A Derivation of Maxwell Equations in Quaternion Space

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Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity) has been defined in a number of papers including [1], and it can be shown that this new theory is capable to describe relativistic motion in elegant and straightforward way. Nonetheless there are subsequent theoretical developments which remains an open question, for instance to derive Maxwell equations in Q-space. Therefore the purpose of the present paper is to derive a consistent description of Maxwell equations in Q- space. First we consider a simplified method similar to the Feynman's derivation of Maxwell equations from Lorentz force. And then we present another derivation method using Dirac decomposition, introduced by Gersten (1999). Further observation is of course recommended in order to refute or verify some implication of this proposition.

1 Introduction

Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity has been defined in a number of papers including [1], and it can be shown that this new theory is capable to describe relativistic motion in elegant and straightforward way. For instance, it can be shown that the Pioneer spacecraft's Doppler shift anomaly can be explained as a relativistic effect of Quaternion Space [11]. The Yang-Mills field also can be shown to be consistent with Quaternion Space [1]. Nonetheless there are subsequent theoretical developments which remains an open issue, for instance to derive Maxwell equations in Q-space [1].

Therefore the purpose of the present article is to derive a consistent description of Maxwell equations in Q-space. First we consider a simplified method similar to the Feynman's derivation of Maxwell equations from Lorentz force. And then we present another method using Dirac decomposition, introduced by Gersten (1999). In the first section we will shortly review the basics of Quaternion space as introduced in [1].

Further observation is of course recommended in order to verify or refute the propositions outlined herein.

2 Basic aspects of Q-relativity physics

In this section, we will review some basic definitions of quaternion number and then discuss their implications to quaternion relativity (Q-relativity) physics [1].

Quaternion number belongs to the group of "very good" algebras: of real, complex, quaternion, and octonion [1], and normally defined as follows [1]:

$$Q \equiv a + bi + cj + dk \tag{1}$$

Where a,b,c,d are real numbers, and i, j, k are imaginary quaternion units. These Q-units can be represented either via 2x2 matrices or 4x4 matrices. There is quaternionic multiplication rule which acquires compact form [1]:

$$1q_k = q_k 1 = q_k,\ q_j q_k = -\delta_{jk} + \epsilon_{jkn} q_n \tag{2}$$

Where δ_{kn} and ϵ_{jkn} represents 3-dimensional symbols of Kronecker and Levi-Civita, respectively.

In the context of Quaternion Space [1], it is also possible to write the dynamics equations of classical mechanics for an inertial observer in constant Q-basis. SO(3,R)- invariance of two vectors allow to represent these dynamics equations in Q-vector form [1]:

$$m\frac{d^2}{dt^2}\left(x_k q_k\right) = F_k q_k. \tag{3}$$

Because of antisymmetry of the connection (generalised angular velocity) the dynamics equations can be written in vector components, by conventional vector notation [1]:

$$m\left(\vec{a} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)\right) = \vec{F} \tag{4}$$

Therefore, from equation (4) one recognizes known types of classical acceleration, i.e. linear, coriolis, angular, centripetal.

From this viewpoint one may consider a generalization of Minkowski metric interval into biquaternion form [1]:

$$dz = \left(dx_k + i dt_k\right) q_k, \tag{5}$$

With some novel properties, i.e.:

- temporal interval is defined by imaginary vector;
- space-time of the model appears to have six dimensions (6D);
- vector of the displacement of the particle and vector of corresponding time change must always be normal to each other, or:

$$dx_k dt_k = 0 \tag{6}$$

One advantage of this Quaternion Space representation is that it enables to describe rotational motion with great clarity.

After this short review of Q-space, next we will discuss a simplified method to derive Maxwell equations from Lorentz force, in a similar way with Feynman's derivation method using commutative relation [2][10].

3 An intuitive approach from Feynman's derivative

A simplified derivation of Maxwell equations will be discussed here using similar approach known as Feynman's derivation [2][3][10].

