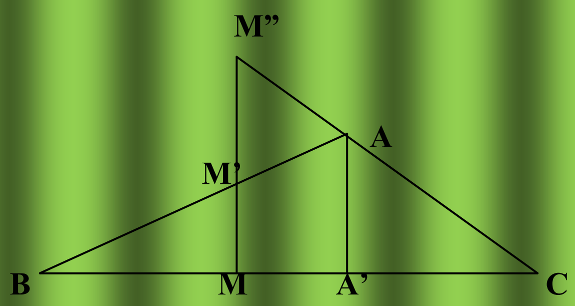
FLORENTIN SMARANDACHE

PROPOSED PROBLEMS of Mathematics

(Vol. II)

Second version - entirely translated to English



Chişinău - 2010

UNIVERSITATEA DE STAT DIN MOLDOVA

CATEDRA DE ALGEBRĂ

FLORENTIN SMARANDACHE

PROPOSED PROBLEMS of Mathematics

(Vol. II)

Second edition - entirely translated to English, 2010.

First edition published in 1997.

Chişinău

FLORENTIN SMARANDACHE, Ph D
University of New Mexico
Department of Mathematics
Gallup, NM 87301, USA

PROPOSED PROBLEMS of Mathematics
(Vol. II), *Second edition - entirely translated to English, 2010.*
First edition published in 1997.
Chişinău, U.S.M, 2010; 73p.

Peer-Reviewers
Prof. Ion Goian and Prof. Vasile Marin,
Universitatea de Stat din Moldova, Chișinău.

ISBN 9781599731292

Table of Contents

PREFACE

The first book of “Problèmes avec et sans … problèmes!” was published in Morocco in 1983.

I collected these problems that I published in various Romanian or foreign magazines (amongst which: “Gazeta Matematică”, magazine which formed me as problem solver, “American Mathematical Monthly”, “Crux Mathematicorum” (Canada), “Elemente der Mathematik” (Switzerland), “Gaceta Matematica” (Spain), “Nieuw voor Archief” (Holland), etc. while others are new proposed problems in this second volume.

These have been created in various periods: when I was working as mathematics professor in Romania (1984-1988), or co-operant professor in Morocco (1982-1984), or emigrant in the USA (1990-1997).

I thank to the Algebra Department of the State University of Moldova for the publication of this book.

AMUSING PROBLEM 1

1. On a fence are 10 crows. A hunter shots 3. How many are left?
   (*Answer: None, because the 3 dead fall on the ground and the rest flew away!*)

2. On a field are 10 crows. A hunter shots 3. How many are left?
   (*Answer: 3, the dead ones because the other 7 flew away!*)

3. In a cage are 10 crows. A hunter shots 3. How many are left?
   (*Answer: 10, the dead ones as well as the live ones, because they couldn't fly away from the cage!)*

4. 10 crows are flying of the sky. A hunter shots 3. How many are left?
   (*Answer: 7(finally!), the ones which are still flying, because the 3 dead fall on the ground*)

("Magazin", Bucureşti)

AMUSING PROBLEM 2

Let ABC a triangle
1. Find the aria of the bisector of angle A.
2. Determine the length of the intersection center of the bisectors.

Solutions
1. Aria = 0
2. The length = 0

MATHEMATICIANS ON THE SOCCER FIELD

In a group of four teams of soccer at the end of the tournament, the grid looks like the following:

Team A	3	1	2	0	3-1	4p
Team B	3	1	2	0	4-3	4p
Team C	3	0	3	0	2-2	3p
Team D	3	0	1	2	0-3	1p

Find the result of all the disputed games in this group (logical reasoning), knowing that for a victory a team gets 2 points, for a tight game it gets 1 point, while for a defeat no point.

Solution
We establish that:
Team C has only null games:

$$C - A = X, C - B = X, C - D = X$$

Team D has only two loses:

$$A - D = 1, B - D = 1$$

The left game:

$$A - B = X$$

Because both teams are without ant loses.

A has +2 point, therefore $A - D = 2 - 0$ (and not 3-1 because D didn't score at all). Similarly $B - D = 1 - 0$, from where $C - D = A - 0$.

By eliminating these cases we can put together an ad-hoc classification as follows:

A	2	0	2	0	1-1	2
B	2	0	2	0	3-3	2
C	2	0	2	0	2-2	2

Then

A-B= 0-0 or, 1-1, it results that A-C= 1-1 or 0-0

Further more

B-C= 3-3 or 2-2

But $B-C \neq 3-3$ because C scored at least 2 points. The first assessment fails, it remains only that:

$$A-B=1-1, A-C=0-0, B-C=2-2 \text{ (unique solution).}$$

["Gazeta Matematică"]

A PROBLEM OF LOGIC

Four people A, B, C, D are called to court.

When the judge asked: "Who is the guilty?" each of then answered as follows:

A: I am not guilty.
B: The guilty one is C.
C: The guilty one is D.
D: That's not true what C said.

Knowing that only one person answered correctly, who is the guilty?

Solution:
A

WHERE IS THE ERROR?

The person X took his vacation of 20 working days starting with the date of December 10, 1980 inclusive (Wednesday) (the non-working day being Sundays, respectively January 1 and 2 J 1981). In the mean time, a state decision declared the day of January 3, 1981 a non-working day, and the makeup day for this would be January 18, 1981.

The person X comes on January 5 to work and also works on January 18 (Sunday). How many vacation or make up days has X to take?

Solution

26 days – 6 holidays = 20 working days.

Because January 3 was declared a free day, it results that X took only 19 working days from his vacation, therefore he is allowed to an extra day from his vacation. Working also on the day of Sunday, January 18, he must take another day (to make up for this Sunday)). Therefore X has to take 2 more days of vacation.

Where is the error?

TEST: Equations and linear systems (VII grade)

Solve the following equations and linear systems:

1) $2x-5=2-5x$
2) $-6x+7y=8$

3) $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=\dfrac{1}{4} \\ \dfrac{x}{3}+\dfrac{y}{4}=\dfrac{1}{5} \end{cases}$

a) substitution method
b) reduction method
c) graphical method

4) $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=0 \\ \dfrac{x}{3}+\dfrac{y}{4}=0 \end{cases}$

5) $\begin{cases} -15x-25y=35 \\ 3x-5y=7 \end{cases}$

6) $\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=4 \\ \dfrac{3}{x}+\dfrac{4}{y}=5 \end{cases}$

7) How many problems of mathematics did Ionescu and Popescu resolve, from the algebra book for grade VII knowing that: If Ionescu would have resolved yet other five problems would have reached Popescu, and if Popescu would have resolved yet other five he would have pass Ionescu three times.

8) $mx+y=0$

$x+my=1$

Discussion.

TEST: Geometry, quadrilaterals, (VII grade)

1) Find the angles of a rhomb knowing that the side is 4 cm and its area is $8\sqrt{2}$ cm^2.

2) Let a parallelogram with the sides of 6 cm and respectively 8 cm, and the height 5 cm. Determine the parallelogram's angles and the angle formed by the height with a diagonal, and the angle formed by the diagonals.

3) Show that the aria of a rhomb is smaller than the aria of a square having the same side.

4) How many circles of a ray equal to 1 cm, at most, tangent among them two by two (or tangent to the sides) can be inserted in a square with the side of 10 cm? Yet, how many right isosceles triangles with the right side of 1 cm (which should have at most a common side)?

GRADE V

Any natural number greater than 3 can be written as a sum of prime numbers.

Solution

If the number n is even, then $n = 2k$, $k \geq 2$, and it can be written $n = \underbrace{2+2+...+2}_{k\ times}$.

If n is odd, it result that $n = 2k+1$, $k \geq 2$ and it can be written: $n = 3+\underbrace{2+2+...+2}_{k\ times}$

($k \geq 2$ always because $n > 3$).

GRADE VI

a) Any set included in N has a minimum?
b) Any set included in Z\N has a maximum? Generalization.
c) Find three sets A, B and C such that $C \subset A$, $C \subset B$ and $C \subset A \setminus B$.

Solutions:

a) Not any set included in N has a minimum. For example, the empty set $\Phi \subset N$, but Φ does not have a minim (does not make sense to say that a null set has a minimum). It is to be emphasized that any $M \neq \Phi$ and $M \subseteq Z \setminus N$ has a minimum.

b) Not any set included in $Z \setminus N$ has a maximum. Similarly, $\Phi \subset Z \setminus N$, but Φ nu are maximum (it doesn't make sense to say that Φ has a maximum or not). Similarly, any set $M \neq \Phi$ and $M \subseteq Z \setminus N$ are maximum. Generalization: Let M o set; not any set $M_0 \subseteq M$ has a minimum and not any set $M_1 \subseteq M$ has a maximum, for any M.

c) If $A \supset C$ and $B \supset C$ then $A \setminus B \supset C$ only if $C = \Phi$. Therefore the three sets are: any A, B, and $C = \Phi$.

"*Mugur Alb*", Revista Şcolii Nr. 19, Clasele I-VIII, Bacău, Anul VI, Nr. 3-4-5, 1985-87, p.61.

GRADE VII

What number smaller than 100, being divided by 4 gives us the remainder $r_1 = 1$ and divided by 3 gives us the remainder $r_2 = 1$?

GRADE VIII

Find the numbers which divided by 52 give the result 3 and the remainder $r_1 > 1$ and divided by 43 give result 3 and the remainder r_2.

1. PROPOSED PROBLEM

Show that the last digit of the number $A = 1981^n + 1982^2 + ... + 1990^n$ is the same with the last digit of the number $A = 1191^n + 1992^n + ... + 2000^n$

["Gamma", Braşov, Anul 8, Nr. 2, Februarie 1986]

2. PROPOSED PROBLEM

Grade VII

How many whole numbers of three digits divided simultaneously to 20, 50 and 70 give the same remainder?

Solution

The LCM of the numbers 20, 50, 70 is 700. For any remainder $r \in \{0,1,2,...,19\}$ there exists only one number of three digits $700 + r$, which divided by those three numbers give the same remainder.

Obviously we cannot gave a remainder >20. In total, therefore, we have 20 numbers that verify the problem: 700, 701, 702,…,719.

3. PROPOSED PROBLEM

Grade VIII

Show that if $p_1^2 - p_1 - 2p_2 = 0$, then $p_1^{2n+1} + p_2^{n+1}$ is divisible by $p_1 + p_2$, $p_1, p_2 \in N^*$.

Solution

$$p_1^{2n+1} + p_2^{n+1} = p_1\left(p_1^2\right)^n + p_2 p_2^n = p_1\left(p_1^2\right)^n - p_1 p_2^n - p_1 p_2^n + p_2 p_2^n =$$
$$= p_1\left[\left(p_1^2\right)^n - p_2^n\right] + p_2^n\left(p_1 + p_2\right) = p_1\left(p_1^2 - p_1\right)\left[\left(p_1^2\right)^{n-1} + \left(p_1^2\right)^{n-2} p_2 + ... +\right.$$
$$\left. + p_1^2 p_2^{n-2} + p_2^{n-1}\right] + p_2^n\left(p_1 + p_2\right) = p_1\left(p_1 + p_2\right)\left[\left(p_1^2\right)^{n-1} + \left(p_1^2\right)^{n-2} p_2 + ... +\right.$$
$$\left. + p_1^2 p_2^{n-2} + p_2^{n-1}\right] + p_2^n\left(p_1 + p_2\right).$$

We observe that we can factor $p_1 + p_2$ and the problem is solved.

Observation

Prove that $(4)^{2n+1} + (6)^{n+1}$ is divisible by 10.

Proof:

We note $p_1 = 4, p_2 = 6$, we observe that the relation is verified: $p_1^2 - p_1 - 2p_2 = 16 - 4 - 12 = 0$.

Prove that $2^{2n+1} + 1$ is divisible by 3.

Proof

We have $p_1 = 2, p_2 = 1$ and then $p_1^2 - p_1 - 2p_2 = 4 - 2 - 2 = 0$

4. PROPOSED PROBLEM

Grade VIII

Find the remainder of the division of $A = \overline{xxx}_y + \overline{yyo}_z + \overline{zzz}_v$ by z, knowing that $x+1 = y, y+1 = z$.

Solution

$$\overline{yyo}_z : z$$

$$\overline{xxy}_y + \overline{zzz}_v = xy^2 + xy + x + z - v^2 + z - v + z$$

Therefore

$$A \equiv x\left(y^2 + y + 1\right) \equiv (z-2)\left(z^2 - 2z + 1 + z - 1 + 1\right) \equiv -2 (\mathrm{mod}\, z).$$

5. PROPOSED PROBLEM

Grade VIII

Determine how many numbers different of zero, with an odd number of digits, which added with their images from the mirror gives a number who's digits are identical, show that these digits cannot be other than 2, 4, 6, 8.

Solution

Let

$$A = k_1 10^{n-1} + k_2 10^{n-2} + ... + k_{n-1} 10 + k_n$$

n odd, $k \in N$.

We note iA the mirror image of A. Then

$$iA = k_n 10^{n-1} + k_{n-1} 10^{n-2} + ... + k_2 10 + k_1$$

$$A + iA = (k_1 + k_n) 10^{n-1} + (k_2 + k_{n-1}) 10^{n-2} + ... + (k_{n-1} + k_2) 10 + (k_1 + k_n) =$$

$$= k10^{n-1} + k10^{n-2} + ... + k10 + k, \quad k = \overline{0.9}$$

Then

$$(*) \begin{cases} k_2 + k_{n-1} = k \\ . \\ . \\ k_{\left[\frac{n}{2}\right]+1} + k_{\left[\frac{n}{2}\right]+1} = k \end{cases}$$

Then

$$k_{\left[\frac{n}{2}\right]+1} = k, \ k_{\left[\frac{n}{2}\right]+1} \in N; \ k = \overline{0.9}$$

Therefore k cannot be but 2, 4, 6, 8 and then $k_{\left[\frac{n}{2}\right]+1}$ cannot be nut 1, 2, 3, 4.

From the system (*) we observe that for a fixed k, the determination of any digit from $k_1, ..., k_{\left[\frac{n}{2}\right]}$ implies also the determination of its mirrored image. Ones established the $k_{\left[\frac{n}{2}\right]+1}$, there exist $k+1$ possibilities for find $k_{\left[\frac{n}{2}\right]+1}$ and for every $k_{\left[\frac{n}{2}\right]}$ found there exist $k+1$ possibilities to find $k_{\left[\frac{n}{2}\right]-1}$, etc.

Therefore for the obtained number to remained with an odd number of digits, for a fixed k, we must eliminate from the $(k+1)^{\left[\frac{n}{2}\right]}$ numbers, those that start with zero and which are in total $(k+1)^{\left[\frac{n}{2}\right]-1}$.

Therefore we find that the number is

$$\sum_{k=2,4,6,8} (k+1)^{\left[\frac{n}{2}\right]} - (k+1)^{\left[\frac{n}{2}\right]-1},\ n \text{ odd}$$

Observation

We noted with $[x]$ the whole part of the number x.

6. PROPOSED PROBLEM

Grade VIII

a) What number smaller than 100 divided by 4 gives us the remainder $r_1 = 1$ and divided by 13 gives us the remainder $r_2 = 1$?

b) Find the numbers which divided by 42 give us 3 and the remainder $r_1 > 1$ and divided by 43 give us the remainder r_2.

("Gazeta Matematică" Anul XCIII, Nr. 5-6, 1988, p. 241)

7. PROPOSED PROBLEM

Show that

$$F = 11^1 + 22^2 + 33^3 + 44^4 + 55^5$$

cannot be a perfect square

Solution

$F \equiv 1+4+7+6+5 \equiv 3 (\bmod 10)$,

That is its last digit is 3 and it is known that there is no perfect square number that ends with this digit.

8. PROPOSED PROBLEM

Grade VIII

What number, smaller than 100, divided by 4 and 26 would give the remainder 1?

Solution

The LCM of The numbers 4 and 26 is 52.
52+1= 53 is the number that satisfies the requested conditions.

9. PROPOSED PROBLEM

A student deposited at the bank the amount of 800 Lei. After how many years he'll have an amount of 926,10 Lei, knowing that the interest is 5%?

Solution

In the first year he'll have $\left(\frac{105}{100}\right)\cdot 800$ Lei. In the second year he'll have $\frac{105}{100}\left(\frac{105}{100}\right)\cdot 800=\left(\frac{105}{100}\right)^2\cdot 800$ Lei. In general, after n years, he'll have $\left(\frac{105}{100}\right)^n\cdot 800$ Lei (it results by induction).

