Nonassociative Octonionic Ternary Gauge Field Theories

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Abstract

A novel (to our knowledge) nonassociative and noncommutative octonionic ternary gauge field theory is explicitly constructed that it is based on a ternary-bracket structure involving the octonion algebra. The ternary bracket was defined earlier by Yamazaki. The field strengths $F_{\mu\nu}$ are given in terms of the 3-bracket $[B_\mu, B_\nu, \Phi]$ involving an auxiliary octonionic-valued scalar field $\Phi = \Phi^a e_a$ which plays the role of a "coupling" function. In the concluding remarks a list of relevant future investigations are briefly outlined.

Keywords: Octonions, ternary algebras, Lie 3-algebras, membranes, nonassociative gauge theories, nonassociative geometry.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see [1], [2], [3], [4], [7], [24], [23], for references, among many others. For instance, the large N limit of Exceptional Jordan Matrix models, advanced by [9], furnished a Chern-Simons membrane action leading to important connections to M and F theory [10]. It was shown in [22] how one could generalize ordinary Relativity into an Extended Relativity theory in Clifford spaces, involving polyvector valued (Clifford-algebra valued) coordinates and fields, where in addition to the speed of light there is also an invariant length scale (set equal to the Planck scale) in the definition of a generalized metric distance in Clifford spaces encoding, lengths, areas, volumes and hyper-volumes metrics. An overview of the basic features of the Extended Relativity in Clifford spaces can be found in [22].

A Chern-Simons E_8 Gauge theory of Gravity was proposed [21] as a unified field theory of a Lanczos-Lovelock Gravitational theory with a E_8 Generalized Yang-Mills field theory and which is defined in the $15D$ boundary of a $16D$ bulk space. The role of Clifford $Cl(16)$ algebras was essential. It was discussed how an E_8 Yang-Mills in $8D$, after a sequence of symmetry breaking processes based on the $non-compact$ forms of the exceptional groups, as follows $E_{8(-24)} \rightarrow E_{7(-5)} \times SU(2) \rightarrow E_{6(-14)} \times SU(3) \rightarrow SO(8,2) \times U(1)$, furnishes a Conformal gravitational theory in $8D$ based on gauging the non-compact conformal group $SO(8,2)$ in $8D$. Upon performing a Kaluza-Klein-Batakis [12] compactification on CP^2, from $8D$ to $4D$ involving a nontrivial torsion, leads to a Conformal Gravity-Yang-Mills unified theory based on the conformal group $SO(4,2)$ and the Standard Model group $SU(3) \times SU(2) \times U(1)$ in $4D$. Other approaches to unification based on Clifford algebras can be found in [13], [16], [14] and on E_8 were proposed long ago by [15].

A Nonassociative Gauge theory based on the Moufang S^7 loop product (not a Lie algebra) has been constructed by [26]. Taking the algebra of octonions with a unit norm as the Moufang S^7-loop, one reproduces a nonassociative octonionic gauge theory which is a generalization of the Maxwell and Yang-Mills gauge theories based on Lie algebras. $BPST$-like instantons solutions in $D = 8$ were also found. These solutions represented the physical degrees of freedom of the transverse 8-dimensions of superstring solitons in $D = 10$ preserving one and two of the 16 spacetime supersymmetries. Nonassociative deformations of Yang-Mills Gauge theories involving the left and right bimodules of the octonionic algebra were presented by [25].

Recently, tremendous activity has been launched by the seminal works of Bagger, Lambert and Gustavsson (BLG) [33], [34] who proposed a Chern-Simons type Lagrangian describing the world-volume theory of multiple $M2$-branes. The original BLG theory requires the algebraic structures of generalized Lie 3-algebras and also of nonassociative algebras. Later developments by [35] provided a $3D$ Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and with gauge groups $U(N) \times U(N)$, $SU(N) \times SU(N)$. The original construction of [35] did not require generalized Lie 3-algebras, but it was later realized that it could be understood as a special class of models based on Hermitian 3-algebras [36], [37]. For more recent developments we refer to [38] and references therein.

The novel (to our knowledge) nonassociative octonionic ternary gauge theory developed in this work differs from the nonassociative gauge theories of [26], [25] in many respects, mainly that it is based on a ternary bracket involving the octonion algebra that was proposed by Yamazaki [28]. It also differs from the work by [33], [34] in that our octonionic-valued gauge fields $B^a_\mu e_a$; $a = 0, 1, 2, ...7$ are not, and cannot be represented, in terms of matrices $\mathbf{A}_\mu = A^{ab}_\mu f_{ab}{}^{cd} = (\tilde{A}_\mu)^{cd}$, defined in terms of $f_{ab}{}^{cd}$ which are the structure constants of the 3-Lie algebra $[t_a, t_b, t^c] = f_{ab}{}^{cd} t_d$. This construction is not unlike writing the matrices $\mathbf{A}_\mu = A^a_\mu f_a{}^{bc} = (A_\mu)^{bc}$ of ordinary Yang-Mills gauge theory in terms of the adjoint representation of the gauge algebra : $[t_a, t_b] = f_{ab}{}^c t_c$. Furthermore, our field strengths $F_{\mu\nu}$ are explicitly defined in terms of a 3-bracket $[B_\mu, B_\nu, \Phi]$

involving an auxiliary octonionic-valued scalar field $\Phi = \Phi^a e_a$ which plays the role of a "coupling" function. Whereas the definition of $F_{\mu\nu}$ by [33], [34] was based on the standard commutator of the matrices $(\tilde{A}_\mu)^a_c(\tilde{A}_\nu)^c_b - (\tilde{A}_\nu)^a_c(\tilde{A}_\mu)^c_b$.

A thorough discussion of the relevance of ternary and nonassociative structures in Physics has been provided in [27], [5], [6]. The earliest example of nonassociative structures in Physics can be found in Einstein's special theory of relativity. Only colinear velocities are commutative and associative, but in general, the addition of non-colinear velocities is non-associative and non-commutative. A putative noncommutative and nonassociative gravity theory for closed strings probing curved backgrounds with non-vanishing three-form flux based on a three-bracket structure were recently discussed by [32] . Nonassociative star product deformations for D-brane world volume in curved backgrounds were studied by [31]. The construction relied in the Kontsevich noncommutative and nonassociative star product.

The complexification of ordinary gravity (not to be confused with Hermitian-Kahler geometry) has been known for a long time. Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ so that now one has $g_{\nu\mu} = (g_{\mu\nu})^*$, which implies that the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\bar{z}_1 \bar{z}_1} = g_{\bar{z}_2 \bar{z}_2}$ must be real. A treatment of a non-Riemannan geometry based on a complex tangent space and involving a symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric component was first proposed by Einstein-Strauss [8] (and later on by [18]) in their unified theory of Electromagentism with gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component.

Borchsenius [17] formulated the quaternionic extension of Einstein-Strauss unified theory of gravitation with EM by incorporating appropriately the $SU(2)$ Yang-Mills field strength into the degrees of freedom of a quaternionc-valued metric. Oliveira and Marques [19] later on provided the Octonionic Gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang-Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions.

