On Nonlinear Quantum Mechanics, Noncommutative Phase Spaces, Fractal-Scale Calculus and Vacuum Energy

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Abstract

A novel (to our knowledge) Generalized Nonlinear Schrödinger equation based on the modifications of Nottale-Cresson's fractal-scale calculus and resulting from the *noncommutativity* of the phase space coordinates is explicitly derived. The modifications to the ground state energy of a harmonic oscillator yields the observed value of the vacuum energy density. In the concluding remarks we discuss how *nonlinear* and *nonlocal* QM wave equations arise naturally from this fractal-scale calculus formalism which may have a key role in the final formulation of Quantum Gravity.

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1 Introduction : QM and Fractal Scale Calculus

Over the years there has been a considerable debate as to whether linear QM can fully describe Quantum Chaos. Despite that the quantum counterparts of classical chaotic systems have been studied via the techniques of linear QM, it is our opinion that Quantum Chaos is truly a new paradigm in physics which is associated with non-unitary and nonlinear QM processes based on non-Hermitian operators (implementing time symmetry breaking). This Quantum Chaotic behavior should be linked more directly to the Nonlinear Schrödinger equation without any reference to the nonlinear behavior of the classical limit. For this reason we analyzed in detail the fractal geometrical features underlying a Nonlinear Schrödinger equation (NLSE) obtained in [15] and that had a diferent form than the nonlinear wave equations of [10], [5], [12].

Nonlinear QM has a practical importance in different fields, like condensed matter, quantum optics and atomic and molecular physics; even quantum gravity may involve nonlinear QM. Another important example is in the modern field of quantum computing. If quantum states exhibit small nonlinearities during their temporal evolution, then quantum computers can be used to solve NP-complete (non polynomial) and #P problems in polynomial time. Abrams and Lloyd [9] proposed logical gates based on non linear Schrödinger equations and suggested that a further step in quantum computing consists in finding physical systems whose evolution is amenable to be described by a NLSE.

On other hand, we consider that Nottale and Ord's formulation of quantum mechanics [1], [2] from first principles based on the combination of scale relativity and fractal space-time is a very promising field of future research. In [15] we extended Nottale and Ord's ideas to derive a nonlinear Schrödinger equation and described the explicit interplay between Fisher Information, Weyl geometry and the Bohm's potential [11] by introducing an action based on a *complex* momentum. The relationship between Bohm's Quantum Potential and the Weyl curvature scalar of the Statistical ensemble of particle-paths (an Abelian fluid) associated to a single particle was initially developed by [35]. A Weyl geometric formulation of the Dirac equation and the nonlinear Klein-Gordon wave equation was provided by [36].

The construction of the NLSE was based on a fractal Brownian motion with a complex diffusion constant. We followed very closely Nottale's derivation of the ordinary Scrödinger equation [1]. Nottale and Celerier [1] following similar methods were able to derive the Dirac equation using bi-quaternions (complex quaternions) and after breaking the parity symmetry $dx^\mu \leftrightarrow -dx^\mu$. Adler has developed a quaternionic QM [6]. Octonionic QM could be the underlying feature behind the recently established relationships between black hole entropy in string theory and the quantum entanglement of qubits and qutrits in quantum information theory [7].

For simplicity, the one-particle case is investigated but the derivation can be extended to many-particle systems. In this approach particles do not follow smooth trajectories but fractal ones, that can be described by a continuous but non-differentiable fractal function $\vec{r}(t)$. The time variable is divided into infinitesimal intervals dt which can be taken as a given scale of the resolution. Then, following the definitions given by Nelson in his stochastic QM approach, Nottale defined the mean backward an forward derivatives as follows,

$$\frac{d_\pm \vec{r}(t)}{dt} = \lim_{\Delta t \to \pm 0} \left\langle \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \right\rangle, \tag{1.1}$$

from which the forward and backward mean velocities are obtained,

$$\frac{d_\pm \vec{r}(t)}{dt} = \vec{b}_\pm. \tag{1.2}$$

For the deduction of the Schrödinger equation from fractal space-time classical mechanics, Nottale starts by defining the complex-time derivative operator

$$\frac{\delta}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - i\frac{1}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right), \tag{1.3}$$

which after some straightforward definitions and transformations takes the following form,

$$\frac{\delta}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla} - i\mathcal{D}\nabla^2. \tag{1.4}$$

$\mathcal{D}$ is a real-valued diffusion constant which is related to the Planck constant. The diffusion constant stems from considering that the scale dependent part of the velocity is a Gaussian stochastic variable with zero mean,

$$\langle d\xi_{\pm i} d\xi_{\pm j}\rangle = \pm 2\mathcal{D}\delta_{ij}dt. \tag{1.5}$$

Afterwards, Nottale defined a set of complex quantities which are generalization of well known classical quantities (Lagrange action, velocity, momentum, etc), in order to be compatible with the introduction of a complex-time derivative operator. The *complex* time-dependent wave function Ψ is expressed in terms of a a complex Lagrange action S as $\Psi = e^{iS/(2m\mathcal{D})}$. S is a complex-valued action. The velocity is related to the momentum, which can be expressed as the gradient of S, $\vec{p} = \vec{\nabla}S$. Then the following known relation is found $\vec{V} = -2i\mathcal{D}\vec{\nabla}\ln\Psi$.

Finally, the Schrödinger equation is obtained from Newton's equation (force = mass times acceleration) by using the expression of $\vec{V}$ in terms of the wave function Ψ,

$$-\vec{\nabla}U \;=\; m\frac{\delta}{dt}\vec{V} \;=\; -\,2im\mathcal{D}\frac{\delta}{dt}\vec{\nabla}\ln\psi. \tag{1.6}$$

Replacing the complex-time derivative operator in Newton's equation gives

$$-\vec{\nabla}U \;=\; -\,2im\left(\mathcal{D}\frac{\partial}{\partial t}\vec{\nabla}\ln\psi\right) \;-\,2\mathcal{D}\vec{\nabla}\left(\mathcal{D}\frac{\nabla^2\psi}{\psi}\right). \tag{1.7}$$

Using the three identities (i): $\vec{\nabla}\nabla^2 = \nabla^2\vec{\nabla}$; (ii): $2(\vec{\nabla}\ln\Psi\cdot\vec{\nabla})(\vec{\nabla}\ln\Psi) = \vec{\nabla}(\vec{\nabla}\ln\Psi)^2$; and (iii): $\nabla^2\ln\Psi = (\nabla^2\Psi/\Psi) - (\vec{\nabla}\ln\Psi)^2$ allows us to integrate such equation above (1.7) yielding after some straightforward algebra

$$\mathcal{D}^2\nabla^2\Psi \;+\; i\mathcal{D}\frac{\partial\Psi}{\partial t} \;-\; \frac{U}{2m}\Psi = 0 \tag{1.8}$$

up to an arbitrary phase factor which may set to zero. Now replacing $\mathcal{D}$ by $\hbar/(2m)$, one arrives at the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} \;=\; -\,\frac{\hbar^2}{2m}\nabla^2\Psi \;+\; U\Psi. \tag{1.9}$$

The Hamiltonian operator is Hermitian, this equation is linear and clearly is homogeneous of degree one under the substitution $\psi \to \lambda\psi$.

