Warp Drive Basic Science Written For "Aficionados". Chapter II - Jose Natario

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Abstract

Natario Warp Drive is one of the most exciting Spacetimes of General Relativity.It was the second Spacetime Metric able to develop Superluminal Velocities.However in the literature about Warp Drives the Natario Spacetime is only marginally quoted.Almost all the available literature covers the Alcubierre Warp Drive. It is our intention to present here the fully developed Natario Warp Drive Spacetime and its very interesting features.Our presentation is given in a more accessible mathematical formalism following the style of the current Warp Drive literature destined to graduate students of physics since the original Natario Warp Drive paper of 2001 was presented in a sophisticated mathematical formalism not accessible to average students. Like the Alcubierre Warp Drive Spaceime that requires a continuous function $f(rs)$ in order to be completely analyzed or described we introduce here the Natario Shape Function $n(r)$ that allows ourselves to study the amazing physical features of the Natario Warp Drive. The non-existence of a continuous Shape Function for the Natario Warp Drive in the original 2001 work was the reason why Natario Warp Drive was not covered by the standard literature in the same degree of coverage dedicated to the Alcubierre Warp Drive. We hope to change the situation because the Natario Warp Drive looks very promising .

1 The Beginning-What exactly is a "Warp Drive"??

The "Warp Drive" as a way to travel faster than light was discovered by the Mexican mathematician Miguel Alcubierre from Universidad Nacional Autonoma de Mexico(UNAM)in 1994. He in that year published a paper that is regarded as one of the cornerstone of modern Physics.The paper was simply called "The Warp Drive:Hyperfast Travel Within General Relativity".Miguel Alcubierre presented a solution of the Einstein Field Equations of General Relativity that allows faster than light Space Travel.This solution is the "Warp Drive".The "Warp Drive" as conceived by Alcubierre was a Bubble of Spacetime with a spaceship inside the Bubble at the rest with its local spacetime feeling no g-forces and no accelerations that otherwise would destroy the ship concerning faster than light velocities and the Bubble would be at the rest with respect to the rest of the Universe.Spacetime behind the Bubble would expand moving away the departure point and Spacetime would contract in the front of the Bubble bringing to the ship the destination point in a way the resembles the Big Bang(or the Big Crunch).A spaceship inside a "Warp Drive" would be able to attain large Superluminal velocities effectively travelling faster than light. Seven years later another revolutionary paper on the "Warp Drive" appeared. In 2001 the Portuguese mathematician Jose Natario from Instituto Superior Tecnico(IST) conceived a "Warp Drive" that do not expands or contracts.The

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ship is still immersed in a "Warp Bubble" and this Bubble is carried out by the Spacetime "stream" at faster than light velocities with the ship at the rest with respect to its local neighborhoods inside the Bubble feeling no g-forces and no accelerations. Imagine an aquarium floating in the course of a river with a fish inside it...the walls of the aquarium are the walls of the Warp Bubble...Imagine that this river is a "rapid" and the aquarium is being carried out by the river stream...the aquarium walls do not expand or contract...an observer in the margin of the river would see the aquarium passing by him at an arbitrarily large speed but inside the aquarium the fish would be protected from g-forces or accelerations generated by the stream...because the fish would be at the rest with respect to its local spacetime inside the aquarium.Jose Natario in 2001 wrote the Revolutionary paper that was called "Warp Drive with Zero Expansion".The Natario Warp Drive Is carried out by the Spacetime stream just like a fish in the stream of a river.The Natario Warp Drive is the main theme of this work.

2 Introducing the Generic Natario Warp Drive Formalism

The Warp Drive Spacetime according to Natario is defined by the following equation(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{1}$$

where X^i is the so-called Shift Vector.This Shift Vector is the responsible for the Warp Drive behavior defined as follows(pg 2 in [2]):

$$X^i = X, Y, Z \curvearrowright i = 1, 2, 3 \tag{2}$$

The Warp Drive spacetime is completely generated by the Natario Vector nX(pg 2 in [2])

$$nX = X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}, \tag{3}$$

Note the difference between the Shift Vector X^i and the Natario Vector $nX = X^i \frac{\partial}{\partial x^i}$
Considering a Natario Warp Drive with motion over the x axis only we would have:

$$ds^2 = dt^2 - (dx^i - X^i dt)^2 \tag{4}$$

$$ds^2 = dt^2 - (dx - X dt)^2 \tag{5}$$

- a)-coordinate

$$x^i = x \tag{6}$$

- b)-Shift Vector

$$X^i = X \tag{7}$$

- c)-Natario Vector

$$nX = X\frac{\partial}{\partial x} \tag{8}$$

Note that the Shift Vector X is used to define the Natario Vector nX.The importance of the Natario work can be outlined in the following statements:

- 1)-The Warp Drive was not a "lucky strike" discovered by Alcubierre in 1994.The Warp Drive is an entire class of solutions of the Einstein Field Equations of General Relativity that allows faster than light Space Travel and the Alcubierre solution is only one of the two known solutions that are chartered.There are many more solutions to be discovered.

- 2)-Any Natario Vector nX generates a Warp Drive Spacetime if $nX = 0$ for a small value of $|x|$ defined by Natario as the interior of the Warp Bubble and $nX = -vs(t)$ or $nX = vs(t)$for a large value of $|x|$ defined by Natario as the exterior of the Warp Bubble with $vs(t)$ being the speed of the Warp Bubble.Again this encompasses many possible solutions.The Warp Drive is an entire family of solutions of the Einstein Field equations of General Relativity. There are at the present moment two solutions already discovered for the Warp Drive Spacetime:Alcubierre(1994) and Natario(2001)

the vector $|x|$ according to Natario is defined as(pg 4 in [2]):

$$|x| = x^i \frac{\partial}{\partial x^i} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \tag{9}$$

Again considering motion over the x axis only we have:

$$|x| = x^i \frac{\partial}{\partial x^i} = x\frac{\partial}{\partial x} \tag{10}$$