The Octonionic Gravity developed by [19] was extended to Noncommutative and Nonassociative Spacetime coordinates associated with octonionic-valued coordinates and momenta by [20]. The octonionic metric $\mathbf{G}_{\mu\nu}$ already encompasses the ordinary spacetime metric $g_{\mu\nu}$, in addition to the Maxwell $U(1)$ and $SU(2)$ Yang-Mills fields such that implements the Kaluza-Klein Grand Unification program *without* introducing extra spacetime dimensions. The color group $SU(3)$ is a subgroup of the exceptional G_2 group which is the automorphism group of the octonion algebra. The flux of the $SU(2)$ Yang-Mills field strength $\vec{\mathcal{F}}_{\mu\nu}$ through the area-momentum $\vec{\Sigma}^{\mu\nu}$ in the *internal isospin space* yields corrections $O(1/M^2_{Planck})$ to the energy-momentum dispersion relations without violating Lorentz invariance as it occurs with Hopf algebraic deformations of the Poincare algebra.

After this brief preamble we proceed with the main results of this work which is the construction, to our knowledge, of a novel nonassociative octonionic ternary gauge field theory. We conclude with a few remarks about the plausible future avenues of research.

2 Octonionic Ternary Gauge Field Theories

Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_m) as

$$\mathbf{X} = x^o\ e_o\ +\ x^m\ e_m, \quad m, n, p = 1, 2, 3, ...7. \tag{2.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_i = e_i e_o = e_i,\ e_i e_j = -\delta_{ij} e_o\ + c_{ijk} e_k,\ i, j, k = 1, 2, 3, ...7. \tag{2.2}$$

where the fully antisymmetric structure constants c_{ijk} are taken to be 1 for the combinations $(124), (235), (346), (457), (561), (672), (713)$ [29]. The octonion conjugate is defined by $\bar{e}_o = e_o$, $\bar{e}_m = -e_m$

$$\bar{\mathbf{X}} = x^o\ e_o\ -\ x^m\ e_m. \tag{2.3}$$

and the norm is

$$N(\mathbf{X})\ =\ <\ \mathbf{X}\ \mathbf{X}\ >\ =\ Real\ (\bar{\mathbf{X}}\ \mathbf{X})\ = (x_o\ x_o\ + x_k\ x_k). \tag{2.4}$$

The inverse

$$\mathbf{X}^{-1}\ =\ \frac{\bar{\mathbf{X}}}{N(\mathbf{X})}, \quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{2.5}$$

The non-vanishing associator is defined by

$$(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{2.6}$$

In particular, the associator

$$(e_i, e_j, e_k)\ =\ (e_i e_j) e_k - e_i (e_j e_k)\ =\ 2\ d_{ijkl}\ e_l$$

$$d_{ijkl}\ =\ \frac{1}{3!}\ \epsilon_{ijklmnp}\ c^{mnp},\ i, j, k.... = 1, 2, 3, ...7 \tag{2.7}$$

The generators of the split-octonionic algebra admit a realization in terms of the 4×4 Zorn matrices (in blocks of 2×2 matrices) by writing

$$u_o = \frac{1}{2}\ (e_o + i e_7), \quad u_o^* = \frac{1}{2}\ (e_o - i e_7)$$

$$u_i = \frac{1}{2}\ (e_i + i e_{i+3}), \quad u_i^* = \frac{1}{2}\ (e_i - i e_{i+3}) \tag{2.8}$$

$$u_o = \begin{pmatrix} 0 & 0 \\ 0 & \omega_o \end{pmatrix} \quad u_o^* = \begin{pmatrix} \omega_o & 0 \\ 0 & 0 \end{pmatrix}$$

$$u_i = \begin{pmatrix} 0 & 0 \\ \omega_i & 0 \end{pmatrix} \quad u_i^* = \begin{pmatrix} 0 & -\omega_i \\ 0 & 0 \end{pmatrix} \tag{2.9}$$

The quaternionic generators $\omega_o,\ \omega_i,\ i = 1,2,3$ obey the algebra $\omega_i \omega_j = \epsilon_{ijk}\omega_k - \delta_{ij}\omega_o$ and are related to the Pauli spin 2×2 matrices by setting $\sigma_i = i\ \omega_i$ and $\omega_o = \mathbf{1}_{2\times 2}$.

The u_i, u_i^* behave like fermionic creation and annihilation operators corresponding to an exceptional (non-associative) Grassmannian algebra

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_j^*\} = -\delta_{ij}. \tag{2.10a}$$

$$\frac{1}{2}[u_i, u_j] = \epsilon_{ijk}\ u_k^*, \quad \frac{1}{2}[u_i^*, u_j^*] = \epsilon_{ijk}\ u_k, \quad u_o^2 = u_o, \quad (u_o^*)^2 = u_o^*. \tag{2.10b}$$

Unlike the octonionic algebra, the split-octonionic algebra contains zero divisors and therefore is not a division algebra.

The automorphism group of the octonionic algebra is the 14-dim exceptional G_2 group that admits a $SU(3)$ subgroup leaving invariant the idempotents u_o, u_o^*. This $SU(3)_c$ was identified as the color group acting on the quarks and antiquarks triplets [11] $\Psi_\alpha = u_i\ \Psi^i_\alpha,\ \bar{\Psi}_\alpha = -u_i^*\ \bar{\Psi}^i_\alpha,\ i = 1,2,3$, respectively. From the split-octonionic algebra multiplication table one learns that *triplet* $\times$ *triplet* $=$ *anti triplet* and *triplet* $\times$ *anti triplet* $=$ *singlet* providing a very natural algebraic interpretation of confinement of 3 quarks.

The multiplication product of the split-octonions generators u_o, u_o^*, u_i, u_i^* is reproduced in this Zorn matrix realization. The Zorn matrix product of

$$\mathbf{A} = \begin{pmatrix} A_o\ \omega_o & -A_i\ \omega^i \\ B_i\ \omega^i & B_o\ \omega_o \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} C_o\ \omega_o & -C_i\ \omega^i \\ D_i\ \omega^i & D_o\ \omega_o \end{pmatrix} \tag{2.11}$$

is defined by

$$\mathbf{A \bullet B} = \begin{pmatrix} (A_o C_o + A_i D_i)\ \omega_o & -(A_o C_k + D_o A_k + \epsilon_{ijk} B_i D_j)\ \omega^k \\ (C_o B_k + B_o D_k + \epsilon_{ijk} A_i C_j)\ \omega^k & (B_o D_o + B_i C_i)\ \omega_o \end{pmatrix} \tag{2.12}$$

where we have used

$$\omega_i\ \omega_j = \epsilon_{ijk}\omega_k - \delta_{ij}\omega_o \ \Rightarrow\ \omega_i\ \omega_i = -\omega_o,\ for\ each\ i = 1,2,3\ \Rightarrow$$

$$\vec{x}.\vec{y} = (x_i\omega_i)\ (y_i\omega_i) = -x_i y_i\ \omega_o. \tag{2.13}$$

In this section we shall focus entirely in the octonion algebra. Yamazaki [28] constructed a realization of a generalized Lie ternary-algebra using the Octonions by defining a three-bracket. It requires using the left and right operators $L_u v = uv,\ R_u v = vu$ and the derivative operator constructed by Nambu [27]

$$D_{u,v}\ x\ =\ (\ [L_u,\ L_v]\ +\ [R_u,\ R_v]\ +\ [L_u,\ R_v]\ )\ x. \tag{2.14}$$

The operator $D_{u,v}$ obeys the analog of Liebnitz rule

$$D_{u,v}\ (xy)\ =\ (D_{u,v}x)y\ +\ x(D_{u,v}y). \tag{2.15}$$

and allows to define the three-bracket as

$$[\,u,\ v,\ x\,]\ \equiv\ D_{u,v}\ x\ =\ \frac{1}{2}\ (\ u(vx) - v(ux)\ +\ (xv)u\ -\ (xu)v\ +\ u(xv)\ -\ (ux)v\ ). \tag{2.16a}$$