One can generalize (1.9) by relaxing the assumption that the diffusion constant is real and using a complex-valued diffusion constant; i.e. a complex-valued $\hbar$. The reader may be biased against such approach because the Hamiltonian ceases to be Hermitian and the energy becomes complex-valued. However this is not always the case. We found plane wave solutions and soliton solutions to the nonlinear and non-Hermitian wave equations that possess real energies and momenta [15]. For a detailed discussion on complex-valued spectral representations in the formulation of quantum chaos and time-symmetry breaking see [4]. Nottale's derivation of the Schrödinger equation required a complex-valued action S stemming from the complex-valued velocities due to the breakdown of symmetry between the forwards and backwards velocities in the fractal zigzagging. If the action S was complex then it is not farfetched to have a complex diffusion constant and consequently a complex-valued $\hbar$ (with same units as the complex-valued action).

Complex energies is not alien in ordinary linear QM. They appear in optical potentials (complex) usually invoked to model the absorption in scattering processes [5] and decay of unstable particles. Complex potentials have also been used to describe decoherence. The accepted way to describe resonant states in atomic and molecular physics is based on the complex scaling approach, which in a natural way deals with complex energies [8]. Before one had

$$\langle d\zeta_\pm d\zeta_\pm \rangle = \pm 2\mathcal{D} dt \tag{1.10}$$

with $\mathcal{D}$ and $2m\mathcal{D} = \hbar$ real. Now one sets

$$\langle d\zeta_\pm d\zeta_\pm \rangle = \pm (\mathcal{D} + \mathcal{D}^*) dt, \tag{1.11}$$

with $\mathcal{D}$ and $2m\mathcal{D} = \hbar = \alpha + i\beta$ complex. The complex-time derivative operator becomes now

$$\frac{\delta}{dt} \;=\; \frac{\partial}{\partial t} \;+\; \vec{V} \cdot \vec{\nabla} \;-\; \frac{i}{2}(\mathcal{D} + \mathcal{D}^*)\nabla^2. \tag{1.12}$$

In the real case $\mathcal{D} = \mathcal{D}^*$, it reduces to the complex-time-derivative operator described previously by Nottale. Writing again the Ψ in terms of the complex action S,

$$\Psi = e^{iS/(2m\mathcal{D})} = e^{iS/\hbar}, \tag{1.13}$$

where S, $\mathcal{D}$ and $\hbar$ are complex-valued, the complex velocity is obtained from the complex momentum $\vec{p} = \vec{\nabla} S$ as $\vec{V} = -2i\mathcal{D}\vec{\nabla} \ln \Psi$. The NLSE is finally obtained after using the generalized Newton's equation (force = mass times acceleration) in terms of Ψ

$$-\vec{\nabla} U \;=\; m\frac{\delta}{dt}\vec{V} = -2im\mathcal{D}\frac{\delta}{dt}\vec{\nabla} \ln \Psi. \tag{1.14}$$

Replacing the complex-time derivative operator in the generalized Newton's equation leads to

$$\vec{\nabla} U \;=\; 2im \left[\mathcal{D}\frac{\partial}{\partial t}\vec{\nabla} \ln \Psi - 2i\mathcal{D}^2(\vec{\nabla} \ln \Psi \cdot \vec{\nabla})(\vec{\nabla} \ln \Psi) - \frac{i}{2}(\mathcal{D} + \mathcal{D}^*)\mathcal{D}\nabla^2(\vec{\nabla} \ln \Psi) \right]. \tag{1.15}$$

Using again the three identities (i): $\vec{\nabla}\nabla^2 = \nabla^2\vec{\nabla}$; (ii): $2(\vec{\nabla}\ln\Psi\cdot\vec{\nabla})(\vec{\nabla}\ln\Psi) = \vec{\nabla}(\vec{\nabla}\ln\Psi)^2$; and (iii): $\nabla^2\ln\Psi = (\nabla^2\Psi/\Psi) - (\vec{\nabla}\ln\Psi)^2$ allows us to integrate such equation above yielding, after some straightforward algebra, the NLSE

$$i\hbar\frac{\partial\Psi}{\partial t} \;=\; -\,\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\nabla^2\Psi \;+\; U\Psi \;-\; i\frac{\hbar^2}{2m}\frac{\beta}{\hbar}\left(\vec{\nabla}\ln\Psi\right)^2\Psi. \tag{1.16}$$

Note the crucial minus sign in front of the kinematic pressure term and that $\hbar = \alpha + i\beta = 2m\mathcal{D}$ is complex. When $\beta = 0$ we recover the linear Schrödinger equation.

The nonlinear potential is now complex-valued in general. Defining

$$W = W(\psi) = -\frac{\hbar^2}{2m}\frac{\beta}{\hbar}\left(\vec{\nabla}\ln\psi\right)^2, \tag{1.17}$$

and U the ordinary potential, then the NLSE can be rewritten as

$$i\hbar\frac{\partial\Psi}{\partial t} \;=\; \left(-\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\nabla^2 \;+\; U \;+\; i\,W\right)\;\Psi. \tag{1.18}$$

This is the nonlinear wave equation found in [15]. It has the form of the ordinary Schrödinger equation with the complex potential $U + iW$ and the complex $\hbar$. The Hamiltonian is no longer Hermitian and the potential $V = U + iW(\psi)$ itself depends on ψ. Nevertheless one had meaningful physical solutions with real valued energies and momenta, like the plane-wave and soliton solutions found in [15]. It was also realized in [15] that the NLSE above *cannot* be obtained by a *naive* scaling of the wavefunction

$$\Psi = e^{iS/\hbar_o} \rightarrow \psi' = e^{iS/\hbar} = e^{(iS/\hbar_o)(\hbar_o/\hbar)} = \Psi^\lambda = \psi^{\hbar_o/\hbar}. \quad \hbar = real. \tag{1.19}$$

related to a scaling of the diffusion constant $\hbar_o = 2m\mathcal{D}_o \rightarrow \hbar = 2m\mathcal{D}$. Upon performing such scaling, the ordinary *linear* Schrödinger equation in the variable Ψ will appear to be *nonlinear* in the new scaled wavefunction Ψ'. The NLSE based on a fractal Brownian motion with a complex valued diffusion constant $2mD = \hbar = \alpha + i\beta$ represents truly a new physical phenomenon and a hallmark of nonlinearity in QM. For other generalizations of QM and experimental tests of quaternionic QM see [6].