The Alcubierre solution of 1994 for the Warp Drive can be retrieved if we define the Shift Vector X as(pg 3 in [2])(pg 4 in [1])

$$X = vsf(rs) \tag{11}$$

$$Y = Z = 0 \tag{12}$$

$$vs = \frac{dxs}{dt} \tag{13}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{14}$$

$$f(rs) = \frac{tanh[@(rs + R)] - tanh[@(rs - R)]}{2tanh(@R)} \tag{15}$$

The $f(rs)$ is the Alcubierre Shape Function with the values $f(rs) = 1$ in the interior of the Warp Bubble and $f(rs) = 0$ in the exterior of the Warp Bubble,xs is the center of the Warp Bubble,R is the radius of the Warp Bubble and @ is the thickness of the Warp Bubble.The Alcubierre Shape Function is very important.We will use it to define later the Natario Shape Function.The Natario Warp Drive with Zero Expansion we will present later is very different than the original Alcubierre Warp Drive with Expansion Contraction but the proof that both Warp Drive Spacetimes are members of the same family of solutions of the Einstein Field Equations of General Relativity is the possibility that the Shape Function defined for one of them can be used to construct the Shape Function for the other.

Considering the Natario Warp Drive Spacetime equation for the x-axis only:

$$ds^2 = dt^2 - (dx - Xdt)^2 \tag{16}$$

we would get the following expressions for the Shift Vector $X = vsf(rs)$:

$$ds^2 = dt^2 - (dx - vsf(rs)dt)^2 \tag{17}$$

Above we retrieved the original Alcubierre Warp Drive solution.The Natario Vector Field nX for the Alcubierre Warp Drive would then be defined as:

$$nX = X\frac{\partial}{\partial x} = vsf(rs)\frac{\partial}{\partial x} \tag{18}$$

The Expansion and Contraction of the Spacetime for the Alcubierre Warp Drive is a consequence of the choice made by Alcubierre in 1994 for the Shift Vector $X = vsf(rs)$ and is defined by the Expansion of the Normal Volume Elements as(pg 3 and 4 in [2])(pg 5 in [1]):

$$\theta = \partial_i X^i \tag{19}$$

$$\theta = \partial_x X = v_s f'(r_s)[\frac{x - x_s}{r_s}] \tag{20}$$

$$\theta = \partial_x X = v_s[\frac{df(rs)}{drs}][\frac{x - x_s}{r_s}] \tag{21}$$

It is easy to see that a different Shift Vector X or a different Shape Function $f(rs)$ would produce a different Expansion of the Normal Volume Elements.Natario in 2001 introduced a different Shift Vector that do not Expands or Contracts the Spacetime in the Warp Bubble.

The Warp Drive as a Dynamical Spacetime requires an amount of energy in order to be generated.The generic expression for the Energy Density for the Warp Drive and the same for the Alcubierre Warp Drive are given below(pg 4 in [2])(pg 8 in [1]):

$$\rho = \frac{1}{16\pi}\left[(\partial_x X)^2 - (\partial_x X)^2 - 2\left(\frac{1}{2}\partial_y X\right)^2 - 2\left(\frac{1}{2}\partial_z X\right)^2\right] \tag{22}$$

$$\rho = -\frac{1}{32\pi}v_s^2\left[f'(r_s)\right]^2[\frac{y^2 + z^2}{r_s^2}] \tag{23}$$

$$\rho = -\frac{1}{32\pi}v_s^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2 + z^2}{r_s^2}] \tag{24}$$

Again note the fact that a different Shift Vector or a different Shape Function would produce a different distribution of the Energy Density.This is very important: a "good" Shift Vector would require less energy in order to be generated then a "bad" Shift Vector. The same applies for the Shape Function.A "good" Shape Function requires less energy than a "bad" or a "evil" Shape Function.Thus is very important because the Warp Drive as a faster than light solution of the Einstein Field Equations requires enormous amounts of energy to be generated and this energy is negative.This is a critical issue.We will address this later in this work.The Warp Drive as a family of solutions of the Einstein Field Equations of General Relativity that allows faster than light Space Travel can have many Shift Vectors and many Shape Functions.The Warp Drive family can have many members.Some of these members of the family are impossible to be generated in a real fashion but other members of the same family look more feasible to be achieved from a realistic physical point of view.This is the first cornerstone of the two very important contributions made to the Warp Drive science by Jose Natario in 2001.We will now examine the second cornerstone.Jose Natario introduced in 2001 a different Shift Vector generating a different Warp Drive Spacetime that behaves very different than the one introduced by Alcubierre in 1994.The Natario Shift Vector have amazing properties and from a realistic point of view looks very promising.The member of the Warp Drive family introduced by Jose Natario in 2001 is known as:"Warp Drive with Zero Expansion"

3 Warp Drive with Zero Expansion:

In 2001 the Portuguese mathematician Jose Natario from Instituto Superior Tecnico(IST) introduced a new Warp Drive spacetime defined using the Canonical Basis of the Hodge Star in spherical coordinates defined as follows(pg 4 in [2])[1]:

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \tag{25}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \tag{26}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \tag{27}$$

Applying the Natario equivalence between spherical and cartezian coordinates as shown below(pg 5 in [2])[2]:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{28}$$

Of course we consider here the following pedagogical approach

$$x = r\cos\theta \tag{29}$$

$$dx = d(r\cos\theta) \tag{30}$$

From the result written above we can review the Natario definition for the Warp Drive:

- 1)-A Natario Vector nX being $nX = 0$ for a small value of $|x|$(interior of the Warp Bubble)
- 2)-A Natario Vector $nX = -vs(t)$ or $nX = vs(t)$for a large value of $|x|$(exterior of the Warp Bubble)
- 3)-$vs(t)$-speed of the Warp Bubble seen by distant observers.