For the octonionic algebra, after a straightforward calculation when the indices span the imaginary elements $a, b, c, d = 1, 2, 3, ..., 7$, one has that

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcd}\ e_d\ =\ -\ [\ d_{abcd}\ -\ \delta_{ac}\ \delta_{bd}\ +\ \delta_{bc}\ \delta_{ad}\ ]\ e_d \tag{2.16b}$$

whereas

$$[\ e_a,\ e_b,\ e_0\ ]\ =\ [\ e_a,\ e_0,\ e_b\ ]\ =\ [\ e_0,\ e_a,\ e_b\ ]\ =\ 0 \tag{2.16c}$$

where the totally antisymmetric associator structure constants d_{abcd} are the 7-dim "duals" to the c_{abc} structure constants as shown by eq-(2.7). Yamazaki [28] has shown that the 3-brackets (2.16a) obey the fundamental identity (2.20) below. This follows from the algebraic properties of derivations based on the analog of Liebnitz rules of differentiation. One should notice that

$$[\ u,\ v,\ x\ ]\ \neq\ \frac{1}{2}\ (\ [\ [\ u,\ v\ ],\ x\ ]\ -\ (u, x, v)\ ) \tag{2.17}$$

where $(u, x, v) = (ux)v - u(xv)$ is the nonvanishing associator. For nonassociative algebras, the Jacobi identity is not obeyed and the Jacobiator is not zero

$$J(x, y, z)\ \equiv\ [\ [\ x,\ y\ ],\ z\ ]\ +\ [\ [\ y,\ z\ ],\ x\ ]\ +\ [\ [\ z,\ x\ ],\ y\ ]\ \neq\ 0 \tag{2.18}$$

In particular, the Jacobiator associated with the octonion algebra is proportional to the associator $J(e_a, e_b, e_c) \sim (e_a, e_b, e_c) = 2d_{abcd}e_d$. The octonion algebra is also a Malcev algebra [30].

The commutator of two generalized derivative operators acting on z is

$$[\ D_{u,v},\ D_{x,y}\ ]\ z\ =\ D_{u,v}\ D_{x,y}\ z\ -\ D_{x,y}\ D_{u,v}\ z\ =$$

$$D_{[u,v,x],y}\ z\ +\ D_{x,[u,v,y]}\ z\ =\ [\ [u,\ v,\ x],\ y,\ z\ ]\ +\ [\ x,\ [u,\ v,\ y],\ z\ ]. \tag{2.19}$$

The result in (2.19) is a direct consequences of the fundamental identity

$$[\ [x,\ u,\ v],\ y,\ z\ ] + [\ x,\ [y,\ u,\ v],\ z\ ] + [\ x,\ y,\ [z,\ u,\ v]\ ]\ =\ [\ [x,\ y,\ z],\ u,\ v\ ] \quad (2.20)$$

which is *obeyed* by the 3-bracket (2.16) [28]. A bilinear positive symmetric product $< u, v > = < v, u >$ is required such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$< [u,\ v,\ x],\ y > \ = \ - \ < [u,\ v,\ y],\ x > \ = \ - \ < x,\ [u,\ v,\ y] > \ \Rightarrow$$

$$D_{u,v}\ < x, y >\ =\ 0 \quad (2.21a)$$

The symmetric product remains invariant under derivations. There is also the additional symmetry condition required by [28]

$$< [u,\ v,\ x],\ y > \ = \ < [x,\ y,\ u],\ y > \quad (2.21b)$$

Okubo [4] constructed an octonionic triple product which is totally antisymmetric in all of its entries and is given by

$$[\ x,\ y,\ z\ ]_{Okubo}\ =\ \frac{1}{2}\ (\ (x,y,z)\ +\ < x, e_o >\ [y,z]\ +\ < y, e_o >\ [z,x]\ )\ +$$

$$\frac{1}{2}\ (\ < z, e_o >\ [x,y]\ -\ <\ z, [x,y]\ >\ e_o\ ). \quad (2.22)$$

where e_o is the Octonion unit element and $(x,y,z) = (xy)z - x(yz)$ is the nonvanishing associator for nonassociative algebras.

The ternary product that we shall be using in this work is the one provided by Yamazaki (2.16) which *obeys* the key fundamental identity (2.20) and leads to the structure constants f_{abcd} that are *pairwise* antisymmetric but are *not* totally antisymmetric in all of their indices : $f_{abcd} = -f_{bacd} = -f_{abdc} = f_{cdab}$; however : $f_{abcd} \neq f_{cabd}$; and $f_{abcd} \neq -\ f_{dbca}$. The associator ternary operation for octonions $(x,y,z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (2.20) as emphasized by [28]. For this reason we cannot use the associator to construct the 3-bracket.

Equipped with the above definition of the 3-bracket eq-(2.16) one may now proceed with the explicit construction of a nonassociative and noncommutative ternary gauge field theory based on the octonions. The building elements are the octonionic-valued gauge field $B_\mu = B^a_\mu\ e_a$ and an *auxiliary* octonionic-valued scalar field $\Phi = \Phi^a e_a$. The ternary infinitesimal gauge transformations for the fields B_μ, Φ are defined respectively in terms of the parameters $\Lambda^{ab} = -\Lambda^{ba}$ and Λ^a as

$$\delta\ (B^m_\mu\ e_m)\ =\ -\ \partial_\mu\ (\Lambda^m\ e_m)\ +\ \Lambda^{ab}\ [e_a\ , e_b,\ \ B^c_\mu\ e_c]\ =$$

$$-\ \partial_\mu\ (\Lambda^m\ e_m)\ +\ \Lambda^{ab}\ B^c_\mu\ f_{abc}{}^m\ e_m \quad (2.23a)$$

$$\delta\ (\Phi^m\ e_m)\ =\ \Lambda^{ab}\ [e_a,\ e_b,\ \Phi^c\ e_c]\ =\ \Lambda^{ab}\ \Phi^c\ f_{abc}{}^m\ e_m \tag{2.23b}$$

The second rank field strength tensor based on the ternary algebra (3-brackets) is defined *differently* from the Yang-Mills case (based on 2-brackets) as follows

$$F_{\mu\nu}\ =\ \partial_\mu B_\nu\ -\ \partial_\nu B_\mu\ +\ [B_\mu,\ B_\nu,\ \Phi]. \tag{2.24}$$

where $\Phi = \Phi^a e_a$ is the *auxiliary* octonionic-valued scalar field without dynamical degrees of freedom and which plays the role of an effective octonionic "coupling" function. By recurring to the infinitesimal transformations (2.23a) one has

$$\delta(F_{\mu\nu})\ =\ \partial_\mu(\delta B_\nu)\ -\ \partial_\nu(\delta B_\mu)\ +\ \delta\ ([B_\mu,\ B_\nu,\ \Phi]) \tag{2.25}$$

The inhomogeneous terms in the infinitesimal gauge transformations $\delta F_{\mu\nu}$ of (2.25) cancel if

$$(\partial_\mu\Lambda^{ab})\ B^c_\nu\ f_{abcm}\ e_m\ -\ (\partial_\nu\Lambda^{ab})\ B^c_\mu\ f_{abcm}\ e_m\ -$$
$$(\partial_\mu\Lambda^a)\ B^b_\nu\ \Phi^c\ f_{abcm}\ e_m\ -\ (\partial_\nu\Lambda^b)\ B^a_\mu\ \Phi^c\ f_{abcm}\ e_m\ =\ 0 \tag{2.26}$$