Therefore $\left(\frac{105}{100}\right)^n\cdot 800$=926.10; from this, using the logarithm function, we obtain that n-3 (years).

10. PROPOSED PROBLEM

Grade VIII

Prove that if the product of n positive numbers is equal to 1, and if their sum is strictly greater that their inverse, then at least one of these numbers is strictly less than 1.

(A generalization of the problem 0:5 GMB 6/1979, p. 253)

11. PROPOSED PROBLEM

Grade VIII

Show that any natural number greater than 1 can be written as a product of prime numbers.

Solution

Certainly, in conformity with the canonical decomposition, any natural number is equal with a product of prime numbers.

If the decomposed natural number is prime, then the product has only one factor.

12. PROPOSED PROBLEM

Grade VIII

During a sport competition of free style wrestling a team member competes with each team member. At the end of the competition the result showed that each member won over three other members. How many members where in the team? Generalization.

13. PROPOSED PROBLEM

Grade VIII

On the three concurrent sides of a cube we take three points M, N, P different of the vertexes.

a) Show that the triangle MNP cannot be a rectangle

b) In which case the triangle is isosceles?

c) In which case the triangle is equilateral?

14. PROPOSED PROBLEM

Grade VIII

Give the parallel lines (e) and (f), intersected by other lines (d_i), $i=\overline{i,n}$, in the points A_i respectively B_i. In each A_i we construct the two bisectors of the interior angles which intersect in C_{i1} and C_{i2} with the other two bisectors of the interior angles constructed in the correspondent point B_i. Then, all points C_{i1} and C_{i2}, $i=\overline{i,n}$ are collinear.

15. PROPOSED PROBLEM

Grade VIII

Let b the LTG of the numbers $a_1, a_2, ..., a_n$. Sow that there exist the whole numbers $K_1, K_2, ..., K_n$ such that $a_1K_1 + a_2K_2 + ... + a_nK_n = b$

16. PROPOSED PROBLEM

Grade IX

Show that a logical sentence is equivalent to its dual if and only if this sentence is a tautology, or contains logical operators (at least the negation)

17. PROPOSED PROBLEM

Grade IX

Let the logical sentence

$$A(\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow):\ (p \Rightarrow Q) \vee R_1 \wedge ... \wedge R_p \Leftrightarrow \neg(R_1 \wedge ... \wedge R_n) \Rightarrow Q \wedge P.$$

Transform A in a logical sentence equivalent $B(\neg, \wedge, \vee)$. Analyze A and its dual B.

18. PROPOSED PROBLEM

Find the sets A, B, T that simultaneously satisfy the relations:

$$C_T\left(A \cup B\right) = \{7,8,9\}$$

$$C_B A = \{4,5,6\}$$

$$T \subseteq \{1,2,3,4,5,6,7,8,9\}$$

("Gazeta Matematică" Anul XCIII, Nr. 5,6, 1988)

19. PROPOSED PROBLEM

Consider the number $n = \underbrace{\overline{xx...x}}_{m\ times}$, where $x, m \in N$, $m > 1$. Prove that if the number n is prime, then $x = 1$ and m also is prime.

Solution

If $x \neq 1$, then n is divisible by x, therefore n is not any longer prime.

If m is not prime, let $p \cdot q = m$ where $p, q \notin \{1, n\}$. Then n is divisible by $\underbrace{\overline{11...1}}_{p\ times}$, then again it is not prime.

20. PROPOSED PROBLEM

Prove that if $x^2 - x - 2y = 0$ then $x^{2n+1} + y^{n+1}$ is divisible by $x + y$, where $x, y \in N^*$.

21. PROPOSED PROBLEM

If the set of the solutions of an equation is symmetric in rapport to two given variable, then the given equation is also symmetric in rapport to these variables.

22. PROPOSED PROBLEM

Determine x and y from the relation $1433_{(5)} + 17_{(8)}$

$$\overline{3xy5}_{(12)} = \sqrt{15F90_{(16)}} + 2_{(17)} 126D7_{(14)} A_{(14)}$$

23. PROPOSED PROBLEM

Grade IX

Find the sets $A, B,$ knowing that these are non-empty and

a) $A\Delta B = (A \setminus B) \cup (B \setminus A) = \{1,2,3\}$

b) $\{1,2\} \subseteq A$

Solution

Because $\{1,2\} \subseteq A$ it results.

1) Let $3 \in B \Rightarrow 3 \notin A$. Then $A = \{1,2\}$ and $B = \{3\} \bigcup C$, where C is a set with the property $\{1,2\} \bigcap C = \Phi$.

2) Let $3 \in A \Rightarrow 3 \notin B$. Then $A = \{1,2,3\} \bigcup C$ and $B = C$, where C is a non-null set with the property that $\{1,2,3\} \bigcap C = \Phi$.

24. PROPOSED PROBLEM

Grade IX

Solve the inequality

$$4x^2 - 12x + 13 \geq \frac{7}{x^2 - 3x + 4}$$

Solution

We note $\mathrm{t} = x^2 - 3x + 4 \geq 0$, because $\Delta_x < 0$.

We have $4t - 3 \geq \dfrac{7}{t}$ or $4t^2 - 3t - 7 \geq 0$

$$t_{1,2} = \frac{3 \pm \sqrt{121}}{8} = \begin{cases} \dfrac{7}{4} \\ -1 \end{cases}$$

And because $t > 0$ it results that $t \geq \dfrac{7}{4}$, hence $4x^2 - 12x + 9 \geq 0$ which is obvious.

Therefore $x \in R$.

25. PROPOSED PROBLEM

Grade IX

Find the necessary and sufficient condition that the following expression:

$$E(x) = \sum_{i=1}^{n} a_i (x - x_i)^2 + b, \quad a_i, b_i, b \in R \text{, for } 1 \leq i \leq n$$

to accept extremes (solution without derivatives)

26. PROPOSED PROBLEM

Grade IX

Resolve within the natural set the following equations:

a) $2x-3y=6$

b) $5x+3y+7z=25$

Solution

a) $\begin{cases} x=3t+3 \\ y=2t, \ \ t\in N \end{cases}$

has been obtained from the general solution in Z for the associated homogeneous equation: $2x-3y=0$, to which has been added the smallest particular natural solution.

b) Resolved in Z gives the general solution:

$$\begin{cases} x=k_1 \\ y=3k_1+7k_2+6, \ \ k_2\in Z \\ z=2k_1+3k_2+1, \ \ k_2\in Z \end{cases}$$

Because $0\le x\le 5$, k_1, traverses the set $\{0,2,3,4,5\}$, therefore we have the solutions:

$$\begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

in N^3

27. PROPOSED PROBLEM

$$\begin{cases} y^2-x^2-2x=1 \\ x^2+y^2+2x+2y+2xy+1=64 \end{cases}$$

Solution

$y^2-(x+1)^2=0$ or $(y-x-1)(y+x+1)=0$.

Substituting $y=\pm(x+1)$ in the second equation, making the notation $\varepsilon=\pm 1$. After all calculations we obtain

$$\begin{cases} y=\varepsilon(x+1) \\ (1+\varepsilon)x^2+2(1+\varepsilon)x+(-31+\varepsilon)=0 \end{cases}$$

If $\varepsilon=-1$, the system is impossible

If $\varepsilon=+1$, it results from the second degree equation in x that

$x_{1,2}=\begin{cases} 3 \\ -5 \end{cases}$, from where $y_{1,2}=\begin{cases} 4 \\ -4 \end{cases}$, that constitute the solution of the given system.

28. PROPOSED PROBLEM
Grade IX

Let b_1 and b_2 two numeration bases. Determine all values x for which $x_{b_1} = x_{b_2}$

29. PROPOSED PROBLEM
Grade IX

Find an algorithm which will determine the direct conversion of a number from base b_1 to base b_2 (without going through base 10)

30. PROPOSED PROBLEM
Grade IX

Show that any quadrilateral can be transformed in an inscribable quadrilateral by eventually swapping the sides between them.

31. PROPOSED PROBLEM
Find which regulate polygons that have the perimeter p and the circle with the length p have the bigger area?

32. PROPOSED PROBLEM
Grade IX

Construct with only the ruler and the compass the number $\sqrt{n}$, where $n \in N$.

33. PROPOSED PROBLEM
Grade IX

If F is the intersection of the diagonals of a quadrilateral inscribable, $ABCD$ then:

1) $\dfrac{AF}{CF} = \dfrac{BA \cdot AD}{BC \cdot CD}$ and $\dfrac{BF}{DF} = \dfrac{AB \cdot BC}{AD \cdot DC}$

2) $\sin \widehat{A} \sin \widehat{B} = \sin \dfrac{\overset{\frown}{AB}}{2} \sin \dfrac{\overset{\frown}{CD}}{2} + \sin \dfrac{\overset{\frown}{BC}}{2} \sin \dfrac{\overset{\frown}{DA}}{2}$

34. PROPOSED PROBLEM
Grade X

Solve in whole numbers:

$24x+13y+7z-6u=14$

Solution

$u=\dfrac{24x+13y+7z-24}{6}=4x+2y+z-2+\dfrac{y+z-2}{6}\in Z$

Let $Z\ni p=\dfrac{y+z-2}{6}$, then $y=6p-z+2$ and $u=4x+13p-z+2$, with $p,x,z\in Z$

35. PROPOSED PROBLEM

Grade X

The equation $x^n+y^n=x+y$, $n\in N, n\geq 2$, does not have whole nontrivial (different of solutions $x,y\in\{0,1\}$, or $x=-y$ and n odd).

Solution

If n is even then $x^n\geq x,\ y^n\geq y$ from which $x^n+\ y^n\geq x+y$, having equalities only for $x,y\in\{0,1\}$.

Let n odd. Excluding the trivial cases $x=-y\in Z$ or $x,y\in\{0,1\}$, we have that $x,y<0$ let $x>0$ and $t=-y>0$. Therefore $x^n-x=t^n-t$, with $x\neq t$; if $x\geq t+1$, evidently $(t+1)^n-(t+1)>t^n-t$ and similarly when $x\leq t-1$.

36. PROPOSED PROBLEM

Grade X

Given $F(x)=3+\dfrac{4}{x}$. Determine the intervals for which F is bijection . In this case determine F^{-1}.

Solution

Evidently $0\notin D_F=$ the definition domain for F.

Let $R_{\ni}y=3+\dfrac{4}{x}$. It results that $x=-\dfrac{4}{x}-y\in R$, then $y\neq 3$.

Therefore $F:R^*\to R\setminus\{3\}$ is bijective: $x_1\neq x_2\Rightarrow F(x_1)\neq F(x_2)$ because $3+\dfrac{4}{x_1}=3+\dfrac{4}{x_2}$, therefore F is injective; let $y\in R\setminus\{3\}$, there exists $x=\dfrac{-4}{3-y}\in R^*$ such that $F(x)=y$, that is F is surjective .

Therefore F accepts inverse and $F^{-1}:R\setminus\{3\}\to R^*$, $F^{-1}(x)=\dfrac{4}{3-x}$

37. PROPOSED PROBLEM

Grade X

Show that if a and b are coprime then

$a^{\varphi(b)+1} + b^{\varphi(b)+1} \equiv a + b \pmod{ab}$, where φ is Euler's indicator.

38. PROPOSED PROBLEM

Grade X

Let a regular tetrahedron $ABCD$ with the side a, sectioned by a plane φ. Find the section's form and its area when α passes through:

a) the middle of the sides BC, BD, AD

b) the middle of the sides AB, AC, AD

c) the middle of the face BCD and is parallel to AD.

d) the middle of the faces ABC, BCD, CDA

39. PROPOSED PROBLEM

Grade X

Let a quadrilateral regular pyramid $VABCD$ with the side a and the height h, sectioned with a plane α'. Find the section's form and its area when α passes through:

a) A and the middle of the sides VC, VF.

b) the middle of VA and is parallel with BC

c) the middle of VA and is perpendicular on BC

d) A and is perpendicular on VC

40. PROPOSED PROBLEM

Grade X

Let a hexagonal regular pyramid with the side a and the height h, sectioned with a plane α'. Find the section's form and its area when α passes through:

a) A and the middle of sides VC and VF

b) BF and is perpendicular on VD

c) BF and is parallel to VD

d) The points M, N, P such that

$$\frac{VM}{MB} = \frac{1}{2},\ \frac{VN}{NC} = \frac{2}{3},\ \frac{VP}{PD} = 1$$

41. PROPOSED PROBLEM

Grade X

Let a triangular regular prism with the side a and the height h, sectioned with a plane α. Find the section's form and its area when α passes through:

a) the middle of the sides AA', BB', CC'

b) the middle of the faces $ABC, BBC'B', ACC'A'$

c) the middle of the base $A'B'C'$ and the side AB and is parallel with the diagonal $A'B$

d) A and is perpendicular on the side $B'C'$

e) M, N, P such that:

$$\frac{AM}{MB}=\frac{1}{2}, \frac{BN}{NC}=\frac{2}{3}, \frac{CP}{PC'}=\frac{3}{4}$$

42. PROPOSED PROBLEM

Grade X

Given a cube $ABCDA'B'C'D'$ of a side a, sectioned with a plane α. Find the section's form and its area when α passes through:

a) the middle of the sides $AA', BC, C'D'$

b) the middle of AB and is perpendicular on the diagonal AC'

c) the side AD and parallel BD'

d) M, N, P such that:

$$\frac{AM}{MB}=2, \frac{CN}{NC'}=3, \frac{D'P}{PA'}=4$$

43. PROPOSED PROBLEM

Grade X

Prove that points of coordinate $(k-n, k, k+n)$, for any k, and $-k<n<k$ belong to the same plane that passes through the origin of the axes.

44. PROPOSED PROBLEM

Grade X

Let a function f derivative on $[-a,a]$ and with the derivative having the sign constant on this interval, where $a=\sup f(x)$ when $x\in D_r$, and $[-a,a]\subseteq D_r$. Show that the sequence defined by $x_{n+1}=f(x_n)$, with $x_0\in D_r$, $n\in N$, is convergent.

45. PROPOSED PROBLEM

Grade X

Compute

$$\lim_{n\to\infty}\lim_{n\to\infty}\frac{tg(a_1xtg(a_2xtg(a_nxtg(...(a_{n-1}xtg(a_nx))...)}{x^n}$$

knowing that $(a_n)n \in N^*$ is sequence of real numbers, with the property that at a certain rang all the terms of the sequence are in the interval $[-1,+1]$.

(In connection with the problem 18038, GM 12/1979, p. 480)

46. PROPOSED PROBLEM

Using the Euclid algorithm, compute LTG of n numbers using a program written in FORTRAN.

47. PROPOSED PROBLEM

Grade X

Determine the expression $E(n)=\sum_{k=1}^{n}\underbrace{6...6}_{k}{}^2$ to be divisible by 25

Solution

$E(n)\equiv 36+56+56+...+56\equiv 56n-20(\text{mod}\,100)$, because

$$\underbrace{6...6}_{k}{}^2=\left(\underbrace{6...600}_{k-2}+66\right)^2\equiv 66^2\equiv 56(\text{mod}\,100),\ k\geq 3$$

Therefore $56n-20\equiv 0(\text{mod}\,25)$, that is $6n\equiv 20(\text{mod}\,25)$, or $3n\equiv 10(\text{mod}\,25)$, from which $n\equiv 10\cdot 3^{-1}\equiv 10\cdot 17\equiv 20(\text{mod}\,25)$.

$n\in\{25k+20, k\in Z\}$

48. PROPOSED PROBLEM

Grade XII

Find the remainder of the division by 493 of the number $B=1971^{1971}+0^{1972}+1^{1973}$

Solution

$B=(-1)^{1971}+0^{1972}+1^{1973}\equiv 0(\text{mod}\,493)$

49. PROPOSED PROBLEM

Grade XII

Solve : $2x+3y+2z\equiv 1(\text{mod}\,4)$

Solution

$(2,3,2,4)=1|1$ then is congruent and has solution: $1\cdot 4^{3-1}=16$ distinct solutions.