The Revolutionary concept of the Warp Drive with Zero Expansion introduced by Jose Natario in 2001 can be defined by the following Natario Vector nX as follows(pg 5 in [2]):

$$nX \sim -v_s(t)dx \tag{31}$$

or

$$nX \sim v_s(t)dx \tag{32}$$

The Natario Vector nX for the motion only in the x axis was defined originally as follows:

$$nX = X\frac{\partial}{\partial x} \tag{33}$$

[1]See Appendix on Hodge Stars and Differential 1-forms and 2-forms

[2]The Mathematical demonstration of this expression will be given in the Appendix on Hodge Stars and Differential Forms

but according to Natario we can use this approximation(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx \tag{34}$$

Hence the Natario Vector nX can be defined as follows:

$$nX = Xdx \tag{35}$$

The Shift Vector X for the Warp Drive with Zero Expansion is defined simply by:

$$X = -vs(t) \tag{36}$$

or by

$$X = vs(t) \tag{37}$$

And for dx we have(pg 5 in [2]) :

$$dx = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{38}$$

Inserting all this stuff in the Natario Vector nX we would get the following expression(pg 5 in [2])

$$nX = Xdx = vs(t)d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{39}$$

Now a little bit of Warp Drive basics:An observer inside the Warp Bubble feel no accelerations and no g-forces if the Warp Bubble moves with a $vs = 200$(two hundred times faster than light) with respect to a distant observer because he is at the rest with respect to his local spacetime inside the Warp Bubble.In this case the Natario Vector is $nX = 0$.On the other hand an external observer faraway from the Warp Bubble sees the Warp Bubble passing by him at $vs = 200$.The Natario Vector in this case is $nX = -vs(t)$ or $nX = vs(t)$.Then we need a dx that is zero in the ship and 1 far from it.Note immediately the resemblances between this and the Alcubierre Shape Function $f(rs) = 1$ in the ship and $f(rs) = 0$ far from it.This is not a coincidence.Both Warp Drives belongs to the same family of solutions of the Einstein Field Equations.

Rewriting the Natario Vector as(pg 5 in [2])

$$nX = vs(t)d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{40}$$

Note that we replaced $\frac{1}{2}$ by $f(r)$ in order to obtain(pg 5 in [2]):

$$nX = vs(t)d\left(f(r)r^2 \sin^2\theta d\varphi\right) \tag{41}$$

$$nX = -v_s(t)d\left[f(r)r^2 \sin^2\theta d\varphi\right] \sim -2v_s f(r)\cos\theta \mathbf{e}_r + v_s(2f(r) + rf'(r))\sin\theta \mathbf{e}_\theta \tag{42}$$

From now on we will use this pedagogical approach that gives results practically similar the ones depicted in the original Natario Vector shown above[3]

$$nX = -v_s(t)d\left[f(r)r^2 \sin^2\theta d\varphi\right] \sim -2v_s f(r)\cos\theta dr + v_s(2f(r) + rf'(r))r\sin\theta d\theta \tag{43}$$

[3]Again see the Mathematical demonstration of the Natario Vector in the Appendix on Hodge Stars and Differential Forms

In order to make the Natario Warp Drive holds true we need for the Natario Vector nX a continuous Natario Shape Function being $f(r) = \frac{1}{2}$ for large r(outside the Warp Bubble) and $f(r) = 0$ for small r(inside the Warp Bubble) while being $0 <_f (r) < \frac{1}{2}$ in the walls of the Warp Bubble

In order to avoid contusion with the Alcubierre Shape Function $f(rs)$ we will redefine the Natario Shape Function as $n(r)$ and the Natario Vector as shown below

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + rn'(r))r\sin\theta d\theta \tag{44}$$

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{45}$$

Lets analyze the behavior of the Natario Vector nX for the Natario Shape Function $n(r)$ with a Shift Vector $X = -vs(t)$ and a Warp Bubble speed $vs(t) = 200$ with respect to a distant observer.two hundred times faster than light.

- 1)-Inside the Warp Bubble $n(r) = 0$
- 2)-Outside the Warp Bubble $n(r) = \frac{1}{2}$
- 3)-In the Warp Bubble walls $0 < n(r) < \frac{1}{2}$
- A)-Inside the Warp Bubble $n(r) = 0$

  Inside the Warp Bubble $n(r) = 0$ as a constant value.Then the derivatives of $n(r)$ vanishes and we can rewrite the Natario Vector nX as follows:

$$nX = -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{46}$$

$$nX = -2v_s[0]\cos\theta dr + v_s(2[0])r\sin\theta d\theta \tag{47}$$

$$nX = 0 \tag{48}$$

  No Motion at all!!!!:The observer inside the Natario Warp Bubble is completely at the rest with respect to its Local Spacetime neighborhoods and this observer dont feel any acceleration or any g-forces..Then for this observer $\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta = 0$.Remember that the Natario Vector is still $nX = -vs(t)dx$.But the Shift Vector is still $X = -vs(t)$.The Natario Warp Bubble still moves with a speed $vs(t) = 200$ with respect to a distant observer.two hundred times faster than light but the internal observer inside the Warp Bubble is completely at the rest and completely in safe from the g-forces that would kill him moving ar such hyper-fast velocities.

- B)-Outside the Warp Bubble $n(r) = \frac{1}{2}$

  Outside the Warp Bubble $n(r) = \frac{1}{2}$ as a constant value.Then the derivatives of $n(r)$ vanishes and we can rewrite the Natario Vector nX as follows:

$$nX \simeq -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{49}$$

$$nX \simeq -2v_s\frac{1}{2}\cos\theta dr + v_s(2\frac{1}{2})r\sin\theta d\theta \tag{50}$$

$$nX \simeq -v_s\cos\theta dr + v_s r\sin\theta d\theta \tag{51}$$

Remember that in this case we have the Natario Vector as still being $nX = -vs(t)dx$ with the Shift Vector defined as $X = -vs(t)$.But now we have $\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \neq 0$.The Natario Vector do not vanishes and an external observer would see the Natario Warp Bubble passing by him at a $vs(t) = 200$ two hundred times faster than light due to the Shift Vector $X = -vs(t)$

- C)-In the Warp Bubble walls $0 < n(r) < \frac{1}{2}$

  This is the region where the walls of the Natario Warp Bubble resides.It is not a good idea to place an observer here because the energy needed to distort the Spacetime generating the Warp Drive is placed in this region.The Natario Vector nX would then be:

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \tag{52}$$

Now the reader can understand the point of view outlined by Jose Natario in 2001.The Natario Vector $nX = -vs(t)dx = 0$ vanishes inside the Warp Bubble because inside the Warp Bubble there are no motion at all because $dx = 0$ while being $nX = -vs(t)dx \neq 0$ not vanishing outside the Warp Bubble because an external observer sees the Warp Bubble passing by him with a speed defined by the Shift Vector $X = -vs(t)$ or $X = vs(t)$.