A Fractal Scale Calculus description of the NLSE was developed later on by Cresson [16] who obtained on a rigorous mathematical footing the same functional form of our NLSE equation above (although with different complex numerical coefficients) by using an extension of Nottale's fractal scale-calculus and which obeyed a quantum Hopf bi-algebra. Hence, a fractal spacetime is deeply ingrained with *nonlinear* wave equations as we have shown and it was later corroborated by Cresson [16].

Complex-valued viscosity solutions to the Navier-Stokes equations were also analyzed by Nottale leading to the Fokker-Planck equation. Clifford-valued extensions of QM were studied in [13] C-spaces (Clifford-spaces whose enlarged coordinates are polyvectors, i.e antisymmetric tensors) that involved a Clifford-valued number extension of Planck's constant; i.e. the Planck constant was a hypercomplex number. Modified dispersion relations were derived from the underlying QM in Clifford-spaces that lead to faster than light propagation in ordinary spacetime but without violating causality in the more fundamental Clifford spaces. Therefore, one should not exclude the possibility of having complex-extensions of the Planck constant leading to nonlinear wave equations associated with the Brownian motion of a particle in fractal spacetimes.

Notice that the NLSE obeys the homogeneity condition $\Psi \rightarrow \lambda\Psi$ for any constant λ. All the terms in the NLSE are scaled respectively by a factor λ. Moreover, the two parameters α, β are intrinsically connected to a complex Planck constant $\hbar = \alpha + i\beta$ such that $||\hbar|| = \sqrt{\alpha^2 + \beta^2} = \hbar_o$ (observed Planck's constant) rather that being *ah-hoc* constants to be determined experimentally. Thus, the nonlinear QM equation derived from the fractal Brownian motion with complex-valued diffusion coefficient is intrinsically tied up with a non-Hermitian Hamiltonian and with complex-valued energy spectra [4]. Despite having a non-Hermitian Hamiltonian one still could have eigenfunctions with real valued energies and momenta. Non-Hermitian Hamiltonians (pseudo-Hermitian) have captured a lot of interest lately in the so-called PT symmetric *complex* extensions of QM and QFT [37]. Therefore these ideas cannot be ruled out and they are the subject of active investigation nowadays.

2 Generalized Nonlinear Schrödinger equation from Noncommutative Phase Spaces

Noncommutative (exotic)mechanics [17], [18], [19], [20] models have the distinctive feature that the Poisson bracket of the planar coordinates does not vanish, $\{x_1, x_2\} = \theta_{12}$. The physical origin of exotic mechanics and its quantum mechanical counterpart was found soon after its introduction [27] : it is a sort of non-relativistic shadow of (fractional) spin [21], [22], [23]. Such particles can be interpreted as nonrelativistic anyons [24], [25]. The supersymmetric extension of the theory was recently outlined in [27].

Remarkably, similar structures were considered, independently and around the same time, in condensed matter physics, namely for the Bloch electron [26]. These, 3-space dimensional, models are also noncommutative, but the noncommutative parameter is now promoted to a vector-valued function of the quasi-momentum. Exotic Galilean symmetry, strictly linked to two space dimensions, is lost. However, a rich Poisson structure and an intricate interplay with external magnetic fields can be studied. Further developments include the anomalous/spin/optical Hall effects. We refer the reader to the most recent review on Exotic Galilean Symmetry (found also in Moyal field theory) and Noncommutative Mechanics with an extensive list of references provided by [27].

In this section we will derive the *novel* (to our knowledge) Generalized Nonlinear Schrödinger equation based on the modifications of the Nottale-Cresson fractal calculus due to the noncommutativity of the phase space coordinates. Let us begin with the deformed Weyl-Heisenberg algebra

$$[\hat{x}^i,\ \hat{p}^j]\ =\ i\ \hbar_{eff}\ \delta^{ij};\quad [\hat{x}^i,\ \hat{x}^j]\ =\ \tilde{\Theta}^{ij};\quad [\hat{p}^i,\ \hat{p}^j]\ =\ \Theta^{ij}. \tag{2.1}$$

$\tilde{\Theta}^{ij}, \Theta^{ij}$ are *real* antisymmetric tensors (matrices) with constant entries; i.e. they are anti-Hermitian matrices since the commutator of two Hermitian (or anti-Hermitian) operators like $\hat{x}, \hat{p}$ is anti-Hermitian. For this reason there is *no* need to introduce i factors in front of the $\Theta^{ij}, \tilde{\Theta}^{ij}$ terms in eq-(2.1). Θ^{ij} has dimensions of $(momentum)^2$, and $\tilde{\Theta}^{ij}$ has dimensions of $(length)^2$. Afterwards we shall discuss the case when the latter tensors are x, p-dependent. The effective Planck constant is given by

$$\hbar_{eff}\ \equiv\ \hbar\ \xi\ \equiv\ \hbar\ (1\ +\ \frac{\tilde{\Theta}^{ij}\Theta^{ji}}{4d\hbar^2}) \tag{2.2}$$

where the dimension $d \geq 2$. By inspection one can verify that the noncommutative coordinates and momenta operators can be recast

$$\hat{p}^i\ =\ (\hat{\pi}^i\ -\ \frac{i}{2\hbar}\Theta^{ij}\ \hat{q}_j);\quad \hat{x}^i\ =\ (\hat{q}^i\ +\ \frac{i}{2\hbar}\tilde{\Theta}^{ij}\ \hat{\pi}_j). \tag{2.3}$$

in terms of the operators $\hat{\pi}^i, \hat{q}^j$ that obey the standard Weyl-Heisenberg algebra

$$[\hat{q}^i,\ \hat{\pi}^j]\ =\ i\ \hbar\ \delta^{ij};\quad [\hat{q}^i,\ \hat{q}^j]\ =\ 0;\quad [\hat{\pi}^i,\ \hat{\pi}^j]\ =\ 0. \tag{2.4}$$

such that the expressions for $\hat{p}^i, \hat{x}^i$ (2.3) lead to the modified commutation relations in eq-(2.1) associated with the deformed Weyl-Heisenberg algebra.