Because the expression in eq-(2.31) is *not* explicitly anti-symmetric under the exchange of the indices $\mu \leftrightarrow \nu$, one must perform a series of three steps. Exchanging the $b \leftrightarrow c$ indices in the third term $-(\partial_\mu\Lambda^a)B^b_\nu\Phi^c f_{abcm}e_m$ allows to rewrite it as

$$-\ (\partial_\mu\Lambda^a)\ B^b_\nu\ \Phi^c\ f_{abcm}\ e_m\ \rightarrow\ -\ (\partial_\mu\Lambda^a)\ B^c_\nu\ \Phi^b\ f_{acbm}\ e_m. \tag{2.27}$$

Exchanging the $a \leftrightarrow c$ indices in the fourth term of (2.26) $-(\partial_\mu\Lambda^b)B^a_\nu\Phi^c f_{abcm}e_m$ allows to rewrite it as

$$-\ (\partial_\nu\Lambda^b)\ B^c_\mu\ \Phi^a\ f_{cbam}\ e_m\ =\ (\partial_\nu\Lambda^b)\ B^c_\mu\ \Phi^a\ f_{bcam}\ e_m. \tag{2.28}$$

due to the antisymmetry under the exchange of a pair of indices $f_{cbam} = -f_{bcam}$. Exchanging the $b \rightarrow a$ indices in (2.28) yields

$$(\partial_\nu\Lambda^a)\ B^c_\mu\ \Phi^b\ f_{acbm}\ e_m \tag{2.29}$$

Therefore the third and fourth terms of (2.26) can be re-expressed in a manifestly antisymmetric expression under the exchange of the $\mu \leftrightarrow \nu$ indices, as it should

$$-\ (\partial_\mu\Lambda^a)\ B^c_\nu\ \Phi^b\ f_{acbm}\ e_m\ +\ (\partial_\nu\Lambda^a)\ B^c_\mu\ \Phi^b\ f_{acbm}\ e_m. \tag{2.30}$$

Finally, by recurring to (2.30) one can rewrite (2.26) as

$$[\ (\partial_\mu\Lambda^{ab})\ f_{abcm}\ -\ (\partial_\mu\Lambda^a)\ \ \Phi^b\ f_{acbm}\ ]\ B^c_\nu\ e_m\ -$$
$$[\ (\partial_\nu\Lambda^{ab})\ f_{abcm}\ -\ (\partial_\nu\Lambda^a)\ \Phi^b\ f_{acbm}\ ]\ B^c_\mu\ e_m\ =\ 0 \tag{2.31}$$

The expression in eq-(2.31) is now explicitly anti-symmetric under the exchange of the indices $\mu \leftrightarrow \nu$. A solution expressing the gauge parameters Λ^{ab}, Λ^a in terms of the Φ^c components can be found by setting

$$[\ (\partial_\mu\Lambda^{ab})\ f_{abcm}\ -\ (\partial_\mu\Lambda^a)\ \Phi^b\ f_{acbm}\ ]\ B^c_\nu\ e_m\ =\ 0 \tag{2.32a}$$

$$[\ (\partial_\nu \Lambda^{ab})\ f_{abcm}\ -\ (\partial_\nu \Lambda^a)\ \Phi^b\ f_{acbm}\ ]\ B^c_\mu\ e_m\ =\ 0 \qquad (2.32b)$$

Eq-(2.32a) is equivalent to eq-(2.32b) since the latter is obtained from the former by a simple exchange of the μ, ν indices. Therefore, by setting the expression inside the parenthesis of (2.32a) to zero one arrives at the desired relation among the gauge parameters and the auxiliary field Φ^c components

$$(\partial_\mu \Lambda^{ab})\ f_{abcm}\ =\ (\partial_\mu \Lambda^a)\ \ \Phi^b\ f_{acbm} \qquad (2.33)$$

The equivalent eq-(2.32b) yields the same functional relation as (2.33) by a simple exchange of the μ, ν indices. The field components B^c_ν have *decoupled* from eq-(2.33) as it should since one does not wish to impose any constraints among B^c_ν and the gauge parameters.

Because the left hand side of (2.33) contains terms which are explicitly anti-symmetric in the a, b and c, m indices one must decompose the right hand side into a symmetric and antisymmetric pieces leading then to

$$(\partial_\mu \Lambda^{ab})\ [\ d_{abcm}\ -\ \delta_{ac}\ \delta_{bm}\ +\ \delta_{bc}\ \delta_{am}\ ]\ =$$

$$\frac{1}{2}\ [\ (\partial_\mu \Lambda^a)\ \Phi^b\ -\ (\partial_\mu \Lambda^b)\ \Phi^a\ ][\ d_{acbm}\ +\ \delta_{c[b}\ \delta_{a]m}\ ]; \quad \mu = 1, 2, 3, ..., D \quad (2.34)$$

Since the associator structure constants d_{acbm} are totally antisymmetric under the exchange of any pair of indices, the right hand side (2.34) is antisymmetric under the exchange of the c, m indices. While the symmetric components yield the following zero contribution

$$\frac{1}{2}\ [\ (\partial_\mu \Lambda^a)\ \Phi^b\ +\ (\partial_\mu \Lambda^b)\ \Phi^a\ ]\ [\ -\delta_{ab}\ \delta_{cm}\ +\ \delta_{c(b}\ \delta_{a)m}\ ]\ =\ 0 \qquad (2.35)$$

with

$$\delta_{c(b}\ \delta_{a)m}\ =\ \frac{1}{2}\ (\delta_{cb}\ \delta_{am} + \delta_{ca}\ \delta_{bm}); \quad \delta_{c[b}\ \delta_{a]m}\ =\ \frac{1}{2}\ (\delta_{cb}\ \delta_{am}\ -\ \delta_{ca}\ \delta_{bm}) \quad (2.36)$$

An extensive analysis of eqs-(2.34, 2.35) for the indices $a, b, c, m = 1, 2, 3, ..., 7$ is presented in the Appendix. One finds that the gauge parameters $\Lambda^a(x) =$ *constant*, whereas the $\Lambda^{ab}(x) = -\Lambda^{ba}(x)$ obey the "self-duality" equations $\frac{1}{2} d_{abcm} \Lambda^{ab}(x) = \Lambda_{cm}(x)$ and whose *nontrivial* solutions are given in terms of 7 *arbitrary* functions $\xi_1(x^\mu), \xi_2(x^\mu), ..., \xi_7(x^\mu)$ of the spacetime coordinates as explained in the Appendix. Thus the 21 gauge parameters $\Lambda^{ab}(x)$ can all be expressed in terms of these 7 *arbitrary* functions and rather than having 21 arbitrary functions one has only 7.

Therefore, one can ensure that the ternary field strength $F_{\mu\nu}$ defined in terms of the 3-brackets (2.24) transforms properly (homogeneously) under the ternary gauge transformations if eqs-(2.34, 2.35) are obeyed in order to ensure a cancelation of the *inhomogeneous* pieces under infinitesimal ternary gauge transformations. This is permissible because $\Phi = \Phi^c e_c$ is not endowed with any dynamics, it is just another variable that one can interpret as an octonionic-valued "coupling" function and whose components can rotate into each other. The real (scalar) part Φ^0 remains invariant under the transformations $\delta(\Phi^m e_m) = \Lambda^{ab}[e_a, e_b, \Phi^c e_c]$ because $[e_a, e_b, e_0] = 0$.