Applying the Extrinsic Curvature Tensor to the Shift Vector(pg 2 and 3 in [2]):

$$K_{ij} = \frac{1}{2}\left(\partial_i X^j + \partial_j X^i\right) \tag{53}$$

and equalizing both covariant scripts we get:

$$K_{ii} = \partial_i X^i \tag{54}$$

This is the Expansion of the Normal Volume Elements as defined in the previous Section(pg 3 in [2]):

$$\theta = \partial_i X^i \tag{55}$$

Redefining the Natario Vector nX as being the Rate-Of-Strain Tensor of Fluid Mechanics as shown below(pg 5 in [2]):

$$nX = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\varphi\mathbf{e}_\varphi \tag{56}$$

Applying the Extrinsic Curvature for the Shift Vectors contained in the Natario Vector nX above we would get the following results(pg 5 in [2]):

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{57}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{58}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{59}$$

$$K_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\theta}{r}\right) + \frac{1}{r}\frac{\partial X^r}{\partial\theta}\right] = v_s\sin\theta\left(n'(r) + \frac{r}{2}n''(r)\right) \tag{60}$$

$$K_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\varphi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial X^r}{\partial\varphi}\right] = 0 \tag{61}$$

$$K_{\theta\varphi} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{X^\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial X^\theta}{\partial\varphi}\right] = 0 \tag{62}$$

Examining the first three results we can clearly see that(pg 5 in [2]):

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{63}$$

The Expansion of the Normal Volume Elements in the Natario Warp Drive is Zero!!!

A Warp Drive With Zero Expansion.

The observer is still immersed in the Warp Bubble and this Bubble is carried out by the Spacetime "stream" at faster than light velocities with the observer at the rest with respect to its local neighborhoods inside the Bubble feeling no g-forces and no accelerations. Imagine an aquarium floating in the course of a river with a fish inside it...the walls of the aquarium are the walls of the Warp Bubble...Imagine that this river is a "rapid" and the aquarium is being carried out by the river stream...the aquarium walls do not expand or contract...an observer in the margin of the river would see the aquarium passing by him at an arbitrarily large speed but inside the aquarium the fish would be protected from g-forces or accelerations generated by the stream...because the fish would be at the rest with respect to its local Spacetime inside the aquarium.The Warp Drive is being carried out by the Spacetime "stream" like a fish in the stream of a river due to the resemblances between the Natario Vector nX and the Rate-Of-Strain Tensor of Fluid Mechanics

This was the main contribution for the Warp Drive science introduced by Jose Natario in 2001.

The second member of the Warp Drive family of Dynamical Spacetimes.

Spacetime Contraction in one direction(radial) is balanced by the Spacetime Expansion in the remaining direction(perpendicular)(pg 5 in [2]).

The Energy Density in the Natario Warp Drive is given by the following expression(pg 5 in [2]):

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(r))^2\cos^2\theta + \left(n'(r) + \frac{r}{2}n''(r)\right)^2\sin^2\theta\right]. \tag{64}$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{dn(r)}{dr})^2 \cos^2\theta + \left(\frac{dn(r)}{dr} + \frac{r}{2} \frac{d^2 n(r)}{dr^2} \right)^2 \sin^2\theta \right]. \qquad (65)$$

This Energy Density is negative and depends of the Natario Shape Function $n(r)$.In order to generate the Warp Drive as a Dynamical Spacetime large outputs of energy are needed but if we have a "good" Natario Shape Function that behaves better than the "bad" or "evil" Natario Shape Functions the derivatives of the "good" Natario Shape Function will be low and the Energy Density will perhaps remains physically attainable.So everything depends on the form or the behavior of the Natario Shape Function.

The Natario Shape Function $n(r)$ did not appeared in the 2001 original Jose Natario IST paper of the "Warp Drive With Zero Expansion"[4].This is the reason why all the current Warp Drive literature still covers the Alcubierre Warp Drive in depth while giving a surface coverage of the Natario Warp Drive.Without knowing the form of the Shape Function $n(r)$ for the Natario Warp Drive we cannot proceed our analysis.We want to change the status situation of the current literature so we will introduce here the Natario Warp Drive Continuous Shape Function $n(r)$

[4]Did not appeared in an explicit algebraic form allowing ourselves to integrate the energy density for example.The original Natario Warp Drive paper of 2001 was conceived as a generic paper on Warp Drives not focusing particular or special cases

4 The Natario Warp Drive Continuous Shape Function $n(rs)$

We already know that the Natario Vector nX in the Warp Bubble walls where $0 <= n(r) <= \frac{1}{2}$ is given by:

$$nX = -v_s(t)d\left[n(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s n(r)\cos\theta dr + v_s(2n(r) + r[\frac{dn(r)}{dr}])r\sin\theta d\theta \quad (66)$$

The Warp Bubble walls are the Natario Warped Region where the Negative Energy resides.This region also have a radius r and a thickness @ just like the Alcubierre Warped Region.