The classical limit of the deformed Weyl-Heisenberg algebra in eqs-(2.1) is obtained by replacing the operators $\hat{x}^i, \hat{p}^i$ for the classical variables x^i, p^i, and the commutators divided by $i\hbar$ for the following modified Poisson brackets

$$\{x^i,\ p^j\}\ =\ \xi\ \delta^{ij};\ \xi \equiv (1 + \frac{\tilde{\Theta}^{ij}\Theta^{ji}}{4d\hbar^2});\ \{x^i,\ x^j\}\ =\ -i\ \frac{\tilde{\Theta}^{ij}}{\hbar};\ \{p^i,\ p^j\}\ =\ -i\ \frac{\Theta^{ij}}{\hbar}. \tag{2.5}$$

The operator-valued variables in eq-(2.3) correspond to the following classical noncommutative phase space coordinates

$$p^i\ =\ \pi^i\ -\ \frac{i}{2\hbar}\Theta^{ij}\ q_j;\quad x^i\ =\ q^i\ +\ \frac{i}{2\hbar}\tilde{\Theta}^{ij}\ \pi_j. \tag{2.6}$$

that are written in terms of the π^i, q^j phase space coordinates which obey the standard Poisson-bracket algebra

$$\{q^i,\ \pi^j\}\ =\ \delta^{ij},\quad \{q^i,\ q^j\}\ =\ 0;\quad \{\pi^i,\ \pi^j\}\ =\ 0. \tag{2.7}$$

such that eqs-(2.6, 2.7) lead to the modified Poisson brackets in eq-(2.5). In particular, the modified Poisson brackets $\{A(x,p), B(x,p)\}$ of two functions of x, p read now :

$$\{A(x,p), B(x,p)\} \;=\; (\partial_{x^i} A)\;\{x^i, p^j\}\;(\partial_{p^j} B) \;+\; (\partial_{p^i} A)\;\{p^i, x^j\}\;(\partial_{x^j} B) \;+$$
$$(\partial_{x^i} A)\;\{x^i, x^j\}\;(\partial_{x^j} B) \;+\; (\partial_{p^i} A)\;\{p^i, p^j\}\;(\partial_{p^j} B) \;=$$
$$(\partial_{x^i} A)\,\xi\delta^{ij}\,(\partial_{p^j} B) - (\partial_{p^i} A)\,\xi\delta^{ij}\,(\partial_{x^j} B) - (\partial_{x^i} A)\,i\frac{\tilde{\Theta}^{ij}}{\hbar}\,(\partial_{x^j} B) - (\partial_{p^i} A)\,i\frac{\Theta^{ij}}{\hbar}\,(\partial_{p^j} B). \tag{2.8}$$

If the coordinates and momenta were commuting variables the modified brackets will reduce to standard Poisson brackets .

The *modified* dynamical equations of motion (in natural units of $c = 1$) are

$$\frac{dx^j}{dt} \;=\; -\{H, x^j\} \;=\; -\frac{\partial H}{\partial p^i}\{p^i, x^j\} \;-\; \frac{\partial H}{\partial x^i}\{x^i, x^j\} \;=$$
$$\xi\;\frac{\partial H}{\partial p_j}\;+ i\;\frac{\tilde{\Theta}^{ij}}{\hbar}\frac{\partial H}{\partial x^i}. \tag{2.9}$$

$$\frac{dp^j}{dt} \;=\; -\{H, p^j\} \;=\; -\frac{\partial H}{\partial x^i}\{x^i, p^j\} \;-\; \frac{\partial H}{\partial p^i}\{p^i, p^j\} \;=$$
$$-\xi\;\frac{\partial H}{\partial x_j}\;+ i\;\frac{\Theta^{ij}}{\hbar}\frac{\partial H}{\partial p^i} \tag{2.10}$$

Given a (nonrelativistic) Hamiltonian $H = \frac{1}{2m}p_j p^j + U(x^i)$ in d-dimensions the modified equations of motion are

$$\frac{dx^j}{dt} \;=\; \xi\;\frac{p^j}{m}\;+ i\;\frac{\tilde{\Theta}^{ij}}{\hbar}\;\frac{\partial U(x)}{\partial x^i}. \qquad ()$$

$$\frac{dp^j}{dt} \;=\; -\,\xi\;\frac{\partial U(x)}{\partial x_j}\;+ i\;\frac{\Theta^{ij}}{\hbar}\;\frac{p_i}{m} \tag{2.11}$$

from which one can infer that the modified velocity v^j (no longer equal to $\frac{p^j}{m}$) is equal to $\xi\frac{p^j}{m} + \frac{i}{\hbar}\tilde{\Theta}^{ij}(\partial U/\partial x^i)$ and the modified force (no longer equal to $-(\partial U/\partial x^i)$) is equal to $-\xi(\partial U/\partial x^i) + \frac{i}{\hbar}\Theta^{ij}\frac{p_i}{m}$.

As a result of a modified velocity and force, the modified scale derivative operator is now given by

$$\frac{\delta}{dt} \;=\; \frac{\partial}{\partial t} \;+\; (\;\xi\frac{p^j}{m}\;+ i\;\frac{\tilde{\Theta}^{ij}}{\hbar}\frac{\partial U(x)}{\partial x^i}\;)\;\nabla_j \;-\; i\mathcal{D}_{eff}\;\nabla^2. \tag{2.12}$$

and the modified Newtonian law is

$$\frac{\delta p^k}{dt} \;=\; \left(\;\frac{\partial}{\partial t} \;+\; (\;\xi\frac{p^j}{m}\;+ i\;\frac{\tilde{\Theta}^{ij}}{\hbar}\frac{\partial U(x)}{\partial x^i}\;)\;\nabla_j \;-\; i\mathcal{D}_{eff}\nabla^2\right)\;p^k \;=$$

$$- \xi \ \frac{\partial U(x)}{\partial x_k} \ + i \ \frac{\Theta^{ik}}{\hbar} \ \frac{p_i}{m} \tag{2.13}$$