To conclude finally, due to a cancelation of the *inhomogeneous* pieces, under infinitesimal ternary gauge transformations, one can infer that $F_{\mu\nu}$ does transform *homogeneously* under the infinitesimal ternary gauge transformations as

$$\delta(F^m_{\mu\nu}\, e_m) \;=\; \Lambda^{ab}\,[\, e_a,\; e_b,\; F^c_{\mu\nu}\; e_c\,] \;=\; \Lambda^{ab}\, F^c_{\mu\nu}\, f_{abc}{}^m\, e_m \;\Rightarrow\; \delta F^m_{\mu\nu} \;=\; \Lambda^{ab}\, F^c_{\mu\nu}\, f_{abc}{}^m \tag{2.37}$$

The results (2.37) is a direct consequence of the fundamental identity because the 3-bracket (2.16a) is defined as a derivation

$$[\,[e_a,\; e_b,\; B_\mu],\; B_\nu,\; \Phi\,] \;+\; [\, B_\mu,\; [e_a,\; e_b,\; B_\nu],\; \Phi\,] \;+\; [\, B_\mu,\; B_\nu,\; [e_a,\; e_b,\; \Phi]\,] \;=$$

$$[\, e_a,\; e_b,\; [B_\mu,\; B_\nu,\; \Phi]\,] \tag{2.38}$$

The parameter $\Lambda^0(x)$ involved in the transformation $\delta B^0_\mu = -\partial_\mu \Lambda^0(x)$, corresponding to the real (identity) element e_0 of the octonion algebra, leads to $\delta F^0_{\mu\nu} = 0$ where the field strength component is Abelian-Maxwell-like $F^0_{\mu\nu} = \partial_\mu B^0_\nu - \partial_\nu B^0_\mu$.

Similar findings as those obtained in eqs-(2.34, 2.35) among the gauge parameters and the octonionic-valued auxiliary field Φ can be found such that the ternary covariant derivative of an octonionic-valued scalar field $\Theta(x^\mu) = \Theta^a(x^\mu) e_a$ defined as

$$D_\mu \Theta \;=\; \partial_\mu \Theta \;+\; [\; B_\mu,\; \Theta,\; \Phi\;] \tag{2.39}$$

transforms homogeneously under the transformations

$$\delta\Theta \;=\; \Lambda^{ab}[e_a, e_b, \Theta] \;\Rightarrow\; \delta\; (D_\mu \Theta) \;=\; \Lambda^{ab}\;[\; e_a,\; e_b,\; (D_\mu\Theta)\;] \tag{2.40}$$

The same relations as those in (2.34, 2.35) can also be found for the octonionic-valued third rank antisymmetric field strength

$$F_{\mu\nu\rho} \;=\; \partial_\rho\, B_{\mu\nu} + \partial_\mu\, B_{\nu\rho} + \partial_\nu\, B_{\rho\mu} - [B_{\mu\nu},\; B_\rho,\; \Phi] - [B_{\nu\rho},\; B_\mu,\; \Phi] - [B_{\rho\mu},\; B_\nu,\; \Phi]. \tag{2.41a}$$

such that under ternary infinitesimal gauge transformations of the form $\delta B_{\mu\nu} = \Lambda^{ab}[e_a, e_b, B_{\mu\nu}]$, it transforms as

$$\delta(F^m_{\mu\nu\rho}\; e_m) \;=\; \Lambda^{ab}\;[\; e_a,\; e_b,\; F^c_{\mu\nu\rho}\; e_c\;] \tag{2.41b}$$

Similar results follow for the octonionic-valued fourth-rank antisymmetric tensor

$$F_{\mu\nu\rho\tau} \;=\; \partial_\tau\; B_{\mu\nu\rho} \;-\; \partial_\mu\; B_{\nu\rho\tau} \;+\; \partial_\nu\; B_{\rho\tau\mu} \;-\partial_\rho\; B_{\tau\mu\nu} \;-$$

$$[B_{\mu\nu\rho},\; B_\tau,\; \Phi] + [B_{\nu\rho\tau},\; B_\mu,\; \Phi] - [B_{\rho\tau\mu},\; B_\nu,\; \Phi] + [B_{\tau\mu\nu},\; B_\rho,\; \Phi] \;\Rightarrow \tag{2.42a}$$

$$\delta(F^m_{\mu\nu\rho\tau}\; e_m) \;=\; \Lambda^{ab}\;[\; e_a,\; e_b,\; F^c_{\mu\nu\rho\tau}\; e_c\;] \tag{2.42b}$$

and so forth.

Given the octonionic valued field strength $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, with *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ i = 1, 2, 3, ..., 7$, a gauge invariant action under ternary infinitesimal gauge transformations in D-dim is

$$S \ = \ - \ \frac{1}{4\kappa^2} \int d^D x \ < \ F_{\mu\nu} \ F^{\mu\nu} \ > \tag{2.43}$$

κ is a numerical parameter introduced to make the action dimensionless and it can be set to unity for convenience. The $<\quad>$ operation is defined as $< XY >= Real(\bar{X}Y) =< YX >= Real(\bar{Y}X)$. Under infinitesimal ternary gauge transformations of the action one has

$$\delta\ S \ = \ -\frac{1}{4} \int d^D x \ < \ F_{\mu\nu} \ (\delta F^{\mu\nu}) \ + \ (\delta F_{\mu\nu}) \ F^{\mu\nu} \ > \ =$$

$$-\frac{1}{4} \int d^D x \ < \ F^c_{\mu\nu} \ e_c \ \Lambda^{ab} \ [e_a, \ e_b, \ F^{\mu\nu \ n} \ e_n] \ > \ +$$

$$-\frac{1}{4} \int d^D x \ < \ \Lambda^{ab} \ [e_a, \ e_b, \ F^c_{\mu\nu} \ e_c] \ F^{\mu\nu \ n} \ e_n \ > \ =$$

$$-\frac{1}{4} \int d^D x \, \Lambda^{ab} \, F^c_{\mu\nu} \, F^{\mu\nu \ n} \ (\ < \ e_c \ f_{abnk} \ e_k \ > \ + \ < \ f_{abck} \ e_k \ e_n \ > \ ) \ = \ 0. \tag{2.44}$$

since

$$< \ e_c \, f_{abnk} \, e_k \ > + < \ f_{abck} \, e_k \, e_n \ > \ = \ f_{abnk} \, \delta_{ck} + f_{abck} \, \delta_{kn} \ = \ f_{abnc} + f_{abcn} \ =$$

$$- \ [\ d_{abnc} \ - \ \delta_{an} \ \delta_{bc} \ + \ \delta_{bn} \ \delta_{ac} \ ] \ - \ [\ d_{abcn} \ - \ \delta_{ac} \ \delta_{bn} \ + \ \delta_{bc} \ \delta_{an} \ ] \ = \ 0 \tag{2.45a}$$

because $d_{abnc}+d_{abcn} = 0$; $d_{nabc}+d_{cabn} = 0$, due to the total antisymmetry of the associator structure constant d_{nabc} under the exchange of any pair of indices. Invariance $\delta S = 0$, only occurs if, and only if, $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c] \ \neq \ \Lambda^{ab}[F^c e_c, e_a, e_b]$. The ordering inside the 3-bracket is crucial. One can check that if one sets $\delta F = \Lambda^{ab}[F^c e_c, e_a, e_b]$, the variation δS leads to a term in the integral which is *not* zero

$$f_{nabc} + f_{cabn} \ = \ - \, [\, d_{nabc} - \delta_{nb} \, \delta_{ac} + \delta_{ab} \, \delta_{nc} \,] - [\, d_{cabn} - \delta_{cb} \, \delta_{an} + \delta_{ab} \, \delta_{cn} \,] \ \neq \ 0 \tag{2.45b}$$

However, under $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c]$, the variation δS is indeed zero as shown. This is a consequence of the fact that $[e_a, e_b, e_c] \neq [e_c, e_a, e_b]$ when the 3-bracket is given by eqs-(2.16).