Now look again to the original form of the Natario Vector nX shown below:

$$nX = -v_s(t)dx \quad (67)$$

or

$$nX = v_s(t)dx \quad (68)$$

We already know that dx can be defined by:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \quad (69)$$

But imagine that we can do this:

$$\frac{\partial}{\partial x} \sim dx = d(rs\cos\theta) = \cos\theta drs - rs\sin\theta d\theta \sim rs^2\sin\theta\cos\theta d\theta\wedge d\varphi + rs\sin^2\theta drs\wedge d\varphi = d\left(\frac{1}{2}rs^2\sin^2\theta d\varphi\right) \quad (70)$$

Redefining the Natario Vector nX in function of this new quantity rs we have:

$$nX = -v_s(t)d\left[n(rs)rs^2\sin^2\theta d\varphi\right] \sim -2v_s n(rs)\cos\theta drs + v_s(2n(rs) + r[\frac{dn(rs)}{drs}])rs\sin\theta d\theta \quad (71)$$

Of course we need now to find a continuous expression for $n(rs)$ that is $n(rs) = 0$ inside the Warp Bubble and $n(rs) = \frac{1}{2}$ outside the Warp Bubble while being $0 <= n(rs) <= \frac{1}{2}$ in the Warp Bubble walls

But does this rs looks familiar?????

The answer is simply.yes!!!

Look again to the expressions for rs and $f(rs)$ introduced by Miguel Alcubierre in 1994

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \quad (72)$$

$$f(rs) = \frac{tanh[@(rs + R)] - tanh[@(rs - R)]}{2tanh(@R)} \quad (73)$$

We defined the Natario Shape Function $n(r)$ in function of a radius r small inside the Warp Bubble giving a $n(r) = 0$ and large outside the Warp Bubble giving a $n(r) = \frac{1}{2}$.Now we are using the Alcubierre

rs and why???

Because we need to know the thickness @ of the Warp Bubble:A Warp Bubble of small thickness implies that the behavior $0 <= n(rs) <= \frac{1}{2}$ is spanned over a small region and the derivatives of $n(rs)$ are high or have high values.Or in short the function $n(rs)$ falls rapidly from $\frac{1}{2}$ to 0 in the Natario Warped Region implying in large values for the derivatives affecting the Negative Energy requirements. If we span the Natario Warp Bubble over a large region the derivatives will fall more smooth from $\frac{1}{2}$ to 0 and derivatives that falls slowly or have low values helps the Negative Energy requirements.Remember our idea of a "good" a "bad" or a "evil" Natario Shape Function $n(rs)$.Then we need a large thickness @ for the Natario Warp Bubble.

But we still need to find out the expression for the Natario Continuous Shape Function $n(rs)$ that is $n(rs) = 0$ inside the Warp Bubble and $n(rs) = \frac{1}{2}$ outside the Warp Bubble.

The Natario Continuous Shape Function $n(rs)$ for the Natario Warp Drive is defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{74}$$

$$n(rs) = \frac{1}{2}[1 - \frac{tanh[@(rs + R)] - tanh[@(rs - R)]}{2tanh(@R)}] \tag{75}$$

Does this looks familiar again????The answer is:yes!!!

Lets examine the original Alcubierre Shape Function that is $f(rs) = 1$ inside the Warp Bubble for a small rs while being $f(rs) = 0$ outside the Warp Bubble for a large rs while being $0 <= f(rs) <= 1$ in the Alcubierre Warped Region and its implications for the behavior of the Natario Shape Function $n(rs)$:

- 1)-$f(rs) = 1$-Inside the Alcubierre Warp Bubble

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{76}$$

$$n(rs) = \frac{1}{2}[1 - 1] \tag{77}$$

$$n(rs) = 0 \tag{78}$$

This matches the requirements of the Natario Warp Drive inside the Natario Warp Bubble

- 2)-$f(rs) = 0$-Outside the Alcubierre Warp Bubble

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{79}$$

$$n(rs) = \frac{1}{2}[1 - 0] \tag{80}$$

$$n(rs) = \frac{1}{2} \tag{81}$$

This matches the requirements of the Natario Warp Drive outside the Natario Warp Bubble

- 3)-0 <= $f(rs)$ <= 1-In the Alcubierre Warped Region

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{82}$$

If:

$$f(rs) = \frac{1}{4} \tag{83}$$

Then:

$$n(rs) = \frac{1}{2}[1 - \frac{1}{4}] = \frac{1}{2}[\frac{3}{4}] = \frac{3}{8} = 0,375 \tag{84}$$

$$0 <= n(rs) <= \frac{1}{2} \curvearrowright 0 <= n(rs) <= 0,5 \tag{85}$$

This matches the requirements of the Natario Warp Drive in the Natario Warped Region

Look to this amazing result!!!!:According to our definition of the Natario Shape Function $n(rs)$ and considering the original Alcubierre Shape Function $f(rs)$ when we have $f(rs) = 1$ we will have a $n(rs) = 0$ and when we have a $f(rs) = 0$ we will have a $n(rs) = \frac{1}{2}$

We derived the requirements of the Natario Shape Function $n(rs)$ for the Natario Warp Drive in function of the original Alcubierre 1994 Shape Function $f(rs)$

Is this a coincidence????

No its not!!!

We pointed out before that the Warp Drive was not a "lucky strike" discovered by Miguel Alcubierre in 1994.The Warp Drive is an entire class of Dynamical Spacetimes that are solutions of the Einstein Field Equations of General Relativity like the Wormholes or Black Holes.The Warp Drive family have many members.The Natario Warp Drive although many different from the Alcubierre Warp Drive is a member of the same family.This is the reason why we were able to use the Continuous Shape Function of Alcubierre $f(rs)$ to create the Natario Continuous Shape Function $n/rs)$.

We can now proceed with the study of the cartography of the Natario Warp Drive

5 Energy Density Requirements and the Continuous Shape Function $n(rs)$ for the Natario Warp Drive

We know that the Warp Drive requires an enormous amount of energy in order to be generated and to make the things worst this energy is negative.