We can introduce now the Lorentz force into equation (4), to become:

$$m\left(\frac{d\vec{v}}{dt} + 2\vec{\Omega}\times\vec{v} + \vec{\Omega}\times\vec{r} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right) = q_{\otimes}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \tag{7}$$

Or

$$\left(\frac{d\vec{v}}{dt}\right) = \frac{q_{\otimes}}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) - 2\vec{\Omega}\times\vec{v} - \vec{\Omega}\times\vec{r} - \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right). \tag{8}$$

We note here that q variable here denotes electric charge, not quaternion number.

Interestingly, equation (4) can be compared directly to equation (8) in [2]:

$$m\ddot{x} = F - m\left(\frac{d\vec{v}}{dt}\right) + m\vec{r}\times\vec{\Omega} + m2\dot{x}\times\vec{\Omega} + m\vec{\Omega}\times\left(\vec{r}\times\vec{\Omega}\right), \tag{9}$$

In other words, we find an exact correspondence between quaternion version of Newton second law (3) and equation (9), i.e. the equation of motion for particle of mass m in a frame of reference whose origin has linear acceleration a and an angular velocity $\vec{\Omega}$ with respect to the reference frame [2].

Since we want to find out an "electromagnetic analogy" for the inertial forces, then we can set F=0. The equation of motion (9) then can be derived from Lagrangian L=T-V, where T is the kinetic energy and V is a velocity-dependent generalized potential [2]:

$$V(x,\dot{x},t) = ma\cdot x - m\dot{x}\cdot\vec{\Omega}\times x - \frac{m}{2}\left(\vec{\Omega}\times x\right)^2, \tag{10}$$

Which is a linear function of the velocities. We now may consider that the right hand side of equation (10) consists of a scalar potential [2]:

$$\phi(x,t) = ma\cdot x - \frac{m}{2}\left(\vec{\Omega}\times x\right)^2, \tag{11}$$

And a vector potential:

$$A(x,t) \equiv m\dot{x}\cdot\vec{\Omega}\times x, \tag{12}$$

So that

$$V(x,\dot{x},t) = \phi(x,t) - \dot{x}\cdot A(x,t). \tag{13}$$

Then the equation of motion (9) may now be written in Lorentz form as follows [2]:

$$m\ddot{x} = E(x,t) + x\times H(x,t) \tag{14}$$

With

$$E = -\frac{\partial A}{\partial t} - \nabla\phi = -m\Omega\times x - ma + m\Omega\times(x\times\Omega), \tag{15}$$

And

$$H = \nabla\times A = 2m\Omega. \tag{16}$$

At this point we may note [2, p. 303] that Maxwell equations are satisfied by virtue of equations (15) and (16). The correspondence between Coriolis force and magnetic force, is known from Larmor method. What is interesting to remark here, is that the same result can be expected directly from the basic equation of Quaternion Space (3) [1]. The aforementioned simplified approach indicates that it is indeed possible to find out Maxwell equations in Quaternion space, in particular based on our intuition of the direct link between Newton second law in Q-space and Lorentz force (We can remark that this parallel between classical mechanics and electromagnetic field appears to be more profound compared to simple similarity between Coulomb and Newton force).

As an added note, we can mention here, that the aforementioned Feynman's derivation of Maxwell equations is based on commutator relation which has classical analogue in the form of Poisson bracket. Then there can be a plausible way to extend directly this 'classical' dynamics to quaternion extension of Poisson bracket [14], by assuming the dynamics as element of the type: $r \in H \wedge H$ of the type: $r = ai\wedge j + bi\wedge k + cj\wedge k$, from which we can define Poisson bracket on H. But in the present paper we don't explore yet such a possibility.

In the next section we will discuss more detailed derivation of Maxwell equations in Q-space, by virtue of Gersten's method of Dirac decomposition [4].

4 A new derivation of Maxwell equations in Quaternion Space by virtue of Dirac decomposition.

In this section we present a derivation of Maxwell equations in Quaternion space based on Gersten's method to derive Maxwell equations from one photon equation by virtue of Dirac decomposition [4]. It can be noted here that there are other methods to derive such a 'quantum Maxwell equations' (i.e. to find link between photon equation and Maxwell equations), for instance by Barut quite a long time ago (see ICTP preprint no. IC/91/255).