To show that the action is invariant under finite ternary gauge transformations requires to follow a few steps. Firstly, one defines

$$< \ x \, y \ > \ \equiv \ Real \, [\, \bar{x} \, y \,] \ = \ \frac{1}{2} \, (\, \bar{x} \, y + \bar{y} \, x \,) \ \Rightarrow \ < \ x \, y \ > \ = \ < \ y \, x \ > \tag{2.46}$$

Despite nonassociativity, the *very special conditions*

$$x(\bar{x}u) \ = \ (x\bar{x})u; \quad x(u\bar{x}) \ = \ (xu)\bar{x}; \quad x(xu) \ = \ (xx)u; \quad x(ux) \ = \ (xu)x \tag{2.47}$$

are obeyed for octonions resulting from the Moufang identities. Despite that $(xy)z \neq x(yz)$ one has that their real parts obey

$$Real\ [\ (x\ y)\ z\ ]\ =\ Real\ [x\ (y\ z)\ ] \tag{2.48}$$

Due to the nonassociativity of the algebra, in general one has that $(gF)g^{-1} \neq g(Fg^{-1})$. However, if and only if $g^{-1} = \bar{g} \Rightarrow \bar{g}g = g\bar{g} = 1$, as a result of the the *very special conditions* (2.47) one has that $F' = (gF)g^{-1} = g(Fg^{-1}) = gFg^{-1} = gF\bar{g}$ is *unambiguously* defined.

Hence, by repeated use of eqs-(2.46, 2.47, 2.48), when $g^{-1} = \bar{g}$, the action density (2.43) is also invariant under finite gauge transformations of the form

$$< F'\ F'\ > \ = \ Re\,[\bar{F}'\ F']\ =\ Re\,[(g\bar{F}g^{-1})\,(gFg^{-1})]\ =\ Re\,[(g\bar{F})\,(\ g^{-1}\,(gF\,g^{-1})\,)]\ =$$

$$Re\ [(g\ \bar{F})\ (g^{-1}\ g)\ (Fg^{-1})]\ =\ Re\ [(g\bar{F})\ (Fg^{-1})]\ =\ Re\ [(Fg^{-1})\ (g\bar{F})]\ =$$

$$Re\,[F\,(\ g^{-1}\,(g\ \bar{F})\,)]\ =\ Re\,[F\,(g^{-1}g)\ \bar{F}]\ =\ Re\,[F\ \bar{F}]\ =\ Re\,[\bar{F}\ F]\ =\ <\ F\ F\ >. \tag{2.49}$$

One can verify that the expression for $g = exp\ (-\alpha\Lambda^{ab}[e_a, e_b]);\ g^{-1} = \bar{g} = exp\ (\alpha\Lambda^{ab}[e_a, e_b])$, where one excludes the identity element e_0 from the above definition of g because it yields the trivial transformation $g = 1$, and $\alpha = \frac{1}{4}$ is a real numerical constant, yields the finite gauge transformations

$$F'\ =\ e^{-\alpha\Lambda^{ab}[\ e_a,\ e_b\ ]}\ (F^c\ t_c)\ e^{\alpha\Lambda^{ab}[\ t_a,\ t_b\ ]}. \tag{2.50}$$

which agree with the *ternary* ones when the real parameters Λ^{ab} are infinitesimals

$$\delta F\ =\ F'\ -F\ =\ \Lambda^{ab}\ F^c\ [\ e_a,\ e_b,\ e_c\ ]\ =\ -\ \alpha\ \Lambda^{ab}\ F^c\ [\ [\ e_a,\ e_b\ ],\ e_c]\ \Rightarrow$$

$$\Lambda^{ab}\ F^c\ f_{abcm}\ e_m\ =\ -\,\alpha\,\Lambda^{ab}\ F^c\ (2c_{abd})\ (2c_{dcm})\ e_m\ \Rightarrow\ -4\,\alpha\ c_{abd}\ c_{dcm}\ =\ f_{abcm}. \tag{2.51}$$

Therefore, by choosing $\alpha = \frac{1}{4}$ one arrives at the condition among the structure constants $c_{abd}\ c_{dcm}\ =\ -\ f_{abcm}$ which is indeed *obeyed* for the octonion algebra as shown in [29]; i.e. the Yamazaki 3-bracket (2.16) satisfies the identity for octonions when $a, b, c, m = 1, 2, 3, ..., 7$

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcm}\ e_m = \ -\ [\ d_{abcm}\ -\ \delta_{ac}\ \delta_{bm}\ +\ \delta_{bc}\ \delta_{am}\ ]\ e_m\ =$$

$$-\ \frac{1}{4}\ [\ [\ e_a,\ e_b\ ],\ e_c\ ]\ =\ -\ c_{abd}\ c_{dcm}\ e_m\ \Rightarrow$$

$$c_{abd}\ c_{dcm}\ =\ d_{abcm}\ -\ \delta_{ac}\ \delta_{bm}\ +\ \delta_{bc}\ \delta_{am} \tag{2.52}$$

d_{abcm} are the associator structure constants given by the duals to the octonion structure constants as shown in eq-(2.7). A series of identities involving the structure constants of octonions can be found in [29]. Therefore, by choosing $\alpha = \frac{1}{4}$, the equality in eq-(2.51) is indeed satisfied for the octonion algebra and such that for infinitesimal real valued parameters Λ^{ab} eq-(2.50) yields to lowest order $\delta F = F' - F = \Lambda^{ab}[e_a, e_b, F]$ recovering the homogeneous ternary infinitesimal gauge transformations for the field strengths as expected.

Since the action (2.43) is invariant under finite and infinitesimal ternary gauge transformations, this means that $S[A^a_\mu; \Phi^a] = S[(A^a_\mu)'; (\Phi^a)' = C^a]$ where $\mathbf{C} = C^a e_a$ is a constant octonionic-valued "coupling" obtained from gauging the octonionic function Φ to a constant $\mathbf{C}$ by performing a finite gauge transformation with $\bar{g} = g^{-1} : \mathbf{C} = g(x)\Phi(x)g^{-1}(x) \Rightarrow \Phi(x) = g^{-1}(x)\mathbf{C}g(x)$ and such that $\Phi^a =< e^a(g^{-1}(x)\mathbf{C}g(x)) >$. Because the real parts $\Phi^0 = C^0$ remain invariant one may identify $\Phi^0 = C^0$ with a physical coupling constant. The physical interpretation of the remaining 7 vector charges/couplings $C^i, i = 1, 2, 3, ...7$ deserves further investigation.