The Energy Density for the Natario Warp Drive is given by:

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(n'(rs))^2 \cos^2\theta + \left(n'(rs) + \frac{rs}{2} n''(rs) \right)^2 \sin^2\theta \right]. \quad (86)$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{dn(rs)}{drs})^2 \cos^2\theta + \left(\frac{dn(rs)}{drs} + \frac{rs}{2} \frac{d^2 n(rs)}{drs^2} \right)^2 \sin^2\theta \right]. \quad (87)$$

With the Natario Continuous Shape Function $n(rs)$ being given by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \quad (88)$$

$$n(rs) = \frac{1}{2}[1 - \frac{tanh[@(rs+R)] - tanh[@(rs-R)]}{2tanh(@R)}] \quad (89)$$

Since the square of the Warp Bubble speed vs appears in the Energy Density and since this makes the things even worst we need very low derivatives in order to keep it at physical attainable levels if we want to have a Warp Bubble speed $vs = 200$.two hundred times faster than light.Assuming a constant Warp Bubble radius R and thickness $@$ we have:

$$n'(rs) = \frac{dn(rs)}{drs} \quad (90)$$

$$n'(rs) = -\frac{1}{4tanh(@R)} \frac{d[tanh[@(rs+R)] - tanh[@(rs-R)]]}{drs} \quad (91)$$

$$n''(rs) = \frac{d^2 n(rs)}{drs^2} \quad (92)$$

$$n''(rs) = -\frac{1}{4tanh(@R)} \frac{d^2[tanh[@(rs+R)] - tanh[@(rs-R)]]}{drs^2} \quad (93)$$

Lets examine first the term $-\frac{1}{4tanh(@R)}$

$$tanh(@R) = \frac{sinh(@R)}{cosh(@R)} = \frac{\varepsilon^{@R} - \varepsilon^{-@R}}{\varepsilon^{@R} + \varepsilon^{-@R}} \cong 1 \quad (94)$$

$$coth(@R) = \frac{cosh(@R)}{sinh(@R)} = \frac{\varepsilon^{@R} + \varepsilon^{-@R}}{\varepsilon^{@R} - \varepsilon^{-@R}} \cong 1 \quad (95)$$

The above assumptions are valid for a Warp Bubble radius $R = 100$ meters because $\varepsilon^{-@R} = \frac{1}{\varepsilon^{@R}} \cong 0$ due to the large value of $\varepsilon^{@R}$

Now lets examine the derivatives:

$$\frac{d[tanh[@(rs+R)]}{drs} = \frac{@}{cosh^2[@(rs+R)]} \tag{96}$$

Making:

$$U = rs + R \tag{97}$$

But since:

$$cosh(@U) = \frac{\varepsilon^{@U} + \varepsilon^{-@U}}{2} \tag{98}$$

We can make the following approximation again taking in mind that $\varepsilon^{-@U} = \frac{1}{\varepsilon^{@U}} \cong 0$ due to the large value of $\varepsilon^{@U}$ for a Warp Bubble radius of $R = 100$

$$cosh(@U) = \frac{\varepsilon^{@U}}{2} \tag{99}$$

$$cosh^2(@U) = \frac{\varepsilon^{2@U}}{4} \tag{100}$$

Look again to the derivative:

$$\frac{d[tanh[@(rs+R)]}{drs} = \frac{@}{cosh^2[@(rs+R)]} \tag{101}$$

The term $\frac{\varepsilon^{2@U}}{4}$ is so big because we are raising $2,718281^{2000}$ for a Warp Bubble Radius of $R = 100$ and thickness $@ = 10$ regardless of a small rs inside the Warp Bubble or a large rs outside the Warp Bubble.This number is enormous.Dividing a Warp Bubble thickness $@$ of a size of $@ = 10$ by such an enormous number will make this derivative close to zero.And since we are concerned in the energy of the Warp Bubble walls we focuses the region where rs approaches R rasing of course the power factor.

Lets examine now the second derivative:

$$\frac{d[tanh[@(rs-R)]}{drs} = \frac{@}{cosh^2[@(rs-R)]} \tag{102}$$

Making:

$$V = rs - R \tag{103}$$

But since:

$$cosh(@V) = \frac{\varepsilon^{@V} + \varepsilon^{-@V}}{2} \tag{104}$$

Here we have a curious situation:Inside the Warp Bubble $rs < R$ and V is negative while outside the Warp Bubble $rs > R$ and V is positive.Note that the change of the signs of the power factor reverse the rules of the exponentials but the result is almost similar to the previous case except that in the point where $rs = R$ the derivative is equal to the Warp Bubble thickness @.
Then for the first order derivative $n'(rs)$ of the Natario Shaoe Function $n(rs)$ the term that counts for the integral is:

$$n'(rs) = +\frac{1}{4tanh(@R)}\frac{d[tanh[@(rs-R)]]}{drs} \tag{105}$$

Because we dropped the first hyperbolic tangent derivative keeping the one that also possesses a negative sign.
But we already know that the term $tanh(@R) \cong 1$ .Then we have

$$n'(rs) = +\frac{1}{4}\frac{@}{cosh^2[@(rs-R)]} \tag{106}$$

Note again that this expression have low values when $rs \neq R$ achieving the value of the Warp Bubble thickness @ as a maximum value when $rs = R$.
Remember also that the Energy Density in the Natario Warp Drive uses the square of the derivative of the Natario Shape Function.
Then we must integrate the following expression:

$$(n'(rs))^2 = +\frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]} \tag{107}$$

The integrals of this function are known from tables of integrals of hyperbolic functions and are given by:

$$\int \frac{dx}{cosh^n(ax)} = \frac{sinh(ax)}{a(n-1)cosh^{n-1}(ax)} + \frac{n-2}{n-1}\int \frac{dx}{cosh^{n-2}(ax)} \tag{108}$$

Making $a = @$ and $x = V$ being $V = rs - R$ we have:

$$\int \frac{dV}{cosh^4(@V)} = \frac{sinh(@V)}{3@cosh^3(@V)} + \frac{2}{3}\int \frac{dV}{cosh^2(@V)} \tag{109}$$

$$\int \frac{dV}{cosh^4(@V)} = \frac{8(\varepsilon^{@V} - \varepsilon^{-@V})}{3@(\varepsilon^{@V} + \varepsilon^{-@V})^3} + \frac{2}{3}\int \frac{dV}{cosh^2(@V)} \tag{110}$$

$$\int \frac{dV}{cosh^2(@V)} = \frac{sinh(@V)}{@cosh(@V)} + \int \frac{dV}{cosh(@V)} \tag{111}$$