We know that Dirac deduces his equation from the relativistic condition linking the Energy E, the mass m and the momentum p [5]:

$$\left(E^2 - c^2\vec{p}^{\,2} - m^2c^4\right) I^{(4)}\Psi = 0 \tag{17}$$

Where $I^{(4)}$ is the 4x4 unit matrix and Ψ is a 4-component column (bispinor) wavefunction. Dirac then decomposes equation (17) by assuming them as a quadratic equation:

$$\left(A^2 - B^2\right)\Psi = 0 \tag{18}$$

Where

$$A = E, \tag{19}$$

$$B = c\vec{p} + mc^2 \tag{20}$$

The decomposition of equation (18) is well known, i.e. (A+B)(A-B)=0, which is the basic of Dirac's decomposition method into 2x2 unit matrix and Pauli matrix [4][12].

By virtue of the same method with Dirac, Gersten found in 1999 [4] a decomposition of one photon equation from relativistic energy condition (for massless photon [5]):

$$\left(\frac{E^2}{c^2} - \vec{p}^{\,2}\right) I^{(3)}\Psi = 0 \tag{21}$$

Where $I^{(3)}$ is the 3x3 unit matrix and Ψ is a 3-component column wavefunction. Gersten then found [4] equation (21) decomposes into the form:

$$\left[\frac{E}{c}I^{(3)} - \vec{p}\cdot\vec{S}\right]\left[\frac{E}{c}I^{(3)} + \vec{p}.\vec{S}\right]\vec{\Psi} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}\left(\vec{p}\cdot\vec{\Psi}\right) = 0 \tag{22}$$

where $\vec{S}$ is a spin one vector matrix with components [4]:

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \tag{23}$$

$$S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{24}$$

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{25}$$

And with the properties:

$$[S_x, S_y] = iS_z,\ [S_x, S_z] = iS_y,\ [S_y, S_z] = iS_x,\ \vec{S}^2 = 2I^{(3)} \tag{26}$$

Gersten asserts that equation (22) will be satisfied if the two equations [4][5]:

$$\left[\frac{E}{c}I^{(3)} + \vec{p}\cdot\vec{S}\right]\vec{\Psi} = 0 \tag{27}$$

$$\vec{p}\cdot\vec{\Psi} = 0 \tag{28}$$

are simultaneously satisfied. The Maxwell equations [9] will be obtained by substitution of E and p with the ordinary quantum operators (see for instance Bethe, *Field Theory*):

$$E \rightarrow i\hbar\frac{\partial}{\partial t}, \tag{29}$$

and

$$p \rightarrow -ih\nabla \tag{30}$$

And the wavefunction substitution:

$$\vec{\Psi} = \vec{E} - i\vec{B} \tag{31}$$

Where E and B are electric and magnetic elds, respectively. With the identity:

$$\left(\vec{p}\cdot\vec{S}\right)\vec{\Psi} = \hbar\nabla\times\vec{\Psi} \tag{32}$$

Then from equation (27) and (28) one will obtain:

$$i\frac{\hbar}{c}\frac{\partial\left(\vec{E} - i\vec{B}\right)}{\partial t} = -\hbar\nabla\times\left(\vec{E} - i\vec{B}\right), \tag{33}$$

$$\nabla\cdot\left(\vec{E} - i\vec{B}\right) = 0 \tag{34}$$

Which are the Maxwell equations if the electric and magnetic fields are real [4][5].

We can remark here that the combination of E and B as introduced in (31) is quite well known in literature [6][7]. For instance, if we use positive signature in (31), then it is known as Bateman representation of Maxwell equations $\left(div\vec{\epsilon} = 0\,;\ rot\vec{\epsilon} = \frac{\partial\epsilon}{\partial t}\,;\ \epsilon = \vec{E} + i\vec{B}\right)$.
But the equation (31) with negative signature represents the *complex nature* of Electromagnetic fields [6], which indicates that these fields can also be represented in quaternion form.