Resolved in whole numbers the equation $2x+3y+2z-1=4t$ gives:

$$\begin{cases} x=3k_1-k_2-2k_3-1=3k_1+3k_2+2k_3+3(\text{mod}\,4) \\ y=-2k_1 \qquad\qquad\quad = \ 3k_1+1(\text{mod}\,4) \\ z=k_2 \qquad\qquad\qquad = k_2(\text{mod}\,4) \end{cases}$$

For any $k_j \in Z$, k_j parameters

$$k_3=0 \Rightarrow \begin{pmatrix} 3k_1+3k_2+3 \\ 2k_1 \qquad +1 \\ k_2 \end{pmatrix}, \quad k_3=1 \Rightarrow \begin{pmatrix} 3k_1+3k_2+1 \\ 2k_1 \qquad +1 \\ k_2 \end{pmatrix}$$

Obviously for $k_3=2$ and 3 it result similarly, and we don't compute it.

$$k_1=0 \Rightarrow \begin{pmatrix} 3k_2+3 \\ +1 \\ k_2 \end{pmatrix}, \quad \begin{pmatrix} 3k_2+1 \\ +1 \\ k_2 \end{pmatrix}$$

$$k_1=1 \Rightarrow \begin{pmatrix} 3k_2+2 \\ +3 \\ k_2 \end{pmatrix}, \quad \begin{pmatrix} 3k_2 \\ 3 \\ k_2 \end{pmatrix}$$

$$k_2=0,1,2,3 \Rightarrow \begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\3\\0\end{pmatrix},\begin{pmatrix}0\\3\\0\end{pmatrix}$$

$$\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\3\\1\end{pmatrix},\begin{pmatrix}3\\3\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}3\\1\\2\end{pmatrix},\begin{pmatrix}0\\3\\2\end{pmatrix},\begin{pmatrix}2\\3\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\3\end{pmatrix},\begin{pmatrix}2\\1\\3\end{pmatrix},\begin{pmatrix}3\\3\\3\end{pmatrix},\begin{pmatrix}1\\3\\3\end{pmatrix}$$

which represents all distinct solutions of the congruency.

50. PROPOSED PROBLEM

Grade XII

Let $f:R\to R$ a function continue, and odd. Then

$$\int_{-p}^{p} f(x)dx = 0, (\forall p \in R)$$

(A generalization of problem 14847, GMBP/1975, p.97)

Solution

Because f is odd we have that $f(x) = -f(-x), (\forall) x \in [-p, p]$.

$$\int_{-p}^{p} f(x)dx = F(p) - F(-p),$$

where F is a primitive of f on the interval $[-p, p]$.

$$\int_{-p}^{p} -f(-x)dx = \int_{-p}^{p} -f(-x)d(-x) = \int_{-p}^{p} f(u)d(u) = F(p) - F(-p)$$

From where: $F(p) - F(-p) = 0$

Remark: for $f(x) = \dfrac{\sin x}{\ln(2+x^2)}$ we obtain 14847.

51. PROPOSED PROBLEM

Find a condition necessary and sufficient for $\sum_{i=1}^{n} a_i^x > \sum_{j=1}^{m} b_j^x$, $(\forall) x \in R$.

(A generalization of problem 1, p. 46 GMA 1/1981)

52. PROPOSED PROBLEM

Let the polynomials

$$P(x) = a_n + \sum_{k=1}^{m} a_{2k-1} x^{2k-1}$$

and

$$Q(x) = \sum_{k=m}^{n} a_{2k} x^{2k}$$

With coefficients positive and not nulls, $1 \le m \le n$. Prove that the equation $P(x) + Q(x)$ has exactly two real solutions.

Solution

The polynomial $R(x) = Q(x) - P(x)$ has only one sign variation (in the Descards sense, that is: there are two consecutive terms a_{i_1} and a_{i_2} of the polynomial such that $a_{i_1} \cdot a_{i_2} < 0$, with $a_{2m}(-a_{2m+1}) < 0$).

If $V = 1$. Let p the number of the positive solutions of the polynomial

$R(x)$. Using the Descartes theorem (that states that $V-p$ is an even number non-negative) it results that $p=1$,
Let r be the number of the negative solution for $R(x)$. That means that r represents the number of the positive solutions of the polynomial $R(-x)$, which has only one variation of sign, $a_0 \cdot (-a_1) < 0$. The same if $r=1$. The equation has one positive solution as a single negative solution. In total, two real solutions.

References:

Alain Bouvier, Michel George, François Le Lyonnais, "Dictionnaire des Mathématiques", Presses Universitaires de France, Paris, 1979.

53. PROPOSED PROBLEM

Let $\sigma = \begin{pmatrix} 12345 \\ 45132 \end{pmatrix}$ a circular permutation. Determine all the circular permutations X such that:

a) $\sigma X = \sigma^{-1} X$
b) $\sigma X \sigma = X$
c) $X \sigma X = X$

Solution

Let $X = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix}$

$\sigma^{-1} = \begin{pmatrix} 12345 \\ 35412 \end{pmatrix}$

a) $\sigma^2 X = X$, but $\sigma^2 = \sigma \cdot \sigma = \begin{pmatrix} 12345 \\ 32415 \end{pmatrix}$

$$\begin{pmatrix} 12345 \\ 32415 \end{pmatrix} \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix} = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix}$$

or

$$\begin{pmatrix} 12345 \\ 32415 \end{pmatrix} \begin{pmatrix} x_1 x_2 x_3 x_4 x_5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix} = e = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 1\ 2\ 3\ 4\ 5 \end{pmatrix}$$

(the identical permutation). Which is absurd. Therefore this equation does not admit solutions.

b) $\sigma X \sigma = \sigma \cdot \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix} = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix}$

and it results that $\begin{pmatrix} 12345 \\ 45132 \end{pmatrix} = \begin{pmatrix} x_1 x_2 x_3 x_4 x_5 \\ x_1 x_2 x_3 x_4 x_5 \end{pmatrix}$

1) $x_4 = 1 \Rightarrow x_1 = 4,\ x_3 = 3$ $\begin{cases} x_5 = 2, x_2 = 5 \\ x_5 = 5, x_2 = 2 \end{cases}$

2) $x_4 = 2 \Rightarrow x_1 = 5,\ x_3 = 5$ which does not satisfy

3) $x_4 = 3 \Rightarrow x_1 = 1,\ x_3 = 4$ $\begin{cases} x_5 = 2, x_2 = 5 \\ x_5 = 5, x_2 = 2 \end{cases}$

4) $x_4 = 4 \Rightarrow x_1 = 3,\ x_3 = 1$ $\begin{cases} x_5 = 2, x_2 = 5 \\ x_5 = 5, x_2 = 2 \end{cases}$

5) $x_4 = 4 \Rightarrow x_1 = 3,\ x_3 = 2$ which is not satisfactory.

We have in total six solutions.

c)

$$X\sigma X = X$$
$$\sigma X = X^{-1}X$$
$$\sigma X = c$$
$$X = \sigma^{-1}e = \sigma^{-1} = \begin{pmatrix} 12345 \\ 35412 \end{pmatrix}$$, the unique solution.

54. PROPOSED PROBLEM

Let a, m two integers, $m \neq 0$. We construct the series

$$d_0 = (a, m), d_{i+1} = (d_i, m / d_0 d_1 ... d_i).$$

Show that, if s is the first index for which $d_s = 1$, then $a^{f(m_s)+s} \equiv a^s \pmod{m}$ where $m_s = m / d_0 d_1 ... d_s$ and is the Euler's index.

(A generalization of the Euler's theorem)

55. PROPOSED PROBLEM

$g : R \to R$ a descending function. Is there a function $f : R \to R$ monotone, such that $f \circ f = g$ on R?

Solution

No.

a) We prove that there does not exist an increasing function f.

Let $x_1 < x_2$, $x_1, x_2 \in R$. If f is ascending it results that $f(x_1) \leq f(x_2)$ and then $f(f(x_1)) \leq f(f(x_2))$, that is $g(x_1) \leq g(x_2)$, where $x_1 \geq x_2$, which contradicts the hypothesis $x_1 < x_2$.

b) We'll prove that it does not exist a descending function f.

Let $x_1 < x_2$, $x_1, x_2 \in R$. If f is descending it results that $f(x_1) \geq f(x_2)$, and from here $f(f(x_1)) \leq f(f(x_2))$, therefore $g(x_1) \leq g(x_2)$, and also $x_1 \geq x_2$, which contradicts the hypothesis $x_1 < x_2$. The same thing can be proved for functions strictly monotones.

56. PROPOSED PROBLEM

Let $g: R \to R$ a function non-bijective . Does it exist a function $f: R \to R$, bijective, such that $f \circ f = g$ on R?

Solution
No

a) If g is not injective, it results that it exists $x_1, x_2 \in R$ such that $x_1 \neq x_2$ with $g(x_1) = g(x_2)$, from where

$$f(f(x_1)) = f(f(x_2)).$$

We note $y_1 = f(x_1)$, $y_2 = f(x_2)$.

We have that, $y_1 \neq y_2$, but in this case f is not injective because $fy_1 = f(y_2)$

b) If g is not surjective, it results that there exists $y_0 \in R$ such that for any $x \in R$, $g(x) \neq y$. We find that f is not surjective, because:

- or $f(x) \neq y_0$, $(\forall) x \in R$;
- or $(\exists) x_0 \in R : f(x_0) = y_0$; but in this case, it results that $f(x) \neq x_0$ on R since it exists $x_1 \in R$ with $f(x_1) = x_0$ we would have that $g(x_1) = f(f(x_1)) = f(x_0) = y_0$ which contradicts the suppositions made regarding g. Then, if g is not injective (surjective) it results that f also is not injective (surjective).

57. PROPOSED PROBLEM

Let $P(x) = \left(1 + x = x^2 + ... + x^{k-1}\right)^n$. We note CK_n^h the coefficient in x^h, which we'll call the k-nominal coefficient. (A generalization of the binomial, trinomial coefficients and of the Pascal triangle). Prove that

a) $CK_n^h = \sum_{i=0}^{k-1} CK_{n-1}^{h-1}$, where by convention we'll consider that $CK_r^t = 0$ for $r(k-1) < t < 0$

b) $CK_n^h = CK_n^{n-h}$

c) $\sum_{h=0}^{h(k-1)} CK_n^h = K^n$

d) $CK_n^h = \sum_{h=0}^{h(k-1)} (-1)^h CK_n^h = \begin{cases} 1 \ if \ k \text{ is odd} \\ 0 \ if \ k \text{ is even} \end{cases}$

e) $\sum_{i=0}^{k-1} CK_n^i \cdot CK_m^{h-1} = CK_{n+m}^h$

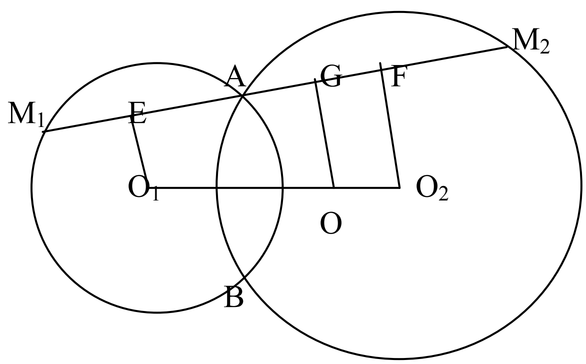
f) $\sum_{h=0}^{n(k-1)} \left(CK_n^h\right)^2 = CK_{2n}^n$

58. PROPOSED PROBLEM

Through the intersection point of two circle we construct a variable line that intersects for the second time the circles in the points M_1 and respectively M_2. Find the locus of the point M that divides the segment M_1M_2 in the rapport k.

(A generalization of Problem IV. The examination test for admission to the Polytechnic Institute, 1987, Romania)

Solution



Let $O_1E \perp M_1M_2$ and $O_2F \perp M_1M_2$. Let $O \in O_1O_2$ such that $O_1O = K \cdot OO_2$ and $M \in M_1M_2$, or $M_1M = K \cdot MM_2$. Construct $OG \perp M_1M_2$.

We note:

$$M_1E \equiv EA = X \text{ and } AF \equiv FM_2 = y.$$

Then $AG \equiv GM$, because

$$AG = EG - EA = \frac{k}{k+1}(x+y) - x = \frac{-x+ky}{k+1}, \text{ and}$$

$$GM = M_1M - M_1A - AG = \frac{k}{k+1}(2x+2y) - 2x - \frac{-x+ky}{k+1} = \frac{-x+ky}{k+1}$$

Then also $OM \equiv OA$

The locus is a circle of center O and a ray OA, without the points A and B.

Reciprocally

If $M \in G(O, OA) \setminus \{A, B\}$, the line AM cuts the two circles in M_1, respectively M_2.

We project the points O_1, O_2, O on the line M_1M_2 in E, F respectively G. Since $O_1O = K \cdot OO_2$ it results that $EG = k \cdot GF$. We note $M_1E \equiv EA = x$ and $AF \equiv FM_2 = y$ and we obtain

$$M_1M \equiv M_1A + AM = M_1A + 2AG = 2x + 2(EG - EA) = 2x + 2$$
$$\left[\frac{k}{k+1}(x+y) - x\right] = \frac{k}{k+1}(2x+2y) = \frac{k}{k+1}M_1M$$

For $k = 2$ we find the answer.

59. PROPOSED PROBLEM

Let d_A and d_B the diagonals of a quadrilateral inscribable that start from A, respectively B. Prove that

$$d_A \sin\widehat{A} = d_B \sin\widehat{B}.$$

First solution

We note the sides of the quadrilateral:

$$AB = a$$
$$BC = b$$
$$CD = c$$
$$DA = d$$

Because

$$S_{ABC} + S_{ACD} = S_{ABD} + S_{BCD}$$

It results that $ab\sin\widehat{B} + cd\sin\widehat{D} = ad\sin\widehat{A} + bc\sin\widehat{C}$.

The quadrilateral being inscribable, we have

$$\sin\widehat{A} = \sin\widehat{C} \text{ and } \sin\widehat{B} = \sin\widehat{D}.$$

Therefore:

$$\frac{\sin\widehat{A}}{\sin\widehat{B}} = \frac{ab+cd}{ad+bc} = \frac{d_B}{d_A}$$

The last equality is the of the second Ptolomeus theorem, therefore the conclusion.

Second solution

We apply the sinus' theorem in the triangles ABC, ABD, from where we obtain: $\widehat{ACB} = \widehat{ADB}$.

Therefore:

$$\frac{d_A}{\sin\widehat{B}} = \frac{a}{\sin\widehat{ACB}} \text{ and } \frac{d_B}{\sin\widehat{A}} = \frac{a}{\sin\widehat{ADB}}$$

From which the requested relation.

60. PROPOSED PROBLEM

Prove that there exists a right triangle having the legs, the heights, the rays of the inscribed respectively circumscribed, and area expressed in whole numbers.

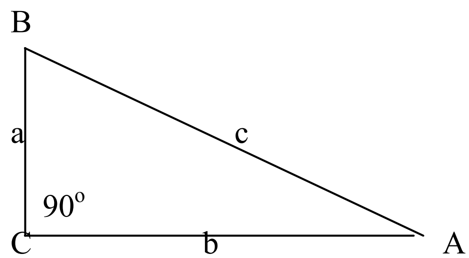
Solution

We consider the right triangle ABC, with the following sides:

$$a = 30n$$
$$b = 40n$$
$$c = 50n,\ n \in N^*$$

Which are Pythagorean numbers:

$$a^2 + b^2 = c^2$$
$$S = \frac{ab}{2} = 600n^2$$



The height h_c from the vertex C is equal to $24n$, because

$$h_c \cdot 50n / 2 = 600n^2$$
$$h_B = 30n$$
$$h_A = 40n$$
$$R = \frac{abc}{4S} = 25n \text{, or } R = \frac{c}{2}$$
$$R = \frac{S}{p} = 10n$$

Where $n =$ semi-perimeter of ABC

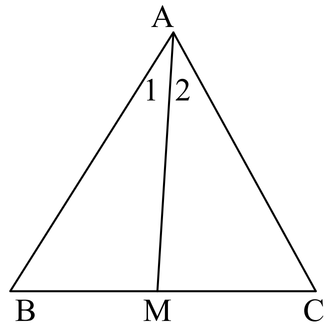
61. PROPOSED PROBLEM

In the triangle ABC we construct the Cevian AM that forms the angles $\widehat{A_1}, \widehat{A_2}$ with the sides AB respectively AC. Prove that:

$$\frac{|BA|}{|CA|} = \frac{|BM|}{|CM|} - \frac{\left|\sin \widehat{A_2}\right|}{\left|\sin \widehat{A_1}\right|}$$

(A generalization of the bisectors' theorem)

Solution



The sinus' theorem in the triangles ABM, ACM gives:

$$\frac{|BA|}{\sin\widehat{AMB}} = \frac{|BM|}{\sin\widehat{A_1}} \quad (1)$$

$$\frac{|CA|}{\sin\widehat{AMB}} = \frac{|CM|}{\sin\widehat{A_2}} \quad (2)$$

By dividing relation (1) by relation (2) we find the conclusion.