From the previous approximations we can write the integral as follows with $dV = drs$ since R is constant:

$$\int \frac{dV}{cosh^2(@V)} = \frac{\varepsilon^{@V} - \varepsilon^{-@V}}{@(\varepsilon^{@V} + \varepsilon^{-@V})} + \int \frac{dV}{cosh(@V)} \tag{112}$$

$$\int \frac{dV}{cosh(@V)} = \frac{2}{@}\arctan(\varepsilon^{@V}) \tag{113}$$

Note that the following term

$$\frac{8(\varepsilon^{@V} - \varepsilon^{-@V})}{3@(\varepsilon^{@V} + \varepsilon^{-@V})^3} \tag{114}$$

Have very low values even in the region where $rs < R$ due to the power of 3 vanishing in the region where $rs = R$.
Note also that the following term

$$\frac{\varepsilon^{@V} - \varepsilon^{-@V}}{@(\varepsilon^{@V} + \varepsilon^{-@V})} \tag{115}$$

Also have low values because we are dividing the exponentials by the thickness of the Bubble @ and also vanished in the region where $rs = R$
Note that a hyperbolic sinh in the upper part of the fractions of the terms that are not integrated helps ourselves because in the regions where $rs < R$ or $rs > R$ these terms have extremely low values and these terms vanished in the region where $rs = R$.
So we are left with only this integral

$$\int \frac{dV}{cosh(@V)} = \frac{2}{@} \arctan(\varepsilon^{@V}) \tag{116}$$

And this is a happy ending because arctan is a periodic function.
We lowered the terms in the integral of the Natario Shape Function first order derivative
But we must concern ourselves with the square of the Warp Bubble speed vs that appears in the Natario Warp Drive Energy Density and for a Warp Bubble speed $vs = 200$ two hundred times faster than light this means a $vs = 6 \times 10^{10}$.Its square would then be:

$$vs^2 = 3,6 \times 10^{21} \tag{117}$$

Examining the derivative of second order of the Natario Shape Function:

$$n''(rs) = -\frac{1}{4tanh(@R)} \frac{d^2[tanh[@(rs + R)] - tanh[@(rs - R)]]}{drs^2} \tag{118}$$

We know that we can make the following approach:

$$n''(rs) = -\frac{1}{4} \frac{d^2[tanh[@(rs + R)] - tanh[@(rs - R)]]}{drs^2} \tag{119}$$

Computing the derivatives using $U = rs + R$ and $V = rs - R$ we would get the following result:

$$n''(rs) = -\frac{1}{4}[-2@^2 \frac{sinh(@U)}{cosh^3(@U)} + 2@^2 \frac{sinh(@V)}{cosh^3(@V)}] \tag{120}$$

From the approximations we developed before we can see that this term is so low that almost vanishes due to the powers of $\varepsilon^{3@U}$ or $\varepsilon^{3@V}$.Remember that our study is a first order approximation and remember also that $R = 100$,$@ = 10$ so regardless of the value of rs that will make the term $@V$ vanish completely when $rs = R$ we are raising ε to a power of 3000.So by now we will neglect the derivative of second order of the Natario Shape Function.

From the Natario Warp Drive Energy Density we can make this approach considering only the first derivative of the Natario Shape Function $n(rs)$:

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs)\right)^2\sin^2\theta\right]. \qquad (121)$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(\frac{dn(rs)}{drs})^2\cos^2\theta + \left(\frac{dn(rs)}{drs}\right)^2\sin^2\theta\right]. \qquad (122)$$

$$(n'(rs))^2 = +\frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]} \qquad (123)$$

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\frac{3}{16}\frac{@^2}{cosh^4[@(rs-R)]}\cos^2\theta + \frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]}\sin^2\theta\right]. \qquad (124)$$

Note that in the direction parallel to the motion(the front of the ship) $\cos\theta = 1$ and $\sin\theta = 0$ so in front of the ship the energy is approximately

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\frac{3}{16}\frac{@^2}{cosh^4[@(rs-R)]}\cos^2\theta\right]. \qquad (125)$$

This is very important we have Negative Energy Density in front of the ship.More on this in the next section.
On the other hand in the direction perpendicular to the motion(above the ship) $\cos\theta = 0$ and $\sin\theta = 1$ so above the ship the energy is approximately

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[\frac{1}{16}\frac{@^2}{cosh^4[@(rs-R)]}\sin^2\theta\right]. \qquad (126)$$

So we can see that the ship is completely covered by the Negatrive Energy of Warp Bubble in the Natario Warp Drive that is being carried out by the Spacetime Stream.Just like a fish inside an aquarium and the aquarium is floating and being carried away by the stream of a river

6 Conclusion

Almost all the existing literature covers the Alcubierre Warp Drive and almost all the literature points out some problems related to the Warp Drive as a Dynamical Spacetime.We kept ourselves with only the original papers of Alcubierre and Natario[5].The Causally Disconnected portions of spacetime also known as The Horizon Problem:It is true that the observer inside the Warp Bubble wether in an Alcubierre or Natario Warp Drive cannot send signals to the front at light speed.The Horizons Problem must be postponed to a more advanced theory that encompasses both General Relativity and Quantum Mechanics specially the Non-Local effects of the Quantum Entanglements eg Einstein-Podolsky-Rosen Paradox or the experiments of Raymond Chiao or Alain Aspect with Superluminal Instantaneous Communication between pairs of polarized photons at a great distance from each other.

The Negative Energy Problem:It is true that we can create small amounts of Negative Energy by the Casimir Effect as noted by Alcubierre himself while the Negative Energy Requirements for a Warp Drive are large specially considering the square of the Warp Bubble Speed that appears in the equations.We need to discover new ways to create Negative Energy again perhaps with a new theory of Quantum Gravity

The Doppler Blueshift Problem:Space is not empty.It is fulfilled with the photons of Cosmic Background Radiation and a Warp Bubble at two hundred times faster than light would impact some of these Doppler Blueshifted to the wavelengths of Synchrotron Radiation which is lethal. Note that Natario Warp Drive have Negative Energy covering all the ship and Negative Energy means a repulsive Spacetime Curvature of General Relativity that could deflect the photons.

