (3.1), thus it is a numeral. And then, it can be taken as a grosspower, thus giving the numeral $< 1 >^{<1>}$. Continuing in this manner any finite number of times, one can obtain numerals given by any *finite* exponential tower

$$(3.4) \qquad < 1 >^{<1>^{<1>^{<1>^{\vdots}}}}$$

consequently, far larger infinitely large numerals are available than the grossone itself.

4. Arithmetic Operations

In [10], it is shown in clear detail that way the usual operations in a field can be performed within the positional numeral system in (3.1).

5. Further Developments

A number of other operations with elements of the positional numeral system in (3.1) are presented in [10], albeit some of them suffer from an insufficient clarity.

6. Ultrapower Fields and the Grossone

Let us relate the above to the well known ultrapower fields, a particular case of which constitutes the subject of Nonstandard Analysis, [11,3-9].

As a first step, let us recall the seemingly less familiar fact that Nonstandard Analysis constructs the field $^*\mathbb{R}$ of nonstandard real numbers by a rather simple and general ultrapower construction, a construction which only requires undergraduate algebra, namely, familiarity with the concepts of ring, ideal, quotient of a ring by an ideal, as well as with the rather elementary set theoretic concept of filter, and its particular case of ultrafilter.

The difference between various ultrapower fields, and on the other hand, the nonstandard field $^*\mathbb{R}$ is in the fact that the latter is accompanied by the so called *transfer principle.*

However, upon a more careful consideration, it may appear that the price paid in the considerable technical complications needed for securing that transfer principle is not rewarded sufficiently. Indeed, by far most of the properties of interest do not transfer from the standard case to the nonstandard one, since they cannot be formulated in first order predicate logic.
And whether or not it may indeed be the case that the complications do not compensate for the rewards, the fact remains that the vast majority of mathematicians have chosen not to use nonstandard methods, ever since their emergence in the 1960s.
On the other hand, that statistical fact cannot in any way be held against Nonstandard Analysis which, as a mathematical accomplishment in its own, can be seen as ranking among the most important, and in fact, revolutionary ones, achieved in the 20th century, among others for the fact that it presents the first ever rigorous, systematic and wide ranging foundation for the concept of *infinitesimals* introduced by Leibniz.

Consequently, in the sequel we shall only relate the emerging "Grossone Theory" to ultrapower fields. For that purpose we briefly recall the construction of such ultrapower fields, further details being presented in [11,3-9].

Let Λ be any infinite set. Below, for convenience, we shall take $\Lambda = \mathbb{N}$. Further, let $\mathcal{F}$ be a free ultrafilter on Λ. We note that the existence of such free ultrafilters on infinite sets follows from the Axiom of Choice.

Obviously, the set $\mathbb{R}^\Lambda$ of functions $x : \Lambda \longrightarrow \mathbb{R}$ is a unital commutative algebra over $\mathbb{R}$. We define in this algebra the maximal ideal

$$(6.1) \qquad \mathcal{I}_\mathcal{F} \ = \ \{\, x \in \mathbb{R}^\Lambda \mid Z(x) \in \mathcal{F} \,\}$$

where $Z(x) = \{\, \lambda \in \Lambda \mid x(\lambda) = 0 \,\}$.

And now we obtain the ultrapower field

(6.2) $\quad \mathbb{R}_{\mathcal{F}} \; = \; \mathbb{R}^{\Lambda}/\mathcal{I}_{\mathcal{F}}$

Further, we have the embedding of fields

(6.3) $\quad \mathbb{R} \ni r \;\longmapsto\; u(r) + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}}$

where $u(r) \in \mathbb{R}^{\Lambda}$ is defined by $(u(r))(\lambda) = r$, for $\lambda \in \Lambda$.