We should remark that when $a = 1, 2, 3, ..., 7$ (excluding the unit element), having $g = exp(\Lambda^a e_a);\ g^{-1} = \bar{g} = exp(-\Lambda^a e_a)$, a finite gauge transformation of the form $F'' = gFg^{-1} = gF\bar{g}$ leads also to an invariant action (2.43). The infinitesimal transformations are in this case $\delta F = F'' - F = \Lambda^a[e_a, F^c\ e_c]$ which leave the action (2.43) *invariant*. However we must emphasize that we must *not* identify the ternary transformations with the ordinary ones based on 2-brackets : $F' \neq F''$ and $\Lambda^{ab}[e_a, e_b, F^c e_c] \neq \Lambda^a[e_a, F^c e_c]$.

We conclude with a few remarks. Wulkenhaar [39] succeeded in formulating another type of geometry which shares some similarities with Connes Noncommutative Geometry (NCG). The theory was coined Nonassociative geometry (NAG). The main difference with the two theories is that NAG is based on a unitary Lie algebra, instead of a unital associative star algebra. A left-right gauge model of Pati-Mohapatra within the context of Nonassociative geometry was provided by [40]. At the tree level they obtained mass relations and mixing angles identical to the ones obtained in $SO(10)$ GUT. It is warranted to explore what kind of phenomenological particle physics models can be developed within the framework of the nonassociative octonionic ternary gauge field theory built in this work. A thorough analysis of Octonionic spinors can be found in [23].

Also, Noncommutative and Nonassociative octonionic gauge field theories of gravity deserve investigation. Comparisons with the standard Octonionic gravity [19], [20] and the E_8 gauge theory of gravity in $8D$ [21] must be made. The split-octonions ternary gauge field theory case should follow naturally. The cubic matrices $\mathbf{A}_\mu = A^a_\mu f_a{}^{bcd} = (A_\mu)^{bcd}$ can be used to construct a ternary product

$$(A \bullet B \bullet C)_{j_1 j_2 j_3} \;=\; A_{i_1 j_1 k_1}\ B_{k_1 j_2 k_2}\ C_{k_2 j_3 i_1}. \tag{2.53}$$

(where the summation is taken over repeated indices) and should play an important role. The ternary $\bullet$ product among the cubic matrices is nonassociative in the sense that

$$A \bullet B \bullet (C \bullet D \bullet E) \;\neq\; A \bullet (B \bullet C \bullet D) \bullet E \;\neq\; (A \bullet B \bullet C) \bullet D \bullet E. \tag{2.54}$$

The quantization program is a challenging task. The non-Desarguesian geometry of the Moufang projective plane to describe Octonionic QM was studied in detail by [11]. This would be a starting point.

APPENDIX

In this appendix we shall study the system of equations (2.34, 2.35)

$$(\partial_\mu \Lambda^{ab}) \; [\; d_{abcm} \; - \; \delta_{ac} \; \delta_{bm} \; + \; \delta_{bc} \; \delta_{am} \;] \; =$$

$$\frac{1}{2} \; [\; (\partial_\mu \Lambda^a) \; \Phi^b \; - \; (\partial_\mu \Lambda^b) \; \Phi^a \;][\; d_{acbm} \; + \; \delta_{c[b} \; \delta_{a]m} \;]; \quad \mu = 1, 2, 3, ..., D \quad (A.1)$$

$$\frac{1}{2} \; [\; (\partial_\mu \Lambda^a) \; \Phi^b \; + \; (\partial_\mu \Lambda^b) \; \Phi^a \;] \; [\; - \delta_{ab} \; \delta_{cm} \; + \; \delta_{c(b} \; \delta_{a)m} \;] \; = \; 0 \qquad (A.2)$$

The solutions to (A.2) are $(\partial_\mu \Lambda^a) \; = \; 0 \;\; \Rightarrow \;\; \Lambda^a(x) \; = \; constant$. Inserting $(\partial_\mu \Lambda^a) = 0$ into eqs-(A.1) yield

$$(\partial_\mu \Lambda^{ab}) \; [\; d_{abcm} \; - \; \delta_{ac} \; \delta_{bm} \; + \; \delta_{bc} \; \delta_{am} \;] \; = \; 0 \; \Rightarrow \; (\partial_\mu \Lambda^{ab}) \; d_{abcm} \; = \; 2 \; (\partial_\mu \Lambda_{cm}) \qquad (A.3)$$

Integrating (A.3) and setting the integration constants to zero gives the "self-duality" equations

$$\frac{1}{2} \; d_{abcm} \; \Lambda^{ab}(x^\mu) \; = \; \Lambda_{cm}(x^\mu). \qquad (A.4)$$

whose general solutions will be found below. In order to solve them one must recall that the indices $a, b, c, m = 1, 2, 3, ..., 7$ are raised and lowered with the metric $-\delta^{ab}, -\delta_{ab}$. Our conventions differ by a change of sign from the conventions of [29] $B_{abcm} = -d_{abcm}$, and $b_{mnp} = -c_{mnp}$ so that the only *nonzero* components of d_{abcm} are

$$d_{1275} \; = \; d_{1236} \; = \; d_{1435} \; = \; d_{1467} \; = \; d_{2473} \; = \; d_{2465} \; = \; d_{3657} \; = \; - \; 1. \quad (A.5)$$

The "self-duality" equations are then given by

$$\frac{1}{2} \; d_{7512} \; \Lambda^{75} \; + \; \frac{1}{2} \; d_{3612} \; \Lambda^{36} \; = \; \Lambda_{12}$$

$$\frac{1}{2} \; d_{3514} \; \Lambda^{35} \; + \; \frac{1}{2} \; d_{6714} \; \Lambda^{67} \; = \; \Lambda_{14}$$

$$\frac{1}{2} \; d_{7324} \; \Lambda^{73} \; + \; \frac{1}{2} \; d_{6524} \; \Lambda^{65} \; = \; \Lambda_{24}$$

$$\frac{1}{2} \; d_{2473} \; \Lambda^{24} \; + \; \frac{1}{2} \; d_{6573} \; \Lambda^{65} \; = \; \Lambda_{73}$$

$$\frac{1}{2} \; d_{2465} \; \Lambda^{24} \; + \; \frac{1}{2} \; d_{3765} \; \Lambda^{37} \; = \; \Lambda_{65}$$

$$\frac{1}{2} \; d_{1275} \; \Lambda^{12} \; + \; \frac{1}{2} \; d_{3675} \; \Lambda^{36} \; = \; \Lambda_{75}$$

$$\frac{1}{2}\ d_{1236}\ \Lambda^{12}\ +\ \frac{1}{2}\ d_{5736}\ \Lambda^{57}\ =\ \Lambda_{36}$$

$$\frac{1}{2}\ d_{1435}\ \Lambda^{14}\ +\ \frac{1}{2}\ d_{6735}\ \Lambda^{67}\ =\ \Lambda_{35}$$

$$\frac{1}{2}\ d_{1467}\ \Lambda^{14}\ +\ \frac{1}{2}\ d_{3567}\ \Lambda^{35}\ =\ \Lambda_{67}$$

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....

$$\frac{1}{2}\ d_{2613}\ \Lambda^{26}\ +\ \frac{1}{2}\ d_{4513}\ \Lambda^{45}\ =\ \Lambda_{13}$$

$$\frac{1}{2}\ d_{1326}\ \Lambda^{13}\ +\ \frac{1}{2}\ d_{4526}\ \Lambda^{45}\ =\ \Lambda_{26}$$

$$\frac{1}{2}\ d_{1345}\ \Lambda^{13}\ +\ \frac{1}{2}\ d_{2645}\ \Lambda^{26}\ =\ \Lambda_{45} \qquad (A.6)$$