We feel that the Natario Warp Drive deserves more coverage in the existing literature.It was the non-existence of a Continuous Shape Function that avoided the coverage of the Natario Warp Drive by the literature.Without a Shape Function the Natario Warp Drive cannot be described.We introduced here the Natario Continuous Shape Function.We hope to change the situation.

To terminate:The Warp Drive as a Dynamical Spacetime was not a "lucky strike" discovered by Alcubierre in 1994.The Warp Drive is an entire family of solutions of the Einstein Field Equations of General Relativity and we feel that the study of the unknown members of this family can be very interesting.This was the main contribution to the Warp Drive Science given by Jose Natario in 2001

[5]References exclusively devoted to the Warp Drive.We have other references on the Hodge Star and Differential Forms

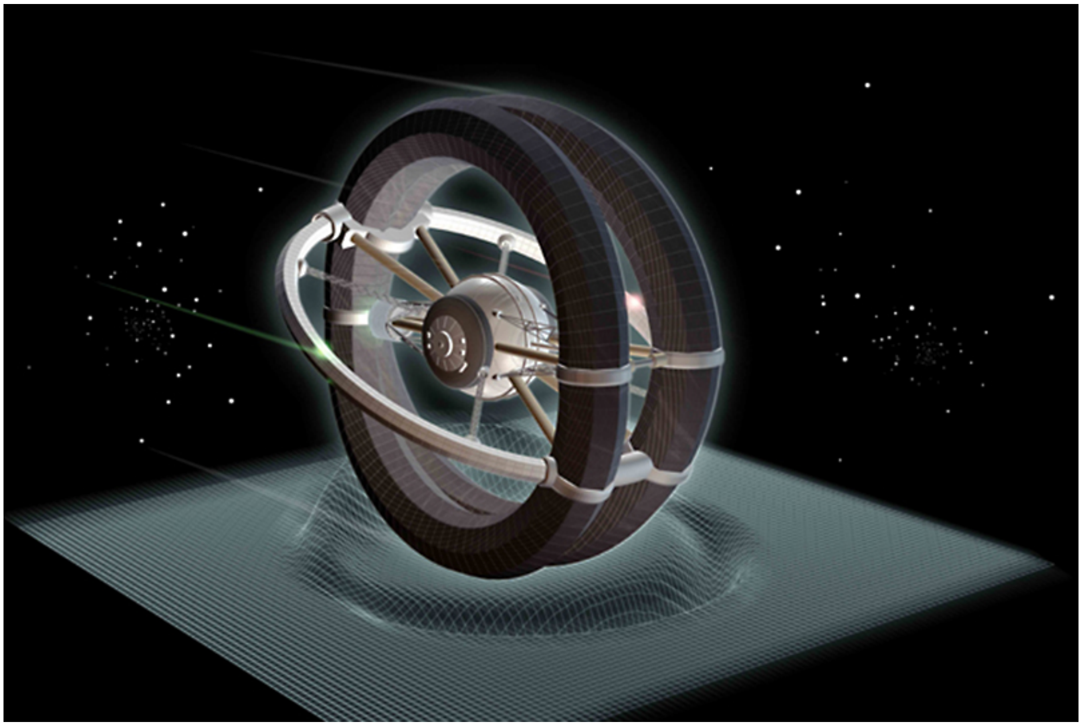

Figure 1: Artistic representation of the Natario Warp Drive Bubble . (Source:Internet)

7 Artistic Graphical Presentation of the Natario Warp Drive

Note that according to the geometry of the Natario Warp Drive the Spacetime Contraction in one direction(radial) is balanced by the Spacetime Expansion in the remaining direction(perpendicular)(pg 5 in [2]). Remember also that the Expansion of the Normal Volume Elements in the Natario Warp Drive is given by the following expression(pg 5 in [2]). :

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{127}$$

If we expand the radial direction the perpendicular direction contracts to keep the Expansion of the Normal Volume Elements equal to zero.
This figure is a pedagogical example of the graphical presentarion of the Natario Warp Drive.The "metal bars" in the figure actually do not exists but were included to illustrate how the expansion in one direction can be counter-balanced by the contraction in the other directions.These "metal bars" keeps the Expansion of the Normal Volume Elements in the Natario Warp Drive equal to zero.
Note also that the graphical presentation of the Alcubierre Warp Drive according to fig 1 pg 10 in [1] is also included

8 Appendix:Differential Forms,Hodge Star and the Natario Vector nX

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{128}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (d\varphi \wedge dr) \tag{129}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{130}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{131}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{132}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{133}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{134}$$

$$*rd\theta = r\sin\theta (d\varphi \wedge dr) \tag{135}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{136}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2} r^2 \sin^2\theta d\varphi\right) \tag{137}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{138}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{139}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{140}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{141}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{142}$$

We know that the following expression holds true(see pg 9 in [3]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{143}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{144}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{145}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{146}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{147}$$

According to pg 10 in [3] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{148}$$

Because and according to pg 10 in [3]:

$$d(\alpha+\beta) = d\alpha + d\beta \tag{149}$$

$$d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{150}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{151}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{152}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{153}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{154}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{155}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{156}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{157}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{158}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta)\wedge(r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{159}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi)\wedge dr \sim r\sin\theta(d\varphi\wedge dr) \sim -r\sin\theta(dr\wedge d\varphi) \tag{160}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{161}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{162}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{163}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{164}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{165}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{166}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{167}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{168}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{169}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{170}$$

9 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[6]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[7][8]

[6] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[7] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[8] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

10 Remarks

References 3 and 4 were taken from Internet although not from a regular available site like the one of references 1 and 2.We can provide the Adobe PDF Acrobat Reader for these references for those interested.

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[4] *Teaching Electromagnetic Field Theory Using Differential Forms,Warnick K.F. Selfridge R. H.,Arnold D. V.,IEEE Transactions On Education Vol* 40 *Num* 1 *Feb* 1997

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