If AM is the bisector, it results that $\widehat{A_1} \equiv \widehat{A_2}$, obtaining the bisector's theorem.

If ABC is right triangle in $\widehat{A}$, then

$$\frac{|BA|}{|CA|} = \frac{|BM|}{|CM|} \cdot tg\widehat{A_2}$$

If AM is the median, then

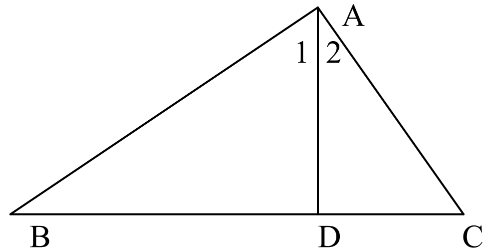
$$\frac{|BA|}{|CA|} = \frac{\left|\sin\widehat{A_2}\right|}{\left|\sin\widehat{A_1}\right|}$$

62. PROPOSED PROBLEM

Show that in a triangle the square of the height is equal to the product of the segment determined by it on the opposite side and the cotangents of the angles formed by the height with the adjoin sides.

(A generalization of the height's theorem)

Solution



Let $AD \perp BC$. We note

$$AD = y$$
$$BD = x_1$$
$$CD = x_2$$

We must prove that

$$y^2 = x_1 \cdot x_2 \cdot ctg\widehat{A_1} \cdot ctg\widehat{A_2}.$$

Applying the sinus' theorem in the triangles ABD, ADC we have

$$\frac{x_1}{\sin\widehat{A_1}} = \frac{y}{\cos\widehat{A_1}} \qquad (1)$$

$$\frac{x_2}{\sin\widehat{A_2}} = \frac{y}{\cos\widehat{A_2}} \qquad (2)$$

Multiplying side by side the relations (1) and (2), we find:

$$y^2 = x_1 \cdot x_2 \cdot \frac{\cos\widehat{A_1}\cos\widehat{A_2}}{\sin\widehat{A_1}\sin\widehat{A_2}} \quad \text{q.e.d.}$$

If the triangle ABC is a right angle triangle in A, then:

$$ctg\widehat{A_1}ctg\widehat{A_2} = ctg\widehat{A_1}ctg(90^\circ - \widehat{A_1}) = 1$$

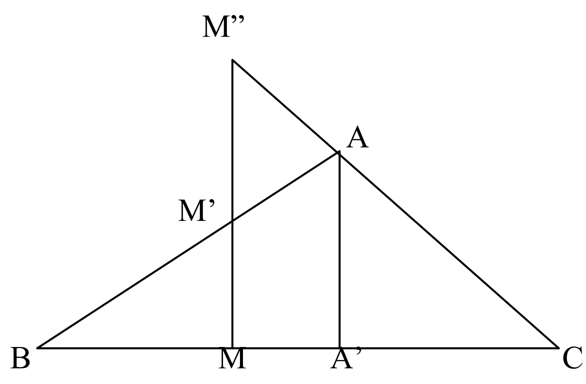
That is the very well known height theorem.

63. PROPOSED PROBLEM

Find, in function of the sides of a triangle. The length of the segment of a median (of a side) between the side and its extension on the other side.

Solution

Let AA' the height of a triangle ABC. We need to determine $|M'M''|$ (see the figure)



1) In A' between M and C (the same when A' is situated between M and B)

$$|AB| = c$$
$$|BC| = b$$
$$|CA| = c$$

because the triangles BMM' and BAA' are similar, we have

$$\frac{|MM'|}{|AA'|} = \frac{|BM'|}{|BA'|}$$

but

$$BA' = \sqrt{c^2 - h_A^2} \text{ with } |AA'| = h_A$$

It results that

$$|MM'| = \frac{a \cdot h_A}{2} \cdot \frac{1}{\sqrt{c^2 - h_A^2}}$$

In the same way, for the similar triangles CMM'' and CAA' we find

$$|MM''| = \frac{a \cdot h_A}{2} \cdot \frac{1}{\sqrt{b^2 - h_B^2}}.$$

2) Then

$$|MM''| = \frac{a \cdot h_A}{2} \cdot \left| \frac{1}{\sqrt{b^2 - h_A^2}} - \frac{1}{\sqrt{c^2 - h_A^2}} \right|$$

with

$$h_A = \frac{2}{a}\sqrt{p(p-a)(p-b)(p-c)}$$

according to Heron's formula and:

$$p = \frac{a+b+c}{2}$$

the semi-perimeter.

We took the absolute value in 2) to understand the two cases (see 1)).

64. PROPOSED PROBLEM

Let $A_1A_2....A_n$ a polygon. On a diagonal A_1A_k we take a point M through which passes a line d_1 which cuts the lines $A_1A_2, A_2A_3, ..., A_{k-1}A_k$ in the points $P_1, P_2, ..., P_{k-1}$ and a line d_2 which cuts the lines $A_kA_{k+1}, ..., A_{n-1}A_n, A_nA_1$ in the points $P_k, ...P_{n-1}P_n$.

Then

$$\prod_{i=1}^{n} \frac{A_1P_i}{A_{f(i)}P_1} = 1,$$

where f is the circular permutation

$$\begin{pmatrix} 12....n-1 & n \\ 23.... \;\; n & 1 \end{pmatrix}$$

(A generalization of Carnot's theorem)

65. PROPOSED PROBLEM

Let a polygon $A_1A_2....A_n$, a point M in its plane, and a circular permutation

$$f=\begin{pmatrix}12....n-1 & n\\ 23.... \ n & 1\end{pmatrix}.$$

We note M_{ij} the intersections of the line A_iM with the lines

$$A_{i+s}A_{i+s+1},...,A_{i+s+t-1}A_{i+s+t}$$

(for all i and j, $j\in\{i+s,...,i+s+t-1\}$).

If $M_{ij}\neq A_n$ for all respective indices, and if $2s+t=n$, then we have

$$\prod_{i,j=1,1+s}^{n,i+s+t-1}\frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{f(j)}}}=(-1)^n$$

where s,t are non-null natural numbers.

(Generalization of Ceva's theorem)

66. PROPOSED PROBLEM

Let a polygon $A_1A_2....A_n$ inscribed in a circle. We'll consider two numbers s,t non-null natural such that $2s+t=n$. Through each vertex A_i we construct a line d_i which cuts the lines $A_{i+s}A_{i+s+1},...,A_{i+s+t-1}A_{i+s+t}$ in the points $M_i^{'}$. Then we have

$$\prod_{i=1}^{n}\prod_{j=i+s}^{i+s+t-1}\frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{f(j)}}}=\prod_{i=1}^{n}\frac{\overline{M_i^{'}A_{i+s}}}{\overline{M_i^{'}A_{i+s+t}}}$$

67. PROPOSED PROBLEM

We consider a convex polyhedral whose faces don't have the same surface. Prove that the sum of the distances of a variable point situated inside of these surfaces of the polyhedral are constant.

Solution

We note with V the volume of the polyhedral and with S the surface of each of the n faces $F_1,F_2,...,F_n$

We'll divide this polyhedral in n pyramids with the vertex in M, where M is any interior point, and the bases $F_1,F_2,...,F_n$. The heights of the pyramids are precisely the distances from M to the polyhedral faces.

Then:

$$\frac{S(F_1)\cdot d_1}{3}+\ldots+\frac{S(F_n)\cdot d_n}{3}=V,$$

or

$$\frac{S}{3}(d_1+\ldots+d_n)=V,$$

then

$$d_1+\ldots+d_n=\frac{3V}{S}=\text{constant}.$$

68. PROPOSED PROBLEM

Let m be a positive integer.
Prove that any positive integer n, with $n \geq m$, can be factorized as follows:

$$n=(\ldots((1\cdot m+x)\cdot m+x)\ldots)\cdot m+x,$$

where x is either 0 or 1, or 2, or …, or $m-1$.

69. PROPOSED PROBLEM

Does the equation $x^z+y^y=z^z$ have integer solutions?

But the equation $x^z+y^y=\frac{1}{xy}$?

Solution:

I. **No**.

Of course x, y, z are not null.

1) If x, y are positive then z is also positive; we may suppose $0<x\leq y<z$, whence
$x^z+y^y\leq 2y^y<(y+1)^{y+1}\leq z^z$.
2) If x, y are odd (even) negative then z is odd (even) also; our equation becomes $\frac{1}{a^a}+\frac{1}{b^b}=\frac{1}{c^c}$, with a, b, c positive; we may suppose $c<a\leq b$, whence
$$\frac{1}{a^a}+\frac{1}{b^b}\leq\frac{2}{a^a}<\frac{1}{(a+1)^{a+1}}\leq\frac{1}{c^c}$$
3) If x is odd negative and y even negative we may suppose z odd negative (because it multiplies the equation by -1); changing the notations, our equation becomes as in the case 2).
4) When one of x and y is positive and the other negative, it is easy to handle.

II. **Yes**: $(-1,-2),(-2,-4),(-2,-1),(-4,-2)$ only.

Of course $x,\ y$ are no null.

Let's $E(x,y)=xy\left(x^{y}+y^{x}\right)$.

For $x,\ y$ positive integers $E(x,y)>1$.

Let's $x,\ y$ negative.

1) If $x,\ y$ are odd then $E(x,y)<0<1$
2) If $x,\ y$ are even, we have

$$E(a,b)=\frac{b}{a^{b-1}}+\frac{a}{b^{a-1}},$$

with $a=-x;\ b=-y$; $a,b\in\{2,4\}$ because for $a\geq 6$ (or $b\geq 6$) it is easily to prove (by recurrence) that

$$E(a,b)<\frac{1}{2}+\frac{1}{2}$$

Thus for $a=2$ it finds $b=4$ reciprocal. Whence

$$(x,y)=(-2,-4),(-4,-2)$$

3) Let's x odd and y even. Analogically

$$E(a,b)=\frac{b}{a^{b-1}}-\frac{a}{b^{a-1}}$$

For $x=-1$ or ($y=-1$) we have integer solutions: $(-1,-2)$, respectively . $(-2,-1)$

From 2) we have $a<6$, b<6 . But for $(a,b)=(3,2),(3,4),(5,2),(5,4)$ $E(a,b)\neq 1$

70. PROPOSED PROBLEM

Prove that the equation

$$a_1^{\ x}+...+a_p^{\ x}=b_1^{\ x}+...+b_q^{\ x},$$

with all $a_i,b_j>0$, has a finite number of real solutions. Of course, $a_1,...,a_p,b_1,...,b_q$ are different one from another.

Solution:

Let $f(x)=a_1^{\ x}+...+a_p^{\ x}-b_1^{\ x}-...-b_q^{\ x}$, $f:R\to R$, a continuous and derivable function.

Let $\max\limits_{i,j}\{a_i,b_j\}=a_1$ and $\min\limits_{i,j}\{a_i,b_j\}=a_p$.

1) If $a_1>1$ then

$$L=\lim_{x\to\infty} f(x)=a_1{}^x\cdot\left[1+\left(\frac{a_2}{a_1}\right)^x+\ldots+\left(\frac{a_p}{a_1}\right)^x-\left(\frac{b_1}{a_1}\right)^x-\ldots-\left(\frac{b_q}{a_1}\right)^x\right]=\infty,$$

and

$$1-\lim_{x\to\infty} f(x)=a_p{}^x\cdot\left[\left(\frac{a_1}{a_p}\right)^x+\ldots+\left(\frac{a_{p-1}}{a_p}\right)^x+1-\left(\frac{b_1}{a_p}\right)^x-\ldots-\left(\frac{b_1}{a_p}\right)^x\right]=\begin{cases}0, & \text{if } a_p>1\\ +\infty, & \text{if } a_p<1\\ 1, & \text{if } a_p=1\end{cases}$$

Hence f cannot have infinity of zeros.

If $a_1<1$ it finds $a_p<1$ too, hence $L=0$ and $1=+\infty$, the function can't have an infinity of zeros.

We may eliminate the case $a_1=1$ because $a_1{}^x=\text{constant}$

2-3-4) These cases are analogously studied, namely when the maximum and the minimum are respectively: (a_1,b_q) or (b_1,a_p) or (b_1,b_q).

71. PROPOSED PROBLEM

Solve in integer numbers the equation:

$$x^y-7z+3w=8$$

Solution:

$$w=\frac{-x^y+7z+8}{3}=2z+2+\frac{-x^y+z+2}{3}$$

But

$$\frac{-x^y+z+2}{3}=t\in Z$$

Then

$$z=x^y+3t-2,$$

hence

$$w=2x^y+7t-2.$$

The general integer solution is

$$\begin{cases}x=u\\ y=v\\ z=u^v+3t-2\\ w=2u^v+7r\\ z=x^y+3t-2\end{cases}$$

$t\in Z$, with $(u,v)\in(Z^*\text{ x }N)\cup(\{0\}\text{x}N^*)\cup(\{\pm1\}\text{x}Z)$.

72. PROPOSED PROBLEM

Solve the equation $x^3 - 4y = 23$ in integer numbers.

Solution:

$x^3 - 4y = x^3 (\bmod 4)$ and $23 \equiv 3 (\bmod 4)$; hence $x^3 \equiv 3 (\bmod 4)$, or

$$x = 4k + 3,\ k \in Z .$$

$$y = \frac{(4k+3)^3 - 23}{4} = 16k^3 + 36k^2 + 27k + 1$$

73. PROPOSED PROBLEM

Prove that the equation $x^3 - 7x_1 ... x_n = 3$ does not have integer solutions.

Solution:

If the equation has solutions then $x^3 - 3$ is divided by 7. But, for any

$$x = 7k + r,\ \ r \in \{0, 1.2, ..., 6\},\ k \in Z,\ \ x^3 \equiv 3 (\bmod 7) .$$

74. PROPOSED PROBLEM

Prove that the equation: $ax^n + by = c$, where $a, b, c \in Z,\ \ a \cdot b \neq 0$ has integer solutions if and only if there is $\{r \in 0, 1, |b| - 1\}$ such that $ar^n = c (\bmod b)$.

Solution:

$$y = \frac{ax^n - c}{b} \in Z ,$$

We write $x = b - k + r,\ \ k \in Z$ and $r \in \{0, 1, |b| - 1\}$

Hence:

$$y = \frac{a(bk + r)^n - c}{b} = d + \frac{ar^n - c}{b} \in Z$$

We have noted d the integer quotient of the division, $d \in Z$. Then it must $ar^n - c$ be divided by b.

Example: Solve in integers the following equation: $3x^5 - 7 = 6$.

We find

$$\begin{cases} x = 7k + 4 \\ y = \dfrac{3(7k+4)^5 - 6}{7} \end{cases}$$

$k \in Z$.

75. PROPOSED PROBLEM

Find the general positive integer solution of the Diophantine equation:

$$9x^2+6xy-13y^2-6x-16y+20=0$$

First solution: The equation becomes:

$$2u^2-7v^2+45=0 \tag{1}$$

where

$$\begin{aligned} u&=3x+y-1\\ v&=2y+1 \end{aligned} \tag{2}$$

We solve (1) 9see [1]). Thus:

$$\begin{cases} u_{n-1}=15u_n+28v_n \\ v_{n1}=8u_n+15v_n \end{cases} \quad n\in N, \tag{3}$$

with $(u_0,v_0)=(3,3\varepsilon)$, $\varepsilon=\pm 1$.

Clearly, for all $n\in N$, v_n is odd, and u_n as well as v_n are multiple of 3. Hence, there exist x_n, y_n in N, with

$$\begin{cases} x_n=\dfrac{(2u_n-v_n+3)}{6} \\ y_n=\dfrac{(v_n-1)}{2} \end{cases}, \ n\in N \tag{4}$$

(from(2)). Now, we find a closed expression for (3):

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}=A^n\cdot\begin{pmatrix} u_0 \\ v_0 \end{pmatrix},\ n\in N,$$

where $A=\begin{pmatrix} 15 & 28 \\ 8 & 15 \end{pmatrix}$, calculating A^n (see [1]).