Finally, the field $\mathbb{R}^{\Lambda}$ is totally, or linearly ordered by the relation

(6.4) $\quad x \leq y \;\;\Longleftrightarrow\;\; \{\, \lambda \in \Lambda \mid x(\lambda) \leq y(\lambda) \,\} \in \mathcal{F}$

We note that the totally, or linearly ordered field $^*\mathbb{R}$ of nonstandard reals is - as a field - but a particular case of the remarkably simple construction above, and in fact, it can be obtained for $\Lambda = \mathbb{N}$.

As mentioned, we shall take $\Lambda = \mathbb{N}$. Then it is easy to give examples of infinitesimal and infinitely large elements in $\mathbb{R}_{\mathcal{F}}$. Indeed, let any $x \in \mathbb{R}^{\mathbb{N}}$. If

(6.5) $\quad x(n) > 0, \quad n \in \mathbb{N}$

and

(6.6) $\quad \lim_{n \in \mathbb{N}} x(n) \; = \; 0$

then $\tilde{x} = x + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}}$ is an infinitesimal number. On the other hand, if

(6.7) $\quad \lim_{n \in \mathbb{N}} x(n) \; = \; \infty$

then $\tilde{x} = x + \mathcal{I}_{\mathcal{F}} \in \mathbb{R}_{\mathcal{F}}$ is an infinitely large number.

Clearly, given any function $f : \mathbb{R} \longrightarrow \mathbb{R}$, one can define its extension

(6.8) $\quad \tilde{f} : \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}_{\mathcal{F}}$

by

(6.9) $\tilde{f}(x + \mathcal{I}_\mathcal{F}) \;=\; f \circ x + \mathcal{I}_\mathcal{F}, \quad x \in \mathbb{R}^\mathbb{N}$

Indeed, given $x, y \in \mathbb{R}^\mathbb{N}$, such that $y - x \in \mathcal{I}_\mathcal{F}$, then (6.1) gives $\{ n \in \mathbb{N} \mid x(n) = y(n) \} \in \mathcal{F}$, hence $\{ n \in \mathbb{N} \mid f(x(n)) = f(y(n)) \} \in \mathcal{F}$.

In fact, we have the following considerably stronger property. Given any $f : \mathbb{N} \times \mathbb{R} \longrightarrow \mathbb{R}$, we define

(6.10) $\tilde{f}(x + \mathcal{I}_\mathcal{F}) \;=\; f \bigodot x + \mathcal{I}_\mathcal{F}, \quad x \in \mathbb{R}^\mathbb{N}$

where

(6.11) $\mathbb{N} \ni n \;\longmapsto\; f(n, x(n)) \in \mathbb{R}$

since for $x, y \in \mathbb{R}^\mathbb{N}$, such that $y - x \in \mathcal{I}_\mathcal{F}$, we have

$$\{ n \in \mathbb{N} \mid f(n, x(n)) = f(n, y(n)) \} \supseteq \{ n \in \mathbb{N} \mid x(n) = y(n) \} \in \mathcal{F}$$

Now, returning to the grossone, we can take it as

(6.12) $< 1 > \in \mathbb{R}_\mathcal{F}$

given by any infinitely large positive number, and clearly, the conditions in the Identity Axiom will be satisfied.

The question, therefore, arises :

> Is the new numeral system $\mathbb{S}$ given by the gossone nothing else but the subfield $\mathbb{Q}(< 1 >)$ generated in $\mathbb{R}_\mathcal{F}$ by $\mathbb{Q}$ and $< 1 >$ chosen in (6.12) ?

Clearly, in view of section 4 above, and in more details, of section 4 in [10], the numeral system $\mathbb{S}$ contains the numbers in $\mathbb{Q}(< 1 >)$.

What is crucially important, however, regarding the "Grossone Theory" is that we have in fact the strict inclusion

(6.13) $\mathbb{Q}(< 1 >) \;\subsetneqq\; \mathbb{S}$

which follows obviously from (3.4).

7. How about a Multi-Gorssone Theory ?

The fact that the grossone numeral system $\mathbb{S}$ is far larger than any field extension of the usual rational numbers generated by an infinitely large number is remarkable, and follows directly from the positional definition of numerals in (3.1).