Now if we represent in other form $\vec{\epsilon} = \vec{E} - i\vec{B}$ as more conventional notation, then equation (33) and (34) will get a quite simple form:

$$i\frac{\hbar}{c}\frac{\partial\vec{\epsilon}}{\partial t} = -\hbar\nabla\times\vec{\epsilon} \tag{35}$$

$$\nabla\cdot\vec{\epsilon} = 0 \tag{36}$$

Now to consider quaternionic expression of the above results from Gersten [4], one can begin with the same linearization procedure just as in equation (5):

$$dz = (dx_k + idt_k)\, q_k, \tag{37}$$

Which can be viewed as the quaternionic square root of the metric interval dz:

$$dz^2 = dx^2 - dt^2 \tag{38}$$

Now consider the relativistic energy condition (for massless photon [5]) similar to equation (21):

$$E^2 = p^2c^2 \;\Rightarrow\; \left(\frac{E^2}{c^2} - \vec{p}^{\;2}\right) = k^2, \tag{39}$$

It is obvious that equation (39) has the same form with (38), therefore we may find its quaternionic square root too, then we find:

$$k = (E_{qk} + i\vec{p}_{qk})\, q_k, \tag{40}$$

Where q represents the quaternion unit matrix. Therefore the linearized quaternion root decomposition of equation (21) can be written as follows [4]:

$$\begin{aligned}&\left[\frac{E_{qk}q_k}{c}I^{(3)} + i\vec{p}_{qk}q_k\cdot\vec{S}\right]\left[\frac{E_{qk}q_k}{c}I^{(3)} + i\vec{p}_{qk}q_k\cdot\vec{S}\right]\vec{\Psi}\\ &-\begin{pmatrix}p_x\\ p_y\\ p_z\end{pmatrix}\left(i\vec{p}_{qk}q_k\cdot\vec{\Psi}\right) = 0\end{aligned} \tag{41}$$

Accordingly, equation (41) will be satisfied if the two equations:

$$\left[\frac{E_{qk}q_k}{c}I^{(3)} + i\vec{p}_{qk}q_k\cdot\vec{S}\right]\vec{\Psi}_k = 0 \tag{42}$$

$$i\vec{p}_{qk}q_k\cdot\vec{\Psi}_k = 0 \tag{43}$$

are simultaneously satisfied. Now we introduce similar wave-function substitution, but this time in quaternion form:

$$\vec{\Psi}_{qk} = \vec{E}_{qk} - i\vec{B}_{qk} = \vec{\epsilon}_{qk}. \tag{44}$$

And with the identity:

$$\left(\vec{p}_{qk}q_k\cdot\vec{S}\right)\vec{\Psi}_k = \hbar\nabla_k\times\vec{\Psi}_k \tag{45}$$

Then from equation (42) and (43) one will obtain *the Maxwell equations in Quaternion-space* as follows:

$$i\frac{\hbar}{c}\frac{\partial\vec{\epsilon}_q k}{\partial t} = -\hbar\nabla_k\times\vec{\epsilon}_{qk} \tag{46}$$

$$\nabla_k\cdot\vec{\epsilon}_{qk} = 0 \tag{47}$$

Now the remaining question is to define quaternion differential operator in the right hand side of (46) and (47).

In this regards one can choose some definitions of quaternion differential operator, for instance the 'Moisil-Theodoresco operator'[8] :

$$D[\varphi] = grad\varphi = \sum_{k=1}^{3} i_k\partial_k\varphi = i_1\partial_1\varphi + i_2\partial_2\varphi + i_3\partial_3\varphi \tag{48}$$

Where we can define here that $i_1 = i$; $i_2 = j$; $i_3 = k$ to represent 2x2 quaternion unit matrix, for instance. Therefore the differential of equation (44) now can be expressed in similar notation of (48) :

$$\begin{aligned}&D\left[\vec{\Psi}\right] = D[\vec{\epsilon}] = i_1\partial_1E_1 + i_2\partial_2E_2 + i_3\partial_3E_3\\ &-i\left(i_1\partial_1B_1 + i_2\partial_2B_2 + i_3\partial_3B_3\right),\end{aligned} \tag{49}$$

This expression indicates that both electric and magnetic fields can be represented in unified manner in a biquaternion form.