Second solution: We transform (2) as: $u_n=3x_n+y_n-1$ and $v_n=2y_n+1$, $n\in N$.
Using (3) and doing the calculus, we find:

$$\begin{cases} x_{n+1}=11x_n+\dfrac{52}{3}y_n+\dfrac{11}{3} \\ y_{n+1}=12x_n+19y_n+3, \ n\in N \end{cases} \tag{5}$$

With $(x_n,y_n)=(1,1)$ or $(2,-2)$; (two infinitude or integer solutions).
Let

$$A=\begin{pmatrix} 11 & \frac{52}{3} & \frac{11}{3} \\ 12 & 19 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_n \\ y_n \\ 1 \end{pmatrix}=A^n\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x_n \\ y_n \\ 1 \end{pmatrix}=A^n\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad n\in N \qquad (6)$$

From (5) we have $y_{n+1}\equiv y_n\equiv\ldots\equiv y_0\equiv 1 (\bmod 3)$. Hence always $x_n\in Z$.
Of course (4) are equivalent to (6) (see [1]), and they constitute the <u>general</u> solutions.

Reference:

[1] F. Smarandache – A Method to solve the Diophantine Equation $ax^2-by^2+c=0$.

76. PROPOSED PROBLEM

Let $\varepsilon=1$ or -1 and

$$N=\varepsilon\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\cdot 3^{n-2k}\cdot 2^{3k}+\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\cdot 3^{n-2k-1}\cdot 2^{3k+1}$$

Prove that N is a perfect square if and only if $\varepsilon=1$ and $n=0$ or $n=3$.

<u>**Solution:**</u> Using the Diophantine equation $x^2=2y^4-1$ (which has the only solutions $(1,1)$ and $(239,13)$; see [1], [2] or [3] we note $t=y^2$. The general integer solution for $x^2-2t^2+1=0$ is:

$$\begin{cases} x_{n+1}=3x_n+4t_n \\ t_{n+1}=2x_n+3t_n \end{cases},$$

for all $n\in N$, $(x_0,y_0)=(1,\varepsilon)$ (see[4]); or

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix}=\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^n\cdot\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}, \text{ for all } n\in N,$$

where a matrix at the power zero is equal to the unit matrix.
Hence

$$\begin{pmatrix} x_n \\ t_n \end{pmatrix}=\begin{pmatrix} \frac{1+\varepsilon\sqrt{2}}{2}\left(3+2\sqrt{2}\right)^n\cdot\frac{1-\varepsilon\sqrt{2}}{2}\left(3-2\sqrt{2}\right)^n \\ \frac{2r+\sqrt{2}}{4}\left(3+2\sqrt{2}\right)^n\cdot\frac{2\varepsilon-\sqrt{2}}{4}\left(3-2\sqrt{2}\right)^n \end{pmatrix}, \quad n\in N.$$

Whence

$$t_n = \frac{2\varepsilon + \sqrt{2}}{4}\left(3 + 2\sqrt{2}\right)^n \cdot \frac{2\varepsilon - \sqrt{2}}{4}\left(3 - 2\sqrt{2}\right)^n,\ n \in N,$$

or $t_n = N$, $n \in N$. We obtain for $\varepsilon = 1$ and $n = 0$ or 3 that $N = 1$ or 169 respectively. Other values for n there are not yet, because we have only two solutions for the equation in integers.

References:

[1] R.K.Guy – Unsolved Problems in Number Theory, Springer-Verlag, 1981, Problem D6, 84-85.

[2] W. Ljundgren - Zur Theorie der Gleichung $x^2 + 1 = Dy^4$, Avh. Noroske Vid. Akad. Oslo, I 5(1942)#5, 27 pp; MR 8, 6.

[3] W. Ljundgren - Some remarks on the Diophantine equations $x^2 - Dy^4 = 1$ and $x^4 - Dy^4 = 1$, J. London Math. Soc. 41 (1966),542-544.

[4] F. Gh. Smarandache – A Method to solve the Diophantine Equations with two unknowns of second degree, Gamma, Braşov (to appear).

77. PROPOSED PROBLEM

Prove that the general solution in positive integers for the Diophantine Equation $2x^2 - 3y^2 = 5$ is

$$x_n = \frac{4 + \varepsilon\sqrt{6}}{4}\left(5 + 2\sqrt{6}\right)^n \cdot \frac{4 - \varepsilon\sqrt{6}}{4}\left(5 - 2\sqrt{6}\right)^n$$

$$y_n = \frac{3\varepsilon + 2\sqrt{6}}{6}\left(5 + 2\sqrt{6}\right)^n \cdot \frac{2\varepsilon - 2\sqrt{6}}{6}\left(5 - 2\sqrt{6}\right)^n$$

For all $n \in N$, where $\varepsilon = \pm 1$.

Solution: The smallest positive integer solutions are (2,1), (4,3).

Let

$$\begin{cases} x_{n+1} = ax_n + by_n \\ y_{n+1} = cx_n + dy_n \end{cases},$$

for all $n \in N$, and $\left(x_0, y_0\right) = \left(2, \pm 1\right)$. Then

$$2ab = 3cd \qquad (1)$$

$$2a^2 - 3c^2 = 2 \qquad (2)$$

$$2b^2 - 3d^2 = -3 \qquad (3)$$

Whence $a = \pm d$ and $b = \pm\frac{3}{2}c$. Because we look for positive solutions we find $a = d$, $b = \frac{3}{2}c$.

The smallest positive integer solution (with nonzero coordinates) for (2) is (5,4). Hence $a = d = 5$, $c = 4$, $b = 6$.

It results by mathematical induction that:

$$(GS) \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 4 & 5 \end{pmatrix}^n \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad n \in N.$$

Let $A = \begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}$, $\lambda \in R$, I the unit matrix.

$Det(A - 2 \cdot I) = 0$ involves $\lambda_{1,2} = 5 \pm 2\sqrt{6}$ and the proper vectors $v_{1,2} = (\sqrt{6}, \pm 2)$; (that is $Av_i = \lambda v_i$, i=1,2).

We note P=$\begin{pmatrix} \sqrt{6} & \sqrt{6} \\ 2 & -2 \end{pmatrix}$. Then $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, whence

$$A = P \cdot \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \cdot P^{-1} = \begin{pmatrix} \frac{1}{2}(\lambda_1^n + \lambda_2^n) & \frac{\sqrt{6}}{6}(\lambda_1^n - \lambda_2^n) \\ \frac{\sqrt{6}}{6}(\lambda_1^n - \lambda_2^n) & \frac{1}{2}(\lambda_1^n + \lambda_2^n) \end{pmatrix}$$

Taking $(x_0, y_0) = (2, \varepsilon)$ one finds the general integer solution.

We prove (GS) constitutes the general solution in positive integers by reduction ad absurdum: let (α, β) be the integer solution for our equation. Of course $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is other solution with $\alpha_1 < \alpha, \ \beta_1 < \beta$, also $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ with $\alpha_2 < \alpha_1, \ \beta_2 < \beta_1$, etc.

Let (α_i, β_i) be the smallest positive solution greater than or equal to (4,3). If (α_i, β_i) =(4,3) then $(\alpha_i, \beta_i) \in (GS)$; if no calculate $(\alpha_{i+1}, \beta_{i+1})^i = A^{-1} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$.

If $(\alpha_{i+1}, \beta_{i+1}) = (2 \pm 1)$ then it belongs to (GS)l if no, it results $2 < \alpha_{i+1} < 4$, but it is absurd.

Remark:
The method can be generalized for all Diophantine Equations of Second Degree with two Unknowns.

78. PROPOSED PROBLEM

Find the general solution in integers of the Diophantine Equation:

$$2x^2 - 3y^2 - 5 = 0$$

79. PROPOSED PROBLEM

Let p be an odd number. Prove or disprove that p and $p+r$ are simultaneously prime numbers if and only if

$$(p-1)!\left(\frac{1}{p}+\frac{2}{p+r}\right)+\frac{1}{p}+\frac{2}{p+r}$$

is an integer.

[Problem 328, "College Mathematical Journal", Vol. 17, No. 3, 1986, p. 249]

80. PROPOSED PROBLEM

Let $p_{i_1+1},\ldots,p_{i_2},\ldots,p_{i_n+1}$ be coprime integers two by two, and let $a_1,\ldots,a_n$ be integers such that a_j is coprime with $p_{i_1+1},\ldots,p_{i_2},\ldots,p_{i_{n+1}}$ for any j. Consider the conditions (j) $p_{j_1+1},\ldots,p_{j_j+1}$ are simultaneously prime if and only if $c_j \equiv 0(\text{mod } p_{j_1+1},\ldots,p_{j_j+1})$ for any j. Prove that $p_{i_1+1},\ldots,p_{i_2},\ldots,p_{i_{n+1}}$ are simultaneously prime if and only if

$$\sum_{j=1}^{n}\frac{a_j c_j}{p_{j_1+1},\ldots,p_{j_{j+1}}}$$

is an integer.

81. PROPOSED PROBLEM

Let P be a point on median AA' of the triangle ABC. Note B' and C' intersections of BP and CP with CA, respectively AB.

a) Prove that $B'C'$ is parallel to BC.

b) When AA' isn't median, let A'' be intersection of $B'C'$ with BC. Prove that A' and A'' divide BC in an unharmonious rapport

Solution:

a) From Ceva's theorem in ABC we find:

$$\frac{|C'A|}{|C'B|}\cdot\frac{|A'B|}{|A'C|}\cdot\frac{|B'C|}{|B'A|}=1, \qquad (1)$$

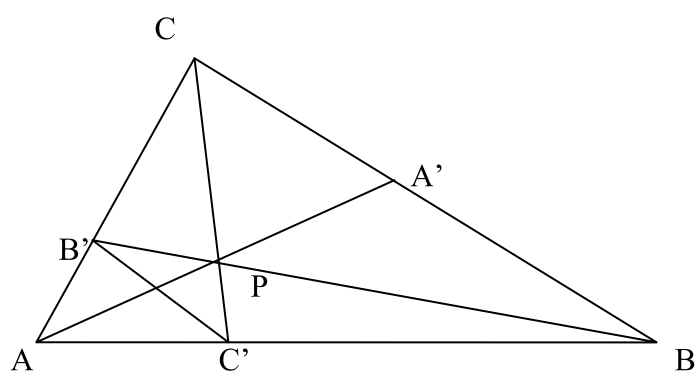


Fig. 1

and because $\dfrac{|A'B|}{|A'C|}=1$,

it results:

$$\frac{|C'A|}{|C'B|}=\frac{|B'C|}{|B'A|},$$

i.e. $B'C'$ is parallel to BC.

b) From Menelaus theorem we find:

$$\frac{|C'A|}{|C'B|}\cdot\frac{|A''B|}{|A''C|}\cdot\frac{|B'C|}{|B'A|}=1, \qquad (2)$$

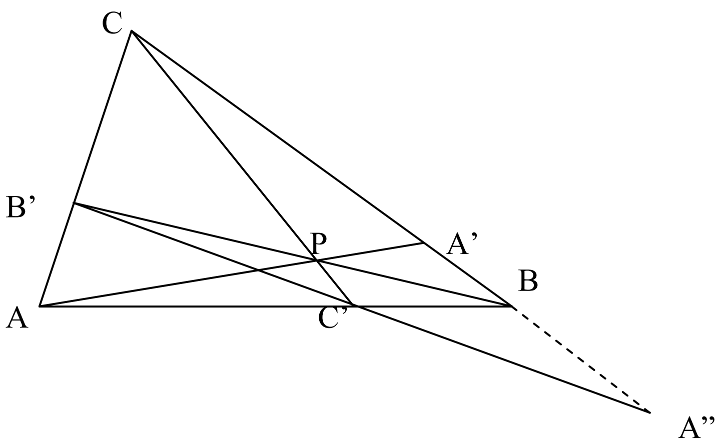


Fig. 2

We divide (1) to (2) and obtain:

$$\frac{|A'B|}{|A'C|}:\frac{|A''B|}{|A''C|}=1.$$

82. PROPOSED PROBLEM

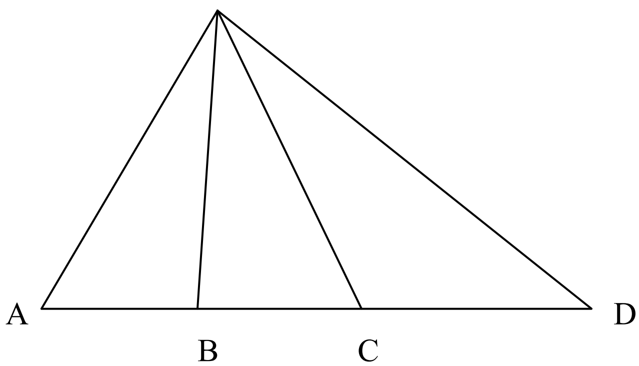
Let A, B, C, D be collinear points and O a point out-worldly of their line. Prove that

$$\left(OA^2-OC^2\right)BD+\left(OD^2-OB^2\right)AC=2AB\cdot BC\cdot CD+\left(AB^2+BC^2+CD^2\right)AD-\left(AB^3+BC^3+CD^3\right)$$

Solution:

Stewart’s theorem is used four times for points: (A, B, C), (B, C, D), (A, C, D) respectively (A, B, D) and it obtains the following relations:

O



$$OA^2 \cdot BC^2 - OB^2 \cdot CA + OC^2 \cdot AB = AB \cdot BC \cdot CA$$
$$OB^2 \cdot CD - OC^2 \cdot DB + OD^2 \cdot BC = BC \cdot CD \cdot DB$$
$$OA^2 \cdot CD - OC^2 \cdot DA + OD^2 \cdot AC = AC \cdot CD \cdot DA$$
$$OA^2 \cdot BD - OB^2 \cdot DA + OD^2 \cdot AB = AB \cdot BD \cdot DA$$

Summing member with member, all these relations, it finds the asked question.
It uses, too, that:

$$AC = AB + BC$$
$$BD = BC + CD$$
$$AD = AB + BC + CD$$

83. PROPOSED PROBLEM

Let $ABCD$ be the tetrahedron and $AA_1 \in CD,\ A_2 \in CB,\ \mathrm{C}_1 \in AD,\ C_2 \in AB$ four coplanar points.

Note $E = BC_1 \cap DC_2$ and $F = BA_1 \cap DA_2$. Prove that the lines AE and CF intersect.
("Crux Mathematicorum", Vol. 14, No. 7, September, 1988, p 203, problem 1368).

Solution

Note $BD \cap AE = A'$ and $BD \cap CF = C'$. We shall prove that A' and C' are identical

Firstly, we project A, B, C, D on plane $(A_1A_2C_1C_2)$ in points A', B', C' respectively D'.
Because we have four couples of similar triangles: $AA'C_2$ with $BB'C_2$ with $CC'A_2$, $CC'A_1$ with $DD'A_1$, and $DD'C_1$ with $AA'C_1$, it involves:

$$\frac{C_2A}{C_2B} = \frac{AA'}{BB'}, \frac{A_2B}{A_2C} = \frac{BB'}{CC'}, \frac{A_1C}{A_1D} = \frac{CC'}{DD'}$$

And respectively

$$\frac{C_1D}{C_1A} = \frac{DD'}{AA'}$$

By multiplication of these four equalities, member with member, we find

$$\frac{C_2A}{C_2B} \cdot \frac{A_2B}{A_2C} \cdot \frac{A_1C}{A_1D} \cdot \frac{C_1D}{C_1A} = 1 \qquad \text{(r)}$$

We apply Ceva's theorem in triangles ABD and BCD:

$$\frac{C_2A}{C_2B}\cdot\frac{C_1D}{C_1A}=\frac{A'D}{A'B}, \text{ respectively } \frac{A_2B}{A_2C}\cdot\frac{A_1C}{A_1D}=\frac{C'B}{C'D}$$

We replace these two equalities into (r) and we find:

$$\frac{A'D}{A'B}=\frac{C'B}{C'D}, \text{ hence } A'\equiv C'$$

because $A"C'\in[BD]$.