After identifying the 21 nonzero antisymmetric $\Lambda^{ab} = -\Lambda^{ba}$ quantities with the 21 variables

$$\Lambda^{12} = \Lambda_{12} = z_1,\ \Lambda^{13} = \Lambda_{13} = z_2,\ \Lambda^{14} = \Lambda_{14} = z_3,$$

$$\Lambda^{15} = \Lambda_{15} = z_4,\ \Lambda^{16} = \Lambda_{16} = z_5,\ \Lambda^{17} = \Lambda_{17} = z_6$$

$$\Lambda^{23} = \Lambda_{23} = z_7,\ \Lambda^{24} = \Lambda_{24} = z_8,\ \Lambda^{25} = \Lambda_{25} = z_9$$

$$\Lambda^{26} = \Lambda_{26} = z_{10},\ \Lambda^{27} = \Lambda_{27} = z_{11}, ...$$

....

$$\Lambda^{56} = \Lambda_{56} = z_{19},\ \Lambda^{57} = \Lambda_{57} = z_{20},\ \Lambda^{67} = \Lambda_{67} = z_{21} \qquad (A.7)$$

the eqs-(A.6) reduce to a system of 21 homogenous linear equations with 21 unknowns $z_1, z_2, ...z_{21}$ of the form $\mathbf{M}_{ij} z_j = 0$ where M_{ij} is a 21×21 matrix. The solutions are *nontrivial* $z_i \neq 0$ if and only if the determinant of the 21×21 matrix M_{ij} is *zero*. One can verify that this is indeed the case by substituting the values of d_{abcm} provided by eq-(A.5) into eqs-(A.6) and taking into account that d_{abcm} are totally antisymmetric under the exchange of any pair of indices. Eqs-(A.6), after the substitutions in eq-(A.7) and using the associator structure constants eq-(A.5), become equivalent to the system of 21 equations that can be assembled into 7 sets of equation-*triplets* for three variables z_i, z_j, z_k of the form

$$-2z_1\ -\ z_{14}\ +\ z_{20}\ =\ 0 \qquad (A.8.1)$$

$$-z_1 \ + \ z_{14} \ + \ 2z_{20} \ = \ 0 \qquad (A.8.2)$$

$$-z_1 \ - \ 2z_{14} \ - \ z_{20} \ = \ 0 \qquad (A.8.3)$$

$$-2z_8 \ + \ z_{15} \ + \ z_{19} \ = \ 0 \qquad (A.8.4)$$

$$-z_8 \ + \ 2z_{15} \ - \ z_{19} \ = \ 0 \qquad (A.8.5)$$

$$-z_8 \ - \ z_{15} \ + \ 2z_{19} \ = \ 0 \qquad (A.8.6)$$

$$-2z_3 \ - \ z_{13} \ - \ z_{21} \ = \ 0 \qquad (A.8.7)$$

$$-z_3 \ - \ 2z_{13} \ + \ z_{21} \ = \ 0 \qquad (A.8.8)$$

$$-z_3 \ + \ z_{13} \ - \ 2z_{21} \ = \ 0 \qquad (A.8.9)$$

....

....

....

$$-2z_2 \ + \ z_{10} \ + \ z_{16} \ = \ 0 \qquad (A.8.19)$$

$$z_2 \ - \ 2z_{10} \ + \ z_{16} \ = \ 0 \qquad (A.8.20)$$

$$z_2 \ + \ z_{10} \ - \ 2z_{16} \ = \ 0 \qquad (A.8.21)$$

The fact that the determinant of M_{ij} is 0 can be established by noticing that equation (A.8.1) is the linear superposition of eq-(A.8.2) and eq-(A.8.3); i.e. eq-(A.8.1) = eq-(A.8.2) + eq-(A.8.3). Similarly one can see that eq-(A.8.4) = eq-(A.8.5) + eq-(A.8.6); .... and finally eq-(A.8.20) plus eq-(A.8.21) equals minus eq-(A.8.19). Therefore the system of the above equations are linearly dependent which implies that the determinant of M_{ij} is 0. Hence, there are *nontrivial* solutions $z_i \neq 0$ such that the 21 nonzero values of Λ^{ab} are given by suitable multiples of 7 *arbitrary* functions $\xi_i(x), \ i = 1, 2, 3, ...7$ as follows.

Identify $z_1 = \xi_1(x)$ and after adding eq-(A.8.1) to eq-(A.8.2) yields

$$-3z_1 \ + \ 3z_{20} \ = \ 0 \ \Rightarrow z_{20} \ = \ z_1 \ = \ \xi_1(x) \ \Rightarrow \ \Lambda^{12} \ = \ \Lambda^{57} \ = \ \xi_1(x) \quad (A.9)$$

subtracting eq-(A.8.3) from eq-(A.8.2) gives

$$3z_{14} + 3z_{20} = 0 \Rightarrow z_{14} = - z_{20} = - z_1 = - \xi_1(x) \Rightarrow \Lambda^{36} = -\xi_1(x) \quad (A.10)$$

Identify $z_8 = \xi_2(x)$ and after adding eq-(A.8.4) to eq-(A.8.5) yields

$$-3z_8 + 3z_{15} = 0 \Rightarrow z_8 = z_{15} = \xi_2(x) \Rightarrow \Lambda^{24} = \Lambda^{37} = \xi_2(x) \quad (A.11)$$

subtracting eq-(A.8.6) from eq-(A.8.5) yields

$$3z_{15} - 3z_{19} = 0 \Rightarrow z_{15} = z_{19} = \xi_2(x) \Rightarrow \Lambda^{56} = \xi_2(x) \qquad (A.12)$$

Proceeding in the same fashion with the remaining 5 sets of equation-triplets one is able to solve for the remaining variables in terms of the *arbitrary* functions $\xi_3(x);\ \xi_4(x); ....\xi_7(x)$. Out of the 21 equations for the 21 variables $\Lambda^{ab}(x)$, assembled into 7 sets of equation-triplets, there are only 14 independent equations since any equation of a given triplet is a linear superposition of the other two equations within the triplet. Hence one has $2 \times 7 = 14$ linearly independent equations involving 21 variables. This is consistent with the fact that the automorphism group of the Octonions is the exceptional group G_2 with 14 generators.

Concluding, the key eqs-(2.34, 2.35) lead to $\Lambda^a(x) = constant$ and to the "self-duality" equations for the gauge parameters $\Lambda^{ab}(x)$ of the form $\frac{1}{2} d_{abcm} \Lambda^{ab}(x^\mu) = \Lambda_{cm}(x^\mu)$ and whose *nontrivial* solutions are given in terms of 7 *arbitrary* functions $\xi_1(x^\mu), \xi_2(x^\mu), ..., \xi_7(x^\mu)$ of the spacetime coordinates as explained above.

The parameter $\Lambda^0(x)$ involved in the transformation $\delta B^0_\mu = -\partial_\mu \Lambda^0(x)$, corresponding to the real (identity) element e_0 of the octonion algebra, leads to $\delta F^0_{\mu\nu} = 0$ for $F^0_{\mu\nu} = \partial_\mu B^0_\nu - \partial_\nu B^0_\mu$.

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