Writing again the Ψ in terms of the complex action S and a *modified* Planck constant $\hbar_{eff}$, which is related to the modified diffusion constant $\mathcal{D}_{eff} = \hbar_{eff}/2m$ as $\Psi = e^{iS/(2mD_{eff})} = e^{iS/\hbar_{eff}}$, the modified complex momentum is $\vec{p} = \vec{\nabla} S = -2im\mathcal{D}_{eff} \vec{\nabla} \ln \Psi = -i\hbar_{eff} \vec{\nabla} \ln \Psi$. Therefore, the modified Newtonian law, upon inserting $\vec{p} = \vec{\nabla} S = -i\hbar_{eff} \vec{\nabla} \ln \Psi$ and $\hbar_{eff}/\hbar = \xi$, after some straightforward algebra leads then to the *third* order vector-valued differential equation

$$i\hbar_{eff} \nabla^i (\frac{1}{\Psi} \frac{\partial \Psi}{\partial t}) \ = \ - \ \frac{\hbar^2_{eff}}{2m} \ \nabla^i [\ (\frac{\nabla^2 \Psi}{\Psi}) \ + \ (\xi - 1)(\nabla ln \Psi)^2 \ ] \ + \xi \ \nabla^i U \ -$$

$$\xi \frac{\Theta^{ij}}{m} \ \nabla_j (ln \Psi) \ + \ \xi \tilde{\Theta}^{kj} (\nabla_j U)(\nabla_k \nabla^i ln \Psi); \quad i = 1, 2, 3, ..., d. \tag{2.14}$$

An important remark is in order about the above equation (2.14). We have taken the ordinary point products of functions $(\nabla ln \Psi)(\nabla ln \Psi)$ instead of the star products $(\nabla ln \Psi) * (\nabla ln \Psi)$. For instance, if one has $\Theta^{ij} = 0$ and $\tilde{\Theta}^{ij} = constants \neq 0$ one can define the noncommutative but associative star product

$$\Psi(x) \ * \ \Psi(x) \ = \ \Psi(x) \ (\ e^{\frac{1}{2} \overleftarrow{\partial}_{x^i} \ \tilde{\Theta}^{ij} \ \overrightarrow{\partial}_{x^j}} \ ) \ \Psi(x) \ =$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} \ \tilde{\Theta}^{i_1 j_1} \ \tilde{\Theta}^{i_2 j_2} \ .... \ \tilde{\Theta}^{i_n j_n} \ \frac{\partial^n \Psi(x)}{\partial x^{i_1} \partial x^{i_2} ..... \partial x^{i_n}} \ \frac{\partial^n \Psi(x)}{\partial x^{j_1} \partial x^{j_2} ..... \partial x^{j_n}} \ . \tag{2.15}$$

Therefore one could have modified eq-(2.14) by replacing ordinary products for star products. For example, $x^i * x^j = x^i x^j + \frac{\tilde{\Theta}^{ij}}{2} \Rightarrow x^i * x^j - x^j * x^i = \tilde{\Theta}^{ij}$. Even further, one could have modified the exponential function (and the logarithm function) by their star-deformed counterparts based in replacing ordinary products in the Taylor series expansion by star products : $e^S_*(x) = 1 + S + \frac{1}{2} S(x) * S(x) + \frac{1}{3!} S(x) * S(x) * S(x)$....... so the star deformed wavefunction is given by $\Psi_* = e_*^{iS/\hbar_{eff}}$ instead of the ordinary exponential. Similary, one could define the star version of the logarithm from its Taylor expansion. At the end of this section we will address this possibility that will modify eq-(2.14) even further.

After this remark, and emphasizing that we are taking the *ordinary* point products in (2.14), upon integrating both sides of eq-(2.14) with respect to each one of the x_i coordinates, $i = 1, 2, 3, ..., d$ and multiplying both sides of eq-(2.14) by Ψ leads to the integro-differential generalized nonlinear Schrödinger equation (GNLSE)

$$i\hbar_{eff} \frac{\partial \Psi}{\partial t} \ =$$

$$- \frac{\hbar^2_{eff}}{2m} \ [\ \nabla^2 \Psi + (\xi - 1)(\nabla ln \Psi)^2 \Psi \ ] + [\ \xi \ U(x) + W_1(x, \Psi) + W_2(x, \Psi) \ ] \ \Psi. \tag{2.16}$$

the ordinary potential $U(x)$ in (2.16) now appears scaled by $\xi U(x)$ and the *induced* (nonlinear) potentials due to the noncommutativity of the momenta and coordinates are respectively given by

$$W_1(x,\Psi) \ = \ - \ \xi \ \frac{\Theta^{ij}}{md} \int \ dx_i \ \nabla_j (ln\Psi); \quad W_1(x,\lambda\Psi) = W_1(x,\Psi), \ \lambda = constant \tag{2.17a}$$

$$W_2(x,\Psi) \ = \ \xi \ \frac{\tilde{\Theta}^{kj}}{d} \int \ dx_i \ (\nabla_j U)(\nabla_k \nabla^i ln\Psi); \quad W_2(x,\lambda\Psi) = W_2(x,\Psi) \tag{2.17b}$$

the standard kinetic operator term $-\frac{\hbar^2_{eff}}{2m}\nabla^2\Psi$ (modulo the replacement of $\hbar$ for $\hbar_{eff}$) is modified by the *nonlinear* term

$$-[\frac{\hbar^2_{eff}}{2m} \ (\xi - 1) \ (\nabla ln\Psi)^2] \ \Psi. \tag{2.18}$$

we may notice that the nonlinear correction (2.18) to the standard kinetic terms has the same functional form as in the nonlinear Schrödinger equation (NLSE) obtained from Nottale's scale derivative operator based on a complex diffusion constant in the previous section, and that the GNLSE (2.16) also obeys the *homogeneity* condition $\Psi \ \rightarrow \ \ \lambda\Psi$ for any constant λ. All the terms in the GNLSE are scaled respectively by a factor λ as they should. The *real* effective $\hbar_{eff}$ in the GNLSE (2.16) is given by $\hbar_{eff} = \hbar\xi = \hbar(1 + \frac{\tilde{\Theta}^{ij}\Theta^{ji}}{4d\hbar^2})$. If Θ^{ij} and/or $\tilde{\Theta}_{ji}$ is zero $\Rightarrow \xi = 1, \hbar_{eff} = \hbar$ and there are no longer nonlinear corrections to the standard kinetic terms.