Then we define quaternion differential operator in the right-hand-side of equation (46) by an extension of the conventional definition of curl:

$$\nabla\times A_{qk} = \begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} \tag{50}$$

To become its quaternion counterpart, where i,j,k represents quaternion matrix as described above. This quaternionic extension of curl operator is based on the known relation of multiplication of two arbitrary complex quaternions q and b as follows:

$$q\cdot b = q_0b_0 - \left\langle\vec{q},\vec{b}\right\rangle + \left[\vec{q}\times\vec{b}\right] + q_0\vec{b} + b_0\vec{q}, \tag{51}$$

where

$$\left\langle\vec{q},\vec{b}\right\rangle := \sum_{k=1}^{3} q_kb_k \;\in C \tag{52}$$

And

$$\left[\vec{q}\times\vec{b}\right] := \begin{vmatrix} i & j & k\\ q_1 & q_2 & q_3\\ b_1 & b_2 & b_3\end{vmatrix}. \tag{53}$$

We can note here that there could be more rigorous approach to define such a quaternionic curl operator.[7]

In the present paper we only discuss derivation of Maxwell equations in Quaternion Space using the decomposition method described by Gersten [4][5]. Further extension to Proca equations in Quaternion Space seems possible too using the same method [5], but it will not be discussed here.

In the next section we will discuss some physical implications of this new derivation of Maxwell equations in Quaternion Space.

5 A few implications: de Broglie's wavelength and spin

In the foregoing section we derived a consistent description of Maxwell equations in Q-Space by virtue of Dirac- Gersten's decomposition. Now we discuss some plausible implications of the new proposition.

First, in accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics [4][5]. The one-to-one correspondence between classical and quantum wave interpretation actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket [2][3]. Furthermore, the proposed quaternion yields to a novel viewpoint of both the wavelength, as discussed below, and also mechanical model of spin [13].

The equation (39) implies that momentum and energy could be expressed in quaternion form. Now by introducing the definition of de Broglie's wavelength
$\left(\lambda_{DB} = \frac{\hbar}{p} \rightarrow p_{DB} = \frac{\hbar}{\lambda}\right)$,
then one obtains an expression in terms of wavelength:

$$k = (E_k + i\vec{p}_k)\, q_k = (E_k q_k + i\vec{p}_k q_k) = \left(E_k q_k + i\frac{\hbar}{\lambda_k^{DB} q_k}\right) \quad (54)$$

In other words, now we can express de Broglie's wavelength in a consistent Q-basis:

$$\lambda_{DB-Q} = \frac{\hbar}{\sum_{k=1}^{3}(p_k)\, q_k} = \frac{\hbar}{v_{group}\sum_{k=1}^{3}(m_k)\, q_k}, \quad (55)$$

Therefore the above equation can be viewed as an Extended De Broglie wavelength in Q-space. This equation means that the mass also can be expressed in Q-basis. In the meantime, a quite similar method to define quaternion mass has also been considered elsewhere (Gupta [13]), but it has not yet been expressed in Dirac equation form as presented here.

Further implications of this new proposition of quaternion de Broglie requires further study, and therefore it is excluded from the present paper.

6 Concluding remarks

In the present paper we derive a consistent description of Maxwell equations in Q-space. First we consider a simplified method similar to the Feynman's derivation of Maxwell equations from Lorentz force. And then we present another method to derive Maxwell equations by virtue of Dirac decomposition, introduced by Gersten (1999).

In accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics. The one-to-one correspondence between classical and quantum wave interpretation asserted here actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket [2][4].

A somewhat unique implication obtained from the above results of Maxwell equations in Quaternion Space, is that it suggests that the DeBroglie wavelength will have quaternionic form. Its further implications, however, are beyond the scope of the present paper.

In the present paper we only discuss derivation of Maxwell equations in Quaternion Space using the decomposition method described by Gersten [4][5]. Further extension to Proca equations in Quaternion Space seems possible too using the same method [5], but it will not be discussed here.

This proposition, however, deserves further theoretical considerations. Further observation is of course recommended in order to refute or verify some implications of this result.

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