Of course, in case postulate P1 is set aside, one is allowed grossonedigits in (3.1) which are arbitrary real numbers, and then (6.13) takes the stronger form

$$(7.1) \qquad \mathbb{R}(< 1 >) \subsetneqq \mathbb{S}$$

However, even without removing P1, or more precisely, a more precisely focused version of it, one may ask the following question :

> How about developing a numeral system based not only on one single grossone, but on a finite number of them ?

In other words, instead of the grossone $< 1 >$ alone, one can start with any finite number of them, say

$$(7.2) \qquad < 1 >, < 2 >, < 3 >, \ldots, < \gamma >$$

and by an obvious direct extension of (3.1), construct a corresponding numeral system $\mathbb{S}(< 1 >, < 2 >, < 3 >, \ldots, < \gamma >)$.

What may happen in such a case is that the mapping

$$(7.3) \qquad \text{numeral} \longmapsto \text{number}$$

which gives the representation of numbers by numerals, may cease to be injective.

A first obvious step in order to avoid such a situation is to introduce the Basic Order Axiom, according to which

$$(7.4) \qquad <1> \;<\; <2> \;<\; <3> \;<\; \ldots \;<\; <\gamma>$$

The difficulty here is that, in view of (3.4), for instance, the gap between successive grossones in (7.4) has to be considerable, if one wants to secure the injectivity of the representation mapping in (7.3).

On the other hand, by only dealing with one single grossone, the corresponding numeral system appear to have no chance at all to represent arbitrarily large infinite numbers in $\mathbb{R}_{\mathcal{F}}$.

Indeed, let us take any grossone $<1>$ in (6.12). The question is the following :

> Is it the case that for every $\tilde{x} \in \mathbb{R}_{\mathcal{F}}$ there exists a numeral $\nu \in \mathbb{S}$ which represents a number $\tilde{y} \in \mathbb{R}_{\mathcal{F}}$ which is at least as large as $\tilde{x}$?

And to be more specific, let us formulate the question in a more particular manner, namely :

> Let be given any $f : \mathbb{N} \times \mathbb{R} \longrightarrow \mathbb{R}$ and the corresponding $\tilde{f}$ in (6.10). Further, let $\tilde{x} = \tilde{f}(<1>) \in \mathbb{R}_{\mathcal{F}}$. Is there a numeral $\nu \in \mathbb{S}$ which represents a number $\tilde{y} \in \mathbb{R}_{\mathcal{F}}$ which is at least as large as $\tilde{x}$?

In view of postulate P1, or for that matter, of certain more focused weaker forms of it, the answer to the above question is likely to be negative.

However, the same appears to be the case of any multi-grossone theory, as long as finite number of grossons are employed.

Here, therefore, can one see one of the possible limitations of "Grossone Theory".

Bibliography

[1] Jin R, Shelah S : Possible Size of an Ultrapower of ω. arxiv:math/9801153v1

[2] Kutateladze S S : www.math.nsc.ru/LBRT/g2/english/ssk/cv.html

[3] Rosinger E E : String Theory : a mere prelude to non-Archimedean Space-Time Structures ? arxiv:physics/0703154

[4] Rosinger E E : How Far Should the Principle of Relativity Go ? arxiv:0710.0226

[5] Rosinger E E : Heisenberg Uncertainty in Reduced Power Algebras. arxiv:0901.4825

[6] Rosinger E E : No-Cloning in Reduced Power Algebras. arxiv:0902.0264

[7] Rosinger E E : Special Relativity in Reduced Power Algebras. arxiv:0903.0296

[8] Rosinger E E : Surprising Properties of Non-Archimedean Field Extensions of the Real Numbers. arxiv:0911.4824

[9] Rosinger E E : Four Departures in Mathematics and Physics. arxiv:1003.0360

[10] Sergeyev Y D : A New Applied Approach for Executing Computations with Infinite and Infinitesimal Quantities. www.grossone.com

[11] Slomson A B, Bell J L : Models and Ultraproducts : An Introduction. Dover, 2006