It is important to emphasize that there is a function of the form

$$F(x_1, x_2, ..., x_d) \ = \ f_1(x_2, x_3, ..., x_d) + f_2(x_1, x_3, ..., x_d) + ... \ f_d(x_1, x_2, ..., x_{d-1}). \tag{2.19}$$

given in terms of d *arbitrary* functions $f_1, f_2, ..., f_d$ that *could* be added in general to the GNLSE in the form of $\frac{1}{d}F(x_1, x_2, ..., x_d)\Psi(x_i)$ as a result of the integration of the third order differential equation (2.14) with respect to the $x_1, x_2, x_3, ..., x_d$ variables, respectively. Cresson has also pointed this out [16].

Solutions to the GNLSE (2.16) when the integration function is set to zero $F(x_i) \ = \ 0$ can be readily found in some special cases. For instance, when the potential $U(x)$ is given by a harmonic isotropic oscillator potential in d spatial dimensions $U(x) \ = \ \frac{1}{2}m\omega x^2 \ = \ \frac{1}{2}m\omega^2 x_k x^k, \ \ k \ = \ 1, 2, 3, ..., d$; after straightforward algebra one can verify that the Gaussian ground state solution $\Psi(x,t) = Ae^{-x^2/\sigma^2}e^{-iE_o t/\hbar_{eff}}$ solves the GNLSE (2.16) if the condition holds

$$\frac{\Theta^{ij}}{m} \ - \ m \ \omega^2 \ \tilde{\Theta}^{ij} \ = \ 0 \ \Rightarrow \ \Theta^{ij} \ = \ m^2 \ \omega^2 \ \tilde{\Theta}^{ij}. \tag{2.20}$$

which implies the cancellation of the mixed terms $x_i \ x_j$ in the GLNSE; while the diagonal terms in the GLNSE will cancel out if

$$[\ \frac{\xi}{2}m\omega^2 \ - \ (\xi - 1)\frac{2\hbar^2_{eff}}{m\sigma^4} \ - \ \frac{2\hbar^2_{eff}}{m\sigma^4}] \ x_k \ x^k \ = \ \xi \ [\ \frac{1}{2}m\omega^2 \ - \ \frac{2\hbar^2_{eff}}{m\sigma^4} \ ] \ x_k \ x^k \ = \ 0 \tag{2.21}$$

If the diagonal terms are zero for all values of x_k one must have the condition relating the Gaussian width σ to the other parameters

$$\frac{1}{2}m\omega^2 \ = \ \frac{2\hbar_{eff}^2}{m\sigma^4} \ \Rightarrow \ \sigma^2 \ = \ 2\frac{\hbar_{eff}}{m\omega}. \tag{2.22}$$

which is exactly the same relationship obtained for the harmonic oscillator potential in ordinary QM with the provision that one replaces $\hbar$ for $\hbar_{eff}$.

For example, in $d = 3$, one may choose

$$\Theta^{ij} \ = \ - \ \Theta^{ji} \ = \ \theta \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{2.23a}$$

$$\tilde{\Theta}^{ij} \ = \ - \ \tilde{\Theta}^{ji} \ = \ \tilde{\theta} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \tag{2.23b}$$

From eqs-(2.20, 2.21, 2.23a, 2.23b) one infers the important positivity condition, valid in any dimension $d \geq 2$ besides $d = 3$,

$$\tilde{\Theta}^{ij} \ \Theta^{ji} \ = \ d \ \tilde{\theta} \ \theta \ = \ d \ m^2\omega^2 \ \tilde{\theta}^2 \ > \ 0 \tag{2.24}$$

resulting in an increase in the effective Planck constant $\hbar_{eff} > \hbar$, since $\xi > 1$ in (2.2). Therefore, in d dimensions ($d \geq 2$), the value for the effective Planck constant becomes

$$\hbar_{eff} \ = \ \hbar \ (1 + \frac{m^2\omega^2\tilde{\theta}^2}{4\hbar^2}). \tag{2.25}$$

and the modified ground state energy will be

$$E_o \ = \ \frac{d}{2} \ \hbar_{eff} \ \omega \ > \ \frac{d}{2} \ \hbar \ \omega. \tag{2.26}$$

hence, there will be an *increase* in the ground state energy of the harmonic isotropic d-dimensional oscillator resulting from the underlying noncommutativity of phase space. It is important to point out that despite that the Gaussian ground state solution solves the GNLSE (2.16), under the above conditions (2.20,2.21), it is no longer true that the solutions for the excited states Ψ_n given by the Gaussian times the Hermite polynomials $H_n(x)$ will solve (2.16).

It is at this point when we may obtain the same order of magnitude to the observed vacuum energy density. In general, an effective Planck constant $\hbar_{eff}$ leads to an energy *increment* for the ground state energy of an isotropic harmonic oscillator given by

$$\Delta E \ = \ \frac{d}{2} \ (\hbar_{eff} \ - \hbar) \ \omega \ = \ \frac{d\hbar\omega}{2} \ \frac{d\theta\tilde{\theta}}{4d\hbar^2} \ = \ \frac{d\omega\theta\tilde{\theta}}{8\hbar}. \tag{2.27}$$

therefore, the increase in the energy density in a spatial volume of Planck length L_P is given by

$$\frac{\Delta E}{(L_P)^d} = \frac{d\omega\theta\tilde{\theta}}{8\hbar(L_P)^d}. \tag{2.28}$$

In $d = 3$ spatial dimensions, in natural units of $\hbar = c = 1$ one arrives at

$$\frac{\Delta E}{L_P^3} = \frac{3}{8}\,\frac{\omega\theta\tilde{\theta}}{L_P^3}. \tag{2.29}$$

Now we may recur to Born's reciprocal relativity principle [28], [29] to fix the values of θ and $\tilde{\theta}$. θ has dimensions of $(momentum)^2$. A *minimal* momentum is associated with the infrared limit linked to a *maximal* length scale that we will set equal to the Hubble radius R_H and which is of the order of $10^{60}L_{Planck}$. $\tilde{\theta}$ has dimensions of $(length)^2$ and is associated with the minimal Planck scale L_P. Thus, in natural units of $\hbar = c = 1$, $\theta = (1/R_H)^2$ and $\tilde{\theta} = L_P^2$. The angular frequency ω of oscillation over a region of size L_P is $(2\pi/T_P)$ and is associated with the Planck time $T_P = L_P/c = L_p$ ($c = 1$). Therefore, upon inserting these values into eq-(2.29), one arrives in $d = 3$ spatial dimensions to an increment to the ground state energy density given by

$$\frac{\Delta E}{L_P^3} = \frac{3}{8}\,2\pi\,\frac{1}{L_P^2}\,\frac{1}{R_H^2} = \frac{6\pi}{8}\,(\frac{L_P}{R_H})^2\,\frac{1}{L_P^4} \sim 10^{-120}\,(M_{Planck})^4. \tag{2.30}$$

which does agree with the observed vacuum energy density. Similar results (modulo π factors) are found from eq-(2.25) if one sets the mass $m = (1/R_H)$ ($\hbar = c = 1$). Had one chosen $R_H = 10^{61}L_P$ one would have obtained a vacuum energy density of the order of $10^{-122}(M_{Planck})^4$. Hence, it appears that the observed vacuum energy density could be related to the fluctuactions/perturbations to the ground state energy density of a harmonic oscillator over a Planck scale region due to an underlying noncommutativity of *phase* space, and which in turn, is associated with a *modified* Newtonian dynamics in the classical limit. Modified Newtonian dynamics has been proposed as an alternative model to dark matter [38].