84. PROPOSED PROBLEM

Let $A_1, A_2, ..., A_n$ be the pyramid and α a plane which cuts the sides A_i, A_{i+1} in P_i. Prove that:

$$\prod_{i=1}^{n}\frac{P_iA_i}{P_iA_{i+1}}=1$$

Solution

We project $A_1, A_2, ..., A_n$ on α in $A_1', A_2', ...$ respectively A_n'. Right triangles $P_iA_iA_i'$ and $P_iA_{i+1}A'_{i+1}$ are similar, hence

$$\frac{P_iA_i}{P_iA_{i+1}}=\frac{A_iA_i'}{A_{i+1}A'_{i+1}}.$$

Our product becomes:

$$\frac{A_1A_1'}{A_2A_2'}\cdot\frac{A_2A_2'}{A_3A_3'}\cdot...\cdot\frac{A_nA_n'}{A_1A_1'}=1$$

85. PROPOSED PROBLEM

How many non-coplanar points in space can be drawn at integral distances each from other?

86. PROPOSED PROBLEM

Let n points be in the space such that the maximum distance among two of them is α.

Prove that there exists a sphere of radius $r\le\alpha\sqrt{\frac{6}{4}}$, which contains in interior or on its surface all these points.

(A generalization in space of Yung's theorem).

87. PROPOSED PROBLEM

Let $f : A \to B$ be a function. Find all functions $h \neq 1_A$ such that $f \circ h = f$.

Solution

If f is injective let $x \in A$ and $f(h(x)) = f(x)$, whence $h(x) = x$ or $h = 1_A$.
Then f may not be injective (necessary condition). It is sufficient:

$$(\exists) x_1 \neq x_2 \text{ such that } f(x_1) = f(x_2)$$

We construct

$$h : A \to A,\ h(x_1) = x_2$$

and

$$h(x) = x \text{ for } x \neq x_1 .$$

Hence $h \neq 1_A$.

Let $C = \{x \in A \mid (\exists) y \in A,\ y \neq x,\ f(x) = f(y)\}$.

All functions h from the problem have the form:

$$h : A \to A,\ h(x) = x \text{ for } x \in A \setminus C',$$

where $T\Phi \neq C' \subset C$, and if $x_1 \in C'$ then $h(x_1) = x_2$ for which $f(x_1) = f(x_2)$.

88. PROPOSED PROBLEM

For any $k, m, n \in N$ prove that:

$$\sum_{i=0}^{k} \binom{k}{i} A_m^1 A_n^{k-1} 2^{n-k+i} = \sum_{j=0}^{n} \binom{n}{j} A_{m+j}^k$$

Solution

$$(1+x)^n \cdot x^m = \binom{n}{0} \cdot x^m + \binom{n}{1} \cdot x^{m+1} + \binom{n}{2} \cdot x^{m+2} + \ldots + \binom{n}{n} \cdot x^{m+n}$$

Now, we derive this equality for k times. Because

$$\left[(1+x)^n\right]^{(k)} = A_n^k (1+x)^{n-k} \text{ and } \left(x^m\right)^{(k)} = A_m^k x^{m-k}$$

for any $k, m, n \in N$, $k \leq m$, and by Leibnitz formula in the left member we find:

$$\binom{k}{0} A_n^k (1+x)^{n-k} \cdot x^m + \binom{k}{1} A_n^{k-1} (1+x)^{n-k+1} \cdot A_{0m}^1 x^{m-1} + \ldots +$$

$$+ \binom{k}{k-1} A_n^1 (1+x)^{n-1} \cdot A_m^{k-1} x^{m-k+1} + \binom{k}{k} (1+x)^n A_m^k x^{m-k} =$$

$$= \binom{n}{0} A_m^k x^{m-k} + \binom{n}{1} A_{m+1}^k x^{m+1-k} + \ldots + \binom{n}{n} A_{m+n}^k x^{m+n-k} .$$

We replace $x = 1$ and obtain:

$$\binom{k}{0}A_n^k 2^{n-k}+\binom{k}{1}A_n^{k-1}2^{n-k+1}\cdot A_m^1 x^{m-1}+\binom{k}{k-1}A_n^1 2^{n-1}\cdot A_m^{k-1}+\binom{k}{k}2^n A_m^k=$$

$$=\binom{n}{0}A_m^k+\binom{n}{1}A_{m+1}^k+...+\binom{n}{n}A_{m+n}^k$$

When $a<b$ we have $\binom{a}{b}=A_a^b=0$.

Reference:

C. Năstăsescu, M. Brandiburu, C. Niţă, D. Joiţa – Exerciţii şi probleme de algebră , EDP, Bucureşti, 1981.

89. PROPOSED PROBLEM

Let f and g be two functions, g has an inverse, and a,b are real numbers. Prove that if the equation:

$$\left[\frac{f(x)}{a}\right]=\left[\frac{g(x)}{b}\right]$$

Admits solutions where $[x]$ is the greatest integer smaller or equal to x, then there exists an integer z such that

$$0\le\frac{\left(f\left(g^{-1}(bz)\right)-az\right)}{a}<1.$$

Solution

Let x_0 be a solution of the equation. We take

$$z=\frac{g(x_0)}{b}\in Z,$$

it involves $x_0=g^{-1}(bz)$.

Hence

$$\frac{f\left(g^{-1}(bz)\right)}{a}=z$$

And in accordance with $[x]\le x<[x]+1$ we find

$$z\le\left[\frac{f\left(g^{-1}(bz)\right)}{a}\right]<z+1$$

whence it involves our double inequality.

90. PROPOSED PROBLEM

Prove that if A and B are two convex sets, then $A\times B$ is a convex set too.

Solution

$A\times B=\{(a,b)\mid a\in A \text{ and } b\in B\}$.

Let $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2) \in A \times B$.

We should prove that $\alpha x_1 + (1-\alpha) x_2 \in A \times B$, for $\alpha \in [0,1]$.

$$\alpha x_1 + (1-\alpha) x_2 = \alpha a_1 + (1-\alpha) a_2 ,\ \alpha b_1 + (1-\alpha) b_2 \in A \times B$$

because $\alpha a_1 + (1-\alpha) a_2 \in A$ and $\alpha b_1 + (1-\alpha) b_2 \in B$.

91. PROPOSED PROBLEM

Let $a_1, a_2, ..., a_i$ be a system of integers with the property that $(\forall) a_i$ $(\exists) a_j$ such that the property

(C) $\quad a_i \equiv -a_j (\operatorname{mod} m)$,

where m is not null integer, and

$$Card\{i/(c)\} = Card\{j/(c)\}.$$

One makes up the sum $S_k = \sum a_{i_1} \cdot a_{i_2} \cdot ... \cdot a_{i_k}$ where summations done after all sequences $i_1, i_2, ..., i_k$ such that $1 \le i_1 < i_2 \le ... < i_k \le t$.

Prove that if t is even number and k odd number, then S_k is multiple of m. (Two partial generalizations of the Problem E3128, 1/1986, American Math Monthly).

The proof is simple. If $t = 2r$, $r \in N^*$ in the development of the congruence:

$$(x - a_1)(x - a_2)...(x - a_i) \equiv 0 (\operatorname{mod} m) \qquad (1)$$

the sums S_k represent the coefficients of x^{t-k}, $k \in \{1,3,5,...t-1\}$, that is all odd powers (vales of x.

But (1) becomes trough a rearrangement, according to the hypothesis:

$$(x - a_{i_1})(x - a_{i_2})...(x - a_{i_r})(x + a_{i_r})...(x + a_{i_2})(x + a_{i_r}) \equiv 0 (\operatorname{mod} m) \qquad (2)$$

or

$$(x^2 - a_{i_1}^2)(x^2 - a_{i_2}^2)...(x^2 - a_{i_r}^2) \equiv 0 (\operatorname{mod} m)$$

In the left part of (2) we have a polynomial which, developed, contains no odd power of x, which means that in (1) all the coefficients of the odd powers of x are multiples of m.

Here are the applications that we obtain:

Corollary 1

We find a nice result when $a_1, a_2, ..., a_t$ constitute a reduced system of rests when $C_1, C_2, ..., C_{\Phi(m)}$ $\operatorname{mod} \text{ulo} m \ne \pm 1, \pm 2$. Then $t = \Phi(m)$ is even number, being the function of Euler, and k is taken as an odd number. S_k will be a multiple of m.

This corollary is a partial generalization of the mentioned problem.

Corollary 2

Analogously, if we take $a_1, a_2, ..., a_t$ as being a complete system of remainders (for example $1, 2, ..., |m| - 1$) modulo m where m and k are odd numbers, S_k will be multiple of m. There appears another partial generalization of the mentioned problem.

92. PROPOSED PROBLEM

A generalization of problem E3037 "American Mathematical Monthly", 2/1984, p. 140.

Let m be an integer. Find all positive integers n such that $m\Phi(n)$ divides n.

Solution

$n=0$ verifies; the case $n=1$ is trivial.

Let $n=p_1^{\alpha_1},...,p_s^{\alpha_s}$, where all p_i are distinct primes and all $\alpha_i \in N^*$, which verify our assumption. Then

$$\frac{n}{m\Phi(n)}=\frac{p_1}{p_1-1}\cdot....\cdot\frac{p_s}{p_s-1}\cdot\frac{1}{m}\in Z \qquad (1)$$

Clearly $2|n$ because all p_i-1 are multiple of 2 (without the case when a $p_i=2$). Hence, from (1) results $1\le s\le 2$, because $(p_2-1)...(p_s-1)$ is a multiple of 2^{s-1}.

a) If $s=1$ we have

$$\frac{n}{m\Phi(n)}=\frac{2}{m}$$

For m a divisor of 2 there exists $n=2^{\alpha},\ \alpha\in N^*$

b) If $s=2$ we have

$$\frac{n}{m\Phi(n)}=2\frac{p_2}{p_2-1}\cdot\frac{1}{m}$$

and we may take $p_2=3$ only. Hence, if m divides 3 there exists $n=2^{\alpha}\cdot 3^{\beta}$, with $\alpha,\beta\in N^*$.

Remarks

1. This division is possible if and only if $m\in\{\pm1,\pm2,\pm3\}$
2. For $m=1$ we find the problem E3037, AMM, 2/84, p.140.

93. PROPOSED PROBLEM

How many solutions has the inequality $\Phi(x)<a$? Find their general form.

Solution

This inequality remembers the Carmichael's equation $\Phi(x)=a$.

We consider $a\in N$ because $\Phi(x)\in N$.

The cases $a=0,\ a=1$, and $a=2$ are trivial.

Let a be a positive integer $\ge$ 3. Let $p_1<p_2<...<p_s\le a$ be prime numbers less or equal to a. If x verifies our inequality then $x=p_1^{\alpha_1}...p_s^{\alpha_s}$, with $\alpha_1\in N$ for all i, because from "p_1 divides x and $p_k>a$ is prime number" we find $\Phi(x)\ge\Phi(p_k)=p_k^{-1}\ge a$:, which is absurd.

Hence

$$\Phi(x) = x\left(\frac{p_1 - 1}{p_1}\right)^{\varepsilon_1} \cdots \left(\frac{p_s - 1}{p_s}\right)^{\varepsilon_s} < a$$

where $\varepsilon_i = \begin{cases} 0, \text{ if } \ \alpha_i = 0 \\ 1, \text{ if } \ \alpha_i \neq 0 \end{cases}$ for all i, or

$$(x) < \left(\frac{p_1}{p_1 - 1}\right)^{\varepsilon_1} \cdots \left(\frac{p_s}{p_s - 1}\right)^{\varepsilon_s} \cdot a$$

It results that the number of solutions of our inequality is finite. With the previous notations we obtain the general form of the solutions:

$$S = \bigcup_{\varepsilon_1 + \ldots + \varepsilon_n = 0}^{s} \left\{ x \mid x = p_1^{\alpha_1} \ldots p_s^{\alpha_s},\ \alpha_i \in N \text{ for all } i,\ (x) < \left(\frac{p_1}{p_1 - 1}\right)^{\varepsilon_1} \cdots \left(\frac{p_s}{p_s - 1}\right)^{\varepsilon_s} \cdot a,\ \varepsilon_i = 0 \text{ when} \right.$$

$$\left. (x) < \left(\frac{p_1}{p_1 - 1}\right)^{\varepsilon_1} \cdots \left(\frac{p_s}{p_s - 1}\right)^{\varepsilon_s} \cdot a,\ \varepsilon_i = 0, \text{ when } \alpha_i = 0 \text{ and } \varepsilon_i = 1 \text{ when } \alpha_i \neq 0 \text{ for all } i \right\} \bigcup \{0\}.$$

The number of the solutions is equal to: $cardS = cardB$, where

$$B = b \mid b = \left[\left(\frac{p_1}{p_1 - 1}\right)^{\varepsilon_1} \cdots \left(\frac{p_s}{p_s - 1}\right)^{\varepsilon_s} \cdot a / \left(p_1^{\alpha_1} \ldots p_s^{\alpha_s}\right) \right],$$

$0 \le \varepsilon_1 + \ldots + \varepsilon_i \le s$, all $\alpha_i \in N$, $\varepsilon_i = 0$, when $\alpha_i = 0$ and $\varepsilon_i = 1$ when $\alpha_i \neq 0$ for all $i\}$

Remark

From this proposal we find that the equation $\Phi(x) = a$ admits a finite number of solutions, because the inequality $\Phi(x) < a - 1$ has a finite number of solutions (on the Carmichael's conjecture).

Example

$\Phi(x) < 10$ has in all 19 solutions:

General form $2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot$		2^{α_1}	3^{α_2}	5^{α_3}	7^{α_4}	$2^{\alpha_1} \cdot 3^{\alpha_2}$	$2^{\alpha_1} \cdot 5^{\alpha_3}$	$2^{\alpha_1} \cdot 7^{\alpha_4}$	$3^{\alpha_2} \cdot 5^{\alpha_3}$	$3^{\alpha_2} \cdot 7^{\alpha_4}$ or $5^{\alpha_3} \cdot 7^{\alpha_4}$	$2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3}$
solutions	0 1	2,4,8,16	3,9	5	7	6,12,18,24	10,20	14	15	-	30

94. PROPOSED PROBLEM

Let f be a function, $f : Z \to Z$.

a) On what terms is there a function $g: R \to Z$ such that $g \circ f = 1_E$, but $f \circ g \neq 1_E$, where 1_E is the identical function $\left(1_E : E \to E \;\; by \;\; 1_E(x) = x, \;\; (\forall) x \in E\right)$?

b) In this case find g.

Solution

a) f isn't an injective function $\Leftrightarrow (\exists) x_1, x_2 \in Z$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2) \Leftrightarrow (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2) \Leftrightarrow g \circ f \neq 1_E$.

Hence f must be an injective function.

b) $g = f^{-1}([x])$ where $f^{-1} : f[Z] \to Z$. Then $(g \circ f)(x) = f^{-1}(f(x)) = x$, $(\forall) x \in Z$, $Z \to^{f} Z \to^{g} Z$; and $(f \circ g)(x) = f^{-1}(f(x)) = [x] \neq x$ because $x \in R$ $R \to^{g} Z \to^{f} Z$.

95. PROPOSED PROBLEM

Let a and b be two elements from a group, the order of a is m and $a^n b^n = b^r a^n$.
If p divides r, calculate the order of b.
("College Mathematical Journal", USA, 1990)

Solution

Let's $r = kp$, $k \in N^*$, and the smallest common multiple $[m,n] = M$, $i = M / n$. We denote the reverse of a with $a^{\bullet}$. One obtains:

$$b^p = \left(a^{\bullet}\right)^n b^{kp} a^n \tag{1}$$

We replace this relation into itself k times.

$$b^p = \left(a^{\bullet}\right)^n \underbrace{b^p \ldots b^p}_{k \; times} a^n = \left(a^{\bullet}\right)^n \underbrace{\left(a^{\bullet}\right)^n b^{kp} a^n}_{1} \underbrace{\left(a^{\bullet}\right)^n b^{kp} a^n}_{2} \ldots \underbrace{\left(a^{\bullet}\right)^n b^{kp} a^n}_{k} a^n = \left(a^{\bullet}\right)^{2n} b^{k^2 p} a^{2n} = \tag{2}$$

(replacing (1) in (2) it results:)

$$= \left(a^{\bullet}\right)^{3n} b^{k^3 p} a^{3n} = \ldots = \left(a^{\bullet}\right)^{in} b^{k^i p} a^{in} = b^{k^i p}$$

(because $a^M = e = \left(a^{\bullet}\right)^M$).

Hence $b^p = b^{k^i p}$ or $b^p = b^{k^i p \cdot p} = e$, whence $ord(b) = p\left(r / p\right)^{[m,n]/n} - 1$.

