# Unambiguous quantization from the maximum classical correspondence that is self-consistent: the slightly stronger canonical commutation rule Dirac missed

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**Abstract**

Dirac's identification of the quantum analog of the Poisson bracket with the commutator is reviewed, as is the threat of self-inconsistent overdetermination of the quantization of classical dynamical variables which drove him to restrict the assumption of correspondence between quantum and classical Poisson brackets to embrace only the Cartesian components of the phase space vector. Dirac's canonical commutation rule fails to determine the order of noncommuting factors within quantized classical dynamical variables, but does imply the quantum/classical correspondence of Poisson brackets between any linear function of phase space and the sum of an arbitrary function of only configuration space with one of only momentum space. Since every linear function of phase space is itself such a sum, it is worth checking whether the assumption of quantum/classical correspondence of Poisson brackets for all such sums is still self-consistent. Not only is that so, but this slightly stronger canonical commutation rule also unambiguously determines the order of noncommuting factors within quantized dynamical variables in accord with the 1925 Born-Jordan quantization surmise, thus replicating the results of the Hamiltonian path integral, a fact first realized by E. H. Kerner. Born-Jordan quantization validates the generalized Ehrenfest theorem, but has no inverse, which disallows the disturbing features of the poorly physically motivated invertible Weyl quantization, i.e., its unique deterministic classical "shadow world" which can manifest negative densities in phase space.

## Introduction

The canonical commutation rule and the Heisenberg equation of motion both give concrete expression to Dirac's profound discovery that