In the case of a free particle, $U(x) = 0$, plane wave solutions $\Psi(x,t) = Ae^{i(\vec{p}.\vec{x}/\hbar_{eff})}e^{-iEt/\hbar_{eff}}$, where $p_k = (p_1, p_2, , ..., p_d)$ are the spatial components of a constant momentum vector $\vec{p}$, solve the GNLSE (2.16) if, and only if, one *includes* the arbitrary integration function term $\frac{1}{d}F(x_1, x_2, ..., x_d)\Psi(x_1, x_2, ..., x_d)$ in (2.16) such that the condition

$$F(x_1, x_2, ..., x_d) - i\,\frac{\xi}{m\hbar_{eff}}\,\Theta^{ij}\,x_i\,p_j = 0 \tag{2.31}$$

holds, *fixing* the functional expression for the integration function

$$F(x_1, x_2, ..., x_d) = i\,\frac{\xi}{m\hbar_{eff}}\,\Theta^{ij}\,x_i\,p_j; \quad i, j = 1, 2, 3, ..., d; \quad p_1, p_2, ..., p_d = constants. \tag{2.32}$$

in terms of the coordinates and the constant momentum components of $\vec{p}$. Therefore, in the free particle case, a modified energy-momentum (dispersion) relation in d spatial dimensions is derived by inserting the plane wave solutions into the GNLSE (2.16) (after implementing the condition (2.31))

$$E \;=\; \frac{p_k p^k}{2m} \;+\; (\xi - 1)\;\frac{p_k p^k}{2m} \;=\; \xi \frac{p_k p^k}{2m} \;>\; \frac{p_k p^k}{2m}; \quad k = 1, 2, 3, ..., d; \quad (2.33)$$

where $\xi \;=\; (1 + \frac{\tilde{\theta}\;\theta}{4\hbar^2}) \;>\; 1$ for $\tilde{\theta}, \theta$ real and positive. Other example worth examining is to insert the ground state solution of the Hydrogen atom (Coulomb potential) into the GNLSE (2.16) (written in spherical coordinates) and finding out what the integration function F must be in this case.

At this stage is important to inquire about the symmetries of the GNLSE (2.16). Like it occurs in Quantum Field Theories defined on Noncommutative spacetimes (with commuting momentum) by having constant entries for $\tilde{\Theta}^{\mu\nu}$ one spoils Lorentz invariance since it selects a preferred frame of reference. In the nonrelativistic case, choosing constant entries for $\tilde{\Theta}^{ij}$ will affect Galilean invariance, a centrally-extended Galilean algebra is required as discussed by [27]. For this reason it is more natural to have an x-dependent expression for $\tilde{\Theta}^{ij}(x)$ which will appear *inside* the integral in eq-(2.17b). Because the Jacobi identities must be obeyed by the modified Poisson brackets the tensor-valued functions $\tilde{\Theta}^{ij}(x)$ cannot be arbitrary but are constrained to satisfy

$$\{x^i, \{x^j, x^k\}\} \;+\; \{x^j, \{x^k, x^i\}\} \;+\; \{x^k, \{x^i, x^j\}\} \;=$$
$$\{x^i, \tilde{\Theta}^{jk}(x)\} \;+\; \{x^j, \tilde{\Theta}^{ki}(x)\} \;+\; \{x^k, \tilde{\Theta}^{ij}(x)\} \;=\; 0. \quad (2.34)$$

Star products in arbitrary Poisson manifolds based on $\tilde{\Theta}^{\mu\nu}(x)$ have been constructed by Kontsevich [30]. Star products for spaces based on a Lie-algebraic noncommutativity $[x^\mu, x^\nu] = f^{\mu\nu}_\rho x^\rho$ have been provided by Gutt [31] using the Baker-Campbell-Hausdorff formula. On the dual of a Lie algebra the Kontsevich star product is equivalent to the Gutt star product as shown by [32]. Star products and Polyvector-valued gauge field theories in Noncommutative Clifford spaces have been recently provided in [39].

The nonrelativistic results found here can be extended to the relativistic case. The ordinary Klein-Gordon, Dirac equation have been derived [1], [3] using the relativistic fractal-scale calculus. The coupling to Electromagnetic fields E, B fields can also be made via the $p_\mu - iA_\mu$ prescription. In the noncommutative phase space case, one would have a modified relativistic fractal-scale derivative operator $(\partial/\partial s) + V^\mu \nabla_\mu - i\lambda\nabla^2$, due to modifications in the four-velocity, and a modification to the relativistic Lorentz force term leading finally to *nonlinear* modifications of the Klein-Gordon and Dirac equations.

The Bopp shifts in eq-(2.3) $\hat{p}^i \;=\; \hat{\pi}^i - \frac{i}{2\hbar}\Theta^{ij}\hat{q}_j$ when $\Theta^{ij} \;=\; 0$, and $\hat{x}^i \;=\; \hat{q}^i + \frac{i}{2\hbar}\tilde{\Theta}^{ij}\hat{\pi}_j$ permits to write a wave equation in noncommutative space

$$\hat{H}\Psi \;=\; \left(\frac{1}{2m}\hat{p}^2 \;+\; U(x)\;\right) \;*\; \Psi(x). \quad (2.35)$$

in the form

$$i\hbar \frac{\partial \Psi}{\partial t} \;=\; \left(\frac{1}{2m} (\; -i\hbar \nabla_i \;)^2 \;+\; U(\; x^i + \frac{1}{2} \tilde{\Theta}^{ij} \nabla_j \;) \; \right) \; \Psi(x). \tag{2.36}$$

after performing the replacement $\hat{p}_i = \hat{\pi}_i \to -i\hbar \nabla_i$ (when $\Theta^{ij} = 0$) and evaluating the star product

$$U(x) \; * \; \Psi(x) \;=\; U(x) \; (\; e^{\frac{1}{2} \overleftarrow{\partial}_{x^i} \; \tilde{\Theta}^{ij} \; \overrightarrow{\partial}_{x^j}} \;) \; \Psi(x) \;=\;$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} \; \tilde{\Theta}^{i_1 j_1} \; \tilde{\Theta}^{i_2 j_2} \; .... \; \tilde{\Theta}^{i_n j_n} \; \frac{\partial^n U(x)}{\partial x^{i_1} \partial x^{i_2} .....\partial x^{i_n}} \; \frac{\partial^n \Psi(x)}{\partial x^{j_1} \partial x^{j_2} .....\partial x^{j_n}} \; . \tag{2.37}$$