Remark

For $m = 5,\; n = p = 1,\; r = 2$ it obtains the Problema Nr. 7, concurs-oposicion a plazas de prof. agregados de Bachillerato 1984, Matematicas, Tribunal No.1, Turno: Libre, R.L.1, R.L.2, Valladolid, Spanish, by Francisco Bellot.

96. PROPOSED PROBLEM

Let a, b be real numbers, $0 < a \neq 1$; prove that there is a continuous, strictly decreasing function $g(x)$ of the real line R into R such that $g\left(g(x)\right) = ax + b$.
(A partial generalization for the problem E3113, Amer. Math. Monthly, 9/1985).

Solution

We construct a function $g : R \to R$,

$$g(x) = \sqrt{a} \cdot x + \frac{b}{1-\sqrt{a}}.$$

Thus $g\left(g(x)\right) = ax + b$, g is continuous and strictly decreasing.

Remark

For $a = b = 2$ we obtain a function of the above mentioned problem.

97. PROPOSED PROBLEM

Let p be a prime number. Prove that a square matrix of integers, having each line and column a unique element non-divisible by p, is nonsingular.

Solution

In accordance with the development of the matrix there exists a unique permutation, i.e. a unique term (=product) having as factors all these non-divisible elements by p.

This term isn't divisible by p, but all other ones are divided by p. Hence, sum (= value of the determinant) isn't divided by p, whence the determinant can't be null.

98. PROPOSED PROBLEM

Let $f(x) = ax^2 + bx + c$ be a function of second degree, $f : R \to R$, such that there are $d, e \in R$ for which $f(d) = e$. Prove that $b^2 + 4ae \geq 4ac$.

Solution

Because $f : R \to R$, it involves that:

$$f(0) = c \in R,$$
$$f(1) = a + b + c \in R$$

and

$$f(-1) = a + b + c \in R$$

or

$$a + b \in R$$

and

$$a - b \in R,$$

Whence

$a \in R$ and $b \in R$.

The second degree equation $ax^2 + bx + c - c = 0$ has at least a real solution $x_1 = d$, and because all coefficients are real, we find $x_2 \in R$.

Hence $\Delta \geq 0$, or $b^2 - 4ac + 4ae \geq 0$.

99. PROPOSED PROBLEM

Find two sequences $\{a_n\}, \{b_n\}$, $(n = 1, 2, ...)$ such that $\sum_1^\infty a_n$, $\sum_1^\infty b_n$, both diverge while $\sum_1^\infty \min\{a_n, b_n\}$ converges.

(On the problem E3125, AMM 1/1986)

Solution

1) For example $a_n = 0,1,0,1,...$ and $b_n = 0,1,0,1,...$ hence

   $\min\{a_n, b_n\} = 0$, for all $n \geq 1$

2) More generally:

   Let $\{c_n\}$ be a sequence such that $\sum_1^\infty c_n$ converges and all $c_n < 1$ (for example $\{1/n^s\}$ $n \geq 1$ with $s > 1$), and $\{d^n\}$ another sequence such that $\sum_1^\infty d_n$ diverges and all $d_n \geq 1$.

   We construct a_n, b_n as bellow:

   $$a_n = d_1, ..., d_{m_1}, c_{m_1+1}, ..., c_{m_1+k_1}, d_{m_1+k_1+1}, ..., d_{m_1+k_1+k_2}, ...$$
   $$b_n = c_1, ..., c_{m_1}, d_{m_1+1}, ..., d_{m_1+k_1}, c_{m_1+k_1+1}, ..., c_{m_1+k_1+k_2}, ...$$

   Of course $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$, diverge because:

   $$\sum_1^\infty a_n > \sum_1^\infty d_n \text{ and } \sum_1^\infty b_n > \sum_1^\infty d_n,$$

and

$$\sum_1^\infty \min\{a_n, b_n\} = \sum_1^\infty c_n \text{ converges.}$$

100. PROPOSED PROBLEM

Let , where p_n is the n[th] prime number.

Does the series $\sum_{n+1}^\infty \frac{1}{d_n}$ converge?

Solution

The series is divergent.

From Tchebishev's theorem, it involves $p_{n+1} - p_n < p_n$

(contrary to this it would involve that p_n and p_{n+1} are not consecutive).

Hence

$$\frac{1}{d_n} > \frac{1}{p_n} \text{ and } \sum_{n+1}^{\infty} \frac{1}{d_n} > \sum_{n}^{\infty} \frac{1}{p_n} = \infty$$

[Published in "The Fibonacci Quarterly", Vol. 30, Nov. 1992, No. 4, pp. 368-0, proposed problem B-726].

101. PROPOSED PROBLEM

Let f be a continuous and positive function on $[a,b]$.

Calculate

$$\lim_{x\to\infty} \left(\int_a^b \left(f(x) \right)^n dx \right)^{\frac{1}{n^2}}, \quad a < b$$

Solution

By mean theorem we find

$$\int_a^b \left(f(x) \right)^n dx = (b-a)\, f(c)^n, \quad c \in f[a,b]$$

Then

$$L = \lim_{x\to\infty} f(c)^{\frac{1}{n}} (b-a)^{\frac{1}{n^2}} = 1 .$$

Remark

If f is positive on $[a,b]$ it is well known that L is equal to 1, too.

102. PROPOSED PROBLEM

Let (S) be a solution in integers for a linear equation with non-null integer coefficients

$$(S) \quad x_i = \sum_{j=-1}^{h} u_{ij} k_j + v_j, \; i \in = \{1, ..., n\};$$

$x_1 a_1 + ... + x_n a_n = b$, where all u_{ij}, v_j are constant in Z, and all k_j are integer parameters. The author has proved (in [1] and [2]) that if (S) is a general solution then:

1) $h = n - 1$;
2) $\left(u_{1j}, ..., u_{nj}\right) = 1$, for all $j \in \{1, ..., n-1\}$;
3) $\left(u_{1j}, ..., u_{in-1}\right) = \left(a_1, ..., a_{i-1}, a_{i+1}, ..., a_n\right)$, for all $i \in = \{1, ..., n\}$

But these three conditions are not sufficient.

Questions

A) Find other condition(s) still, so that (S) together with all these to be a general solution; (reciprocally)

B) A similar question in order that an integer solution of a linear system to be a general solution.

References

[1] Florentin Smarandache – Rezolvarea în numere întregi a ecuaţiilor şi sistemelor de ecuaţii liniare. Lucrare de licenţă, Univ. of Craiova, 1979.

[2] Florentin Smarandache – Whole Number Algorithms to solve Linear Equations and Systems. Ed. Scientifiques, Casablanca, 1984. See Zentrallblatt fur Mathematik (Berlin)

103. PROPOSED PROBLEM

Prove that from a non-constant arithmetical progression of $\left((2m-3)\cdot 3^n+1\right)/2$ terms one can extract $(m-1)\cdot 2^n$ distinct terms such that any m of them do not constitute an arithmetical progression. ("Nieuw Archief voor Wiskunde"), 1988, No.2, problem No.809, p. 171, and "The Mathematical Intelligencer", Vol. 11, No.1, 1989).

*Find the greatest extraction with this property.

<u>Solution</u>

It's sufficient to make as non-constant arithmetical progression $N*$ (see [1], p. 69).

One constructs the sets sequence:

$$A_1=(1,2,...,m-1)$$

$$A_{n+1}=A_n\cup(A_n+2a_n-1),\text{ for } n\geq 1,$$

where $a_p=\max\{a\mid a\in A_p\}$ and $A+x=\{a+x\mid a\in A\}$. Of course $a_1=m-1$, $a_{n+1}=3a_n-1$, for $n\geq 1$, whence $a_{n+1}=1/2\left[(2m-3)\cdot 3^n+1\right]$. Also, $CardA_{n+1}=(m-1)\cdot 2^n$.

Each A_n has the property of the problem, because for A_1 is obvious; if it is true for A_n, then: for $m\geq 4$ we have $(a_{n+1}-b_1)/(m-1)<b_{n+1}-a_n$, where $b_p=\min\{a\mid a\in A_p\}$; for $m=3$, if $x_i\in A_n$ and $x_3\in A_{n+1}\setminus A_n$ then $x_2=\frac{1}{2}(x_1+x_2)\notin A_{n+1}$; for $m=2$ we prove there are not $(x,y)\neq(z,t)$ from A^2_{n+1} such that $|x-y|=|z-t|$m that is $\left|3^a-3^b\right|=\left|3^c-3^d\right|$, or we can suppose $3^a-3^d=3^b-3^c$; let's $a=\min\{a,b,c,d\}$; then $d=\max\{a,b,c,d\}$; then $1+3^{d-a}=3^{b-a}+3^{c-a}$, or $1\equiv 0(\mathrm{mod}\,3)$ impossible.

*The author isn't able to answer to this question.

Remarks

This problem is valid for a non-constant geometrical progression too. (see [1], p.69).

It is a generalization of the Problem 5, Mathematics International Olympiad, Paris, 1983; (When $m = 3$, $(m-1)\cdot 2^n \geq 1983$ and $1/2\left[(2m-3)\cdot 3^n + 1\right] \leq 10^5$).

References

[1] F. Smarandache - “Problèmes avec et sans…problèmes!”, Somipress, Fès, Morocco, 1983!

[2] T. Andreescu & Co. – “Probleme date la concursurile şi examenele din 1983”, Timişoara.

104. PROPOSED PROBLEM

Let n be odd and m be even numbers. Consider a number of points inside a convex n-gon. Is it possible to connect these points (as well as the vertices of the n-gon) by segments that do not cross one another, until the interior is subdivided into smaller disjoint regions that are all m-gons and each given point is always a vertex of any m-gon containing it?

Answer: No.

<u>**Solution**</u>

(after a Honsberger and Klamkin’s idea): against all reason let p be the number of all m-gons, and i the number of interior points.

The sum of the angles of the p smaller m-gons is $\pi(m-2)p$; also, the sum of the angles of the n vertices of the convex n-gon is $\pi(m-2)$. Thus $\pi(m-2)p = 2\pi i + \pi(n-2)$ or $p(m-2) = 2i + n - 2$, but the right part is even while the left part is odd!

Remark

It’s a generalization of [1], and of a problem of [2]

References

[1] Problem 4, solution by M.S.Klamkin, 1079 Michigan Mathematics Prize Competition, Crux mathematicorum, Vol. 11, No. 1, January, 1985, p.7, Canada.

[2] Ross Honsberger, Mathematical Gems, Mathematical Associations of America, 1973, pp. 106, 164.

[3] R.N. Gaskell, M.S.Klamkin, and P. Watson, Triangulations and Pick’s theorem, Math. Magazine, 49(1976), pp. 35-37.

105. PROPOSED PROBLEM

Prove there exist infinity of primes containing the digits $a_1, a_2, ..., a_m$ on positions $i_1, i_2, ...,$ respectively i_m with $i_1 > i_2 > ... > i_m \geq 0$. (Of course, if $i_m = 0$ then a_m must be odd).
("Mathematics Magazine", April 1990)

Solution

a) Let $i_m > 0$. We construct the number $N = \underbrace{a_1 0...0}\underbrace{a_2 0...0}\underbrace{a_3 0...0}\underbrace{a_m 0...1}$ such that each a_j is on positions i_j, another positions being equal to zero, without the last one equal to 1.
We construct an arithmetical progression

$$x_k = N + k \cdot 10^{i_1 - 1}, \quad k = 1,2,3...$$

From Dirichlet's theorem we find our result.

b) $i_m = 0$. Same proof, but on last position of N we take a_m.

106. PROPOSED PROBLEM

Let n be a positive integer of two or more digits, not all equal.

Ranging its digits by decreasing, respectively increasing order, and making their difference, one finds another positive number n_1. One follows this way, obtaining a series: $n_1, n_2, n_3, ...$ (For example, if $n = 56$, then $n_1 = 65 - 56 = 09$, $n_2 = 90 - 09 = 81$, etc.)

Prove that this series is periodical (i.e., of the form: $a_1, ..., a_t, b_1, ..., a_p, b_1, ..., a_p, ...$).
[On the Kaprekar's algorithm.]

Solution

Let m be the number of digits of $n, m \geq 2$ of course, for all $i \in N$, $n_i \in \{1, 2, ..., 10^m - 1\}$.

In this situation, taking 10^m numbers of m digits each one (that means 10^m numbers of the series), it finds at least two equal numbers (from the Dirichlet principle). Let they be n_p and n_t in the series. Since $n_s = n_t$ and the way to obtain n_{t+1} by means of n_t, it involves $n_{s+1} = n_{t+1}$, hence $n_{s+2} = n_{t+2}$, etc.

Example

Let $n = 90000$. Then

$n_1 = 90000 - 00009 = 89991$

$n_2 = 99981 - 18999 = 80982$

$n_3 = 99920 - 02889 = 95931$

$n_4 = 99531 - 13599 = 85932$

$n_5 = 98532 - 23589 = \boxed{74943}$

$\boxed{n_6 = 97443 - 34479 = 62964}$

$n_7 = 96642 - 24669 = 71973$

$n_8 = 97731 - 13779 = 83952$

$n_9 = 98532 - 23589 = \boxed{74943}$ ⟵

$\boxed{n_{10} = 97443 - 34479 = 62964}$ ⟵

……………………………………..

Remarks

For 3 and 4 digits Kaprekar showed that the period length is just one, i.e. the series is stationary always equal to 495, respectively 6174.

Look:

528, 594, 495, 495, 495, …

Respectively

8031,8172,7443,3996,6264,4176,6174,6174,…

Reference

[1] Pierre Berloquin – "Le jardin du Sphnx" – Bordas, Paris, 1981; Problème 37 (L'Algorithme de Kaprekar), pp. 61, 146.

107. PROPOSED PROBLEM

Assume the second degree equation $Ax^2 + Bx + C = 0$ has the real solution x_1 and x_2 and $a < b < c < d$ are real numbers. On what terms must the real coefficients A, B, C be such that

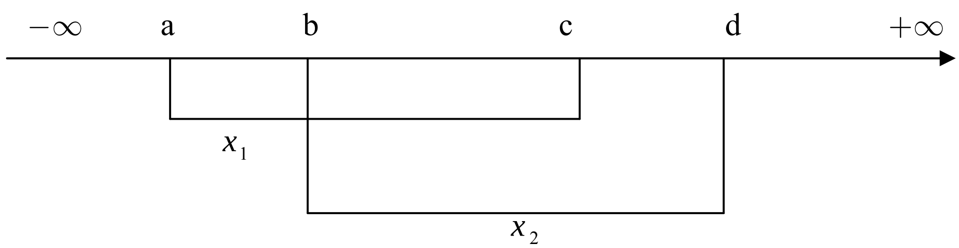
a) $x_1 \in (a,c)$ and $x_2 \in (b,d)$

b) $x_1 \in (b,c)$ and $x_2 \in (a,d)$.

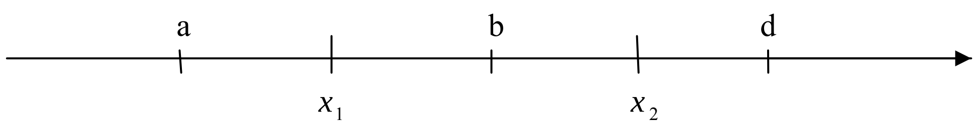
Solution

Let's note $f(x) = Ax^2 + Bx + C$, $\Delta = b^2 - 4AC$ and $S = x_1 + x_2 = -\dfrac{B}{A}$

a)



I. $x_1 \in (a,c)$, $x_2 \in (b,d)$

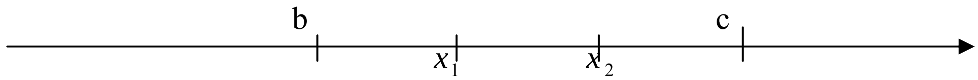


Whence it involves the conditions

(C1): $\Delta > 0,\ Af(a) > 0,\ Af(b) < 0,\ Af(d) > 0$

Or

II. $x_1 \in (b,c)$, $x_2 \in (b,c)$.

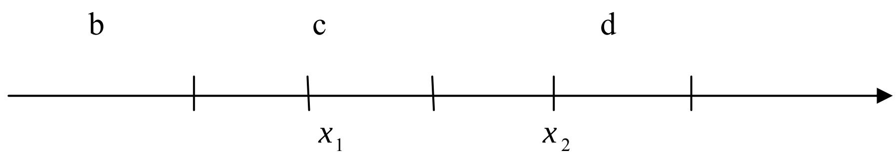


Whence it involves the conditions

(C2) $\Delta \geq 0,\ Af(b) \geq 0,\ Af(c) \geq 0,$ 2b<S<2c ;

Or

III. $x_1 \in (b,c)$, $x_2 \in (c,d)$

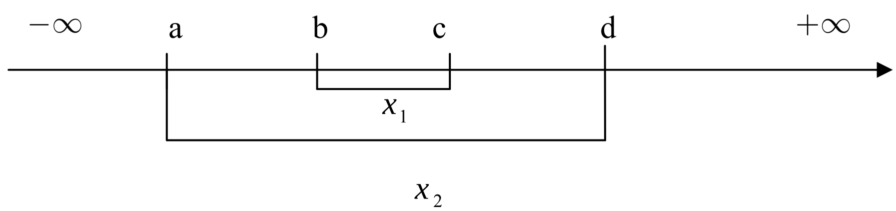


Whence it involves the conditions

(C3) $\Delta > 0,\ Af(b) \geq 0,\ Af(c) < 0,\ Af(d) < 0$;

Hence A, B, C must accomplish the conditions: $(C1) \cup (C2) \cup (C3)$.