A Taylor series expansion of $U(x^i + \frac{1}{2} \tilde{\Theta}^{ij} \nabla_j) \Psi$ in (2.36) agrees with the star product $U(x) * \Psi(x)$ in (2.35). We may notice that $\hbar_{eff} = \hbar$ when $\Theta^{ij} = 0$, for this reason $\hbar$ appears in (2.36).

When $\Theta^{ij} \neq 0$ one can construct a generalized star product in phase spaces

$$A(x,p) \; * \; B(x,p) \;=\; A(x,p) \; (\; e^{\frac{1}{2} \overleftarrow{\partial}_{z^i} \; \Omega^{ij} \; \overrightarrow{\partial}_{z^j}} \;) \; B(x,p). \tag{2.38}$$

where the operator kernel inside the exponential is defined in terms of derivatives w.r.t the phase space coordinates $z^i \equiv x^i, p^i$ as

$$\overleftarrow{\partial}_{z^i} \; \Omega^{ij} \; \overrightarrow{\partial}_{z^j} \;=\;$$

$$i\hbar_{eff} (\overleftarrow{\partial}_{x^i} \; \delta^{ij} \; \overrightarrow{\partial}_{p^j} \; - \; \overleftarrow{\partial}_{p^i} \; \delta^{ij} \; \overrightarrow{\partial}_{x^j}) \; + \; \overleftarrow{\partial}_{p^i} \; \Theta^{ij} \; \overrightarrow{\partial}_{p^j} \; + \; \overleftarrow{\partial}_{x^i} \; \tilde{\Theta}^{ij} \; \overrightarrow{\partial}_{x^j}. \tag{2.39}$$

and despite the noncommutativity of the phase space base coordinates $z^i = x^i, p^i$ one has commuting derivatives in the tangent space

$$[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] \;=\; [\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}] \;=\; [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p^j}] \;=\; 0. \tag{2.40}$$

When Θ^{ij} and $\tilde{\Theta}^{ij} = 0$ one recovers the standard Moyal star product. QM wave equations corresponding to the star products (2.38, 2.39) can be constructed by recurring to the Bopp shifts (2.3) $\hat{p}^i = \hat{\pi}^i - \frac{i}{2\hbar} \Theta^{ij} \hat{q}_j$, $\hat{x}^i = \hat{q}^i + \frac{i}{2\hbar} \tilde{\Theta}^{ij} \hat{\pi}_j$ in the Hamiltonian $H(x,p) = (p^2/2m) + U(x)$; replacing $\hat{\pi}_i \to -i\hbar \nabla_i$ and writing $H\Psi = i\hbar_{eff} (\partial \Psi / \partial t)$.

To sum up, the QM wave equation of eq-(2.36) is *linear* and *nonlocal* for nonpolynomial analytic potentials (a Taylor expansion yields infinite derivatives). It *differs* from the GLNSE (2.16) which is *nonlinear* but *local* since the integrands of eq-(2.17) are local : they are *not* of the form $\int \; dx' \; K(x, x')$, corresponding to a nonlocal kernel. If one had *modified* eq-(2.14) by replacing ordinary products for star products one would have obtained a more complicated wave equation than (2.16) and (2.36) because it would have been both *nonlinear* and *nonlocal*.

Valentini [42] has discussed in detail that many fundamental questions about Quantum Physics still remain unanswered. In de Broglie's pilot-wave theory, Quantum Theory emerges as a special subset of a wider physics, which allows non-local signals and violation of the uncertainty principle. Experimental evidence for this new physics might be found in the cosmological-microwave-background anisotropies and with the detection of relic particles with exotic new properties predicted by the theory. For this reason it is warranted to explore the *nonlinear* and *nonlocal* wave equations resulting from the use of star products in eq-(2.14). Nonlocality can be recaptured in the infinite derivatives associated with a fractional differential operator after performing a Taylor series expansion. The role of fractional Riemann-Liouville derivatives in QM also deserves further investigation.

The GNLSE equation (2.16) obtained from Nottale-Cresson's fractal-scale calculus approach does *not* coincide with the wave equation in (2.36). The NLSE, GLNSE are *not* the same as the Schrödinger-Poisson equations obtained by the coupled differential equations when the potential $U(x)$ in the Schrödinger equation is constrained to satisfy Poisson's equation and such that the mass density ρ is proportional to $\Psi^*\Psi$. One could have extended the modified Newtonian dynamics to one involving acceleration-dependent Lagrangians, typical of Finsler geometries, the rigid particle and the rigid string.

QM wave equations in Yang's noncommutative phase spaces [40], involving a maximal and minimal length, were constructed in [41] and it was argued how QM in noncommutative Yang's spaces of d-dimensions has a one-to-one correspondence to ordinary QM in commutative spaces of $d+2$-dimensions. Noncommutativity is associated with fuzzy spaces and the latter involve higher dimensions as the authors [33] have shown in their formulation of the 8-dimensional Quantum Hall Effect based on Octonions. The noncommutative and nonassociative geometry of Octonionic spacetime, modified dispersion relations and Grand Unification haven been studied in [34].

The formulation of Fuzzy Fractals and a q-Fractal calculus remains open to our knowledge. Its consequences for Quantum Gravity must be important because the asymptotic safety scenario [43] with an ultraviolet non-Gaussian fixed point [44] seems to be operating and leading to $fractal$ $2+\epsilon, 4+\epsilon$ spacetime dimensions in the ultraviolet and infrared region, respectively. Similar values for the fractal spacetime dimensions have been found by [45] based on the Causal Dynamical Triangulations approach to (Euclidean) Quantum Gravity. To conclude, a fractal spacetime and nonlinear QM might be key ingredients in a quantum theory of gravity.

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