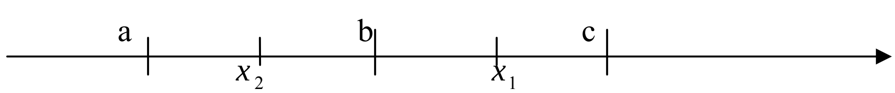
b)



We can have the following cases:

I.

$x_1 \in (b,c)$ $x_2 \in (a,b)$



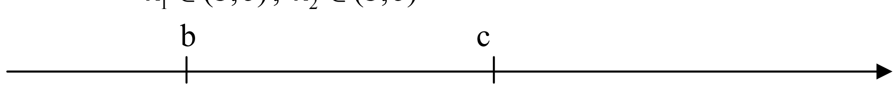
Whence it involves the conditions

(K1): $\Delta > 0,\ Af(a) > 0,\ Af(b) \leq 0,\ Af(c) > 0$;

Or

II.

$x_1 \in (b,c)$, $x_2 \in (b,c)$



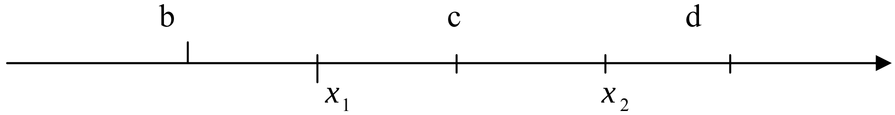
Whence it involves the conditions :

(K2): $\Delta \geq 0,\ Af(b) > 0,\ Af(c) > 0,$ 2b<S<2c

Or

III.

$x_1 \in (b,c)$, $x_2 \in (c,d)$



Whence it involves the conditions:

(K3): $\Delta > 0,\ Af(b) > 0,\ Af(c) \leq 0,\ Af(d) > 0$

Hence A, B, C must accomplish the conditions:

$(K1) \cup (K2) \cup (K3)$

108. PROPOSED PROBLEM

Let $n, m \in N$ and let S be the set of $nm+1$ points in space such that any subset of S consisting of $m+1$ points contains two points with distances less than d (a given positive number)

Prove that there exists a sphere of radius d containing at last $n+1$ points of S in its interior.

(Nieuw Archief voor Wiskunde", Vierde Series deel 5, No. 3, November 1987, proposed problem No. 797, pp.375-6.

Particular case ($d = 1$) in "Crux Mathematicorum", Vol. 14, No. 5, May 1988, problem 1344, p140).

Solution

Let A_1 be a point in space. We choose an A_2 for which $|A_1A_2| \geq d$. If it there isn't, our problem is solved.)

Afterwards we choose an A_3 with the same property ($|A_iA_j| \geq d,\ i \neq j$), etc. The method ends when we can't add another point with this property.

Clearly, we have at most m points $A_1, A_2, ..., A_p$, where $1 \leq p \leq m$, with this property. Distributing the $nm+1$ points in p classes corresponding to $A_1, A_2, ..., A_p$ there exists a sphere of radius d and center A_i which contains at least $n+1$ points in interior (Dirichlet principle).

Remark

This is a generalization of a problem proposed by prof. Mircea Lascu.

109. PROPOSED PROBLEM

How many digits on base b does the n-th prime contain? But $n!$? But n^n ?

("PI MU Epsilon Journal", USA, 1992).

Solution

By Paul T. Bateman, University of Illinois at Urbana-Champaign, USA.

The number of digits of the number N to the base b is the integral part of $1+\log N / \log b$. The number of digits of n^n is $[1+n \cdot \log n / \log b]$.

The known formula

$$n \cdot \log n < p_n < n \cdot \log n(1+\delta(n))$$

and

$$n^n e^n \sqrt{2\pi n} < n! < n^n e^n \sqrt{2\pi n}\left(1+\varepsilon(n)\right),$$

Where $\delta(n)$ and $\varepsilon(n)$ are positive quantities which approach zero when n is large, make it possible to approximate the number of digits in p_n or $n!$ within one unit for large n.

Reference

Paul T. Bateman, Letter to the Author, February 8, 1988.

110. PROPOSED PROBLEM

Let $m \in N$, $m > 1$. Is there a positive integer n having the property: anyhow one chooses n integers sum is divided by m?

Solution

No.

If, against all reason, there would be such n then taking the sets A and B of integers:

$$A = \{a_1, a_2, ..., a_n\} \text{ with } S = \sum_1^n a_i$$

$$B = \{a_1 + 1, a_2, ..., a_n\} \text{ with } S' = S + 1,$$

It would result that $S - S' = 1$ is divided by m.

111. PROPOSED PROBLEM

Prove that anyhow an infinite arithmetical progression is divided into n sets, at least one of the sets contains m arithmetical progressions triplets.

Is this result still true when n (or m) $\to \infty$?

(Nieuw Archief voor Wiskunde", Vierde Serie Deel 12, No. 2, Maart/Juli 1994, p. 93)

Solution

It's sufficient to take as arithmetical progression N^* (see [1], p. 69). One decomposes N^* in $n \cdot m$ arithmetical under-progressions:

$a_h^{(i)} = i + (n \cdot m)h, \quad h = 0, 1, 2, 3, ...$

for all $i \in \{1, 2, 3, ..., n \cdot m\}$

The arbitrary division of N^* is equivalent to the arbitrary division of these under-progressions.

From Van der Waerden's theorem, if each under-progression $a_h^{(i)}$ is divided in two sets $M_1^{(i)}$ and $M_2^{(i)}$, at least one contains an arithmetical progression triplet. In all there are $n \cdot m$ sets which contain each one an arithmetical progression triplet, in the end, these from behind are distributive in n classes (the box principle)
*The author is not able to answer this question.

Remark
This problem is valid for the geometrical progressions, too (see [1], p.69).

Reference

[1] F. Smarandache – "Problèmes avec et sans…problèmes!", Somipress, Fès, Morocco, 1983.

112. PROPOSED PROBLEM

Prove that if $n \geq 3$ then between n and $n!$ there are at least $3\frac{n}{2}-2$ prime numbers.

Solution
a) Let $n = 2k+1$, with $k \geq 1$.
$n! = 2^k \cdot k! \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)n > 2^{3k-2} \cdot k! \cdot n \geq 2^{3k-2} \cdot n$,

And by Bertrand-Tchebishev's theorem, applied of $3k-2$ times, $k = \left[\frac{n}{2}\right]$, we find our result.

b) The same proof when $n = 2k$, with $k > 2$.

113. PROPOSED PROBLEM

Prove that there exist at least five primes of s digits, $s \geq 2$.

Solution
Because for $n \geq 4$ we have that

between n and $\frac{s}{2}n$ there exists at least a prime (see [1]), one obtains that between $\frac{3}{2}n$ and $\frac{3^2}{2^2}n$ there exists at least a prime ;

……………………………………………………………………

between $\frac{3^4}{2^4}n$ and $\frac{3^5}{2^5}n$ there exists at least a prime.

Hence, between n and $\frac{3^5}{2^5}n$ there exists at least five prime numbers (the approximation is very gross).

Taking $n = 10^{s-1}$ we have that between 10^{s-1} and $\frac{3^5}{2^5} \cdot 10^{s-1}$ we have at least five prime, but

$$\frac{3^5}{2^5} \cdot 10^{s-1} < 10^s .$$

Reference

[1] I. Cucuruzeanu – "Probleme de aritmetică şi teoria numerelor" – Ed. Tehnică, Bucureşti, 1976, pp. 108 (Problema III.48*) and 137-140.

114. PROPOSED PROBLEM

Find the n^{th} term of the following sequence:

1,3,5,7,6,4,2,8,10,12,14,13,11,9,15,17,19,…

Generalization.

Solution

Let φ be the following permutation:

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 6 & 4 & 2 \end{pmatrix}$$

Then $a_{7k+r} = 7k + \varphi(r)$, where $0 \le r \le 6,\ k \ge 0$.

Generalization:

For any permutation φ of m elements:

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & \ldots m \\ a_1 & a_2 & a_3 \ldots a_m \end{pmatrix}$$

we have $a_{mk+r} = m \cdot k + \varphi(r)$, where $0 \le r \le m-1,\ k \ge 0$.

115. PROPOSED PROBLEM

Find the general term formula for the following sequence:

1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,6,6,6,6,6,6,7,7,7,7,7,7,7,8,8,8,8,8,8,8,8,…

(the natural sequence where each number n is repeated n times).

Solution

For $r \ge 1$, we have

$$a_{\frac{r(r+1)}{2}-k} = r, \text{ where } 0 \le k \le r-1 .$$

116. PROPOSED PROBLEM

Find a partition on N^* into an infinite number of distinct classes (not all of them finite) such that no classes contain an arithmetic progression of three terms.

<u>Solution</u>

Let M be the set of all positive integers which are not perfect power:

$$M = \{2,3,5,6,7,10,11,12,13,14,17,18,19,20,21,22,...\}$$

For any $m \in M$ we define

$$C_m = \{m^k;\ k = 1,2,3,...\}.$$

Then:

$$N^* = \{1\} \cup \bigcup_{m \in M} C_m$$

There are an infinite number of classes C_m, because M is infinite and no classes C_m contain an arithmetical progression of three terms (or more) – because all C_m are non-constant geometric progressions.

117. PROPOSED PROBLEM

Let $a_1\ a_2\ a_3...a_m$ be digits.

Are there primes, on a base b, which contain the group of digits $\overline{a_1\ a_2\ a_3...a_m}$ into its writing?

(For example, if $a_1 = 0$ and $a_2 = 9$, $b = 10$, there are primes as 109, 409, 709, 809,…)

*The same questions replacing "primes" by numbers of the form $n!$ or n^n.

<u>Solution</u>

Let $N = \overline{a_1\ a_2\ ...a_m 1}$, and the following arithmetical progression:

$$x_k = N + k \cdot b^{m+1}, \quad k = 1,2,3,...$$

Using Dirichlet's theorem on primes in arithmetical progression, we find that there exist an infinity of primes with the required property.

*The author is not able to answer this last question.. He conjectures that generally speaking the answer is <u>*no.*</u>

118. PROPOSED PROBLEM

Find a natural number N such that, if:

1) a q_1^{th} part of it and a_1 more are taken away;

2) a q_2^{th} part of the remainder and a_2 more are taken away;

3) a q_3^{th} part of the second remainder and a_3 more are taken away;

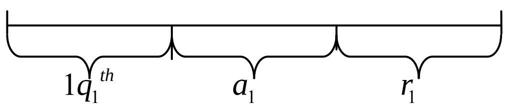
………………………………………………

s) a q_s^{th} part of the remainder and $s-1^{th}$ remainder and a_s more are taken away;

the last remainder is r_s.

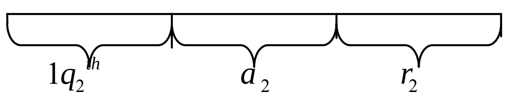
Solution
One uses a backwards arithmetic way.
Consider the problem for the particular case $s = 3$ (only three steps):

I)



$$(a_1 + r_1) \cdot \frac{q_1}{q_1 - 1} = N$$

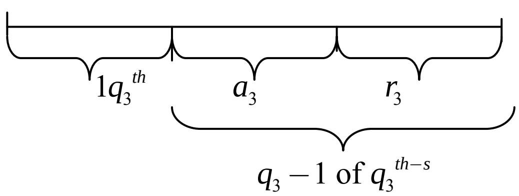
II)



$$(a_2 + r_2) \cdot \frac{q_2}{q_2 - 1} = r_1$$

(by notation)

III)



$$(a_3 + r_3) \cdot \frac{q_3}{q_3 - 1} = r_3$$

(by notation)

Therefore:

$$N = \left(a_1 + \left(a_2 + (a_3 + r_3) \cdot \frac{q_3}{q_3 - 1} \right) \cdot \frac{q_2}{q_2 - 1} \right) \cdot \frac{q_1}{q_1 - 1}.$$

One can generalize and prove by induction that:

$$N = \left(a_1 + \left(a_2 + \ldots + \left(a_{s-1} + (a_s + r_s) \frac{q_s}{q_s - 1} \right) \cdot \frac{q_{s-1}}{q_{s-1} - 1} \right) \cdot \ldots \frac{q_2}{q_2 - 1} \right) \cdot \frac{q_1}{q_1 - 1}$$

119. PROPOSED PROBLEM

[International Congress of Mathematicians, Berkeley, CA, USA, 1986, Section 3, Number Theory]

SMARANDACHE, FLORENTIN GH., University of Craiova, Romania. An infinity of unsolved problems concerning a function in number theory.

We have constructed a function η which associates to each non-null integer m the smallest positive n such that $n!$ is a multiple of m. Let $\eta^{(i)}$ note $\eta^0\eta^0\ldots\eta$ of i times.

Some Unsolved problems concerning η.

1) For each integer $m > 1$ find the smallest k and the constant c for which $\eta^{(k)}(m) = c$.

2) Is there a closed expression for $\eta^{(n)}$?

3) For a fixed non-null integer m does $\eta^{(n)}$ divide $n - m$?

4) Is η an algebraic function?

5) Is 0.0234537465114…, where the sequence of digits is $\eta^{(n)}$, $n \geq 1$, an irrational number?

6) For a fixed integer m, how many primes have the form $\overline{\eta^{(n)}\eta^{(n+1)}\ldots\eta^{(n+m)}}$?

Solve the Diophantine equations and inequalities:

7) $\eta^{(x)y} = x\eta^{(y)}$, for x, y not primes.

8) $\eta^{(x)} - y = x / \eta^{(y)}$, for x, y not primes.

9) Is $0 < \left\{ x / \eta^{(x)} < \eta^{(x)} / x \right\}$ infinitely often? Where $\{a\}$ is the fractional part of a.

10) Find the number of partitions of n as sum of $\eta^{(m)}$, with $2 < m < n$.

By means of η^{*} we construct recursively an infinity of arithmetic functions and unsolved problems.

*This function has been called THE SMARANDACHE FUNCTION (see "Personal Computer World", "Mathematical Reviews", "Fibonacci Quarterly", "Octogon", "Mathematical Spectrum", "Elemente der Mathematik", "Bulletin of Pure and Applied Science", etc. [*Editor's note.*]

Florentin SMARANDACHE

PROPOSED PROBLEMS OF MATHEMATICS
(Vol. II), *Second version - entirely translated to English, 2010.*

Semnat pentru tipar 1.X.10. Format 60x84 1/16.
Rotaprint. Coli de tipar 6,0.
Comanda 268. Preţ contractual

Secţia poligrafie operativă
A U.S.M. 2010. Chişinău,
St. A. Mateevici, 60

I collected these problems that I published in various Romanian or foreign magazines (amongst which: “Gazeta Matematică”, magazine which formed me as problem solver, “American Mathematical Monthly”, “Crux Mathematicorum” (Canada), “Elemente der Mathematik” (Switzerland), “Gaceta Matematica” (Spain), “Nieuw voor Archief” (Holland), etc. while others are new proposed problems in this second volume.

These have been created in various periods: when I was working as mathematics professor in Romania (1984-1988), or co-operant professor in Morocco (1982-1984), or emigrant in the USA (1990-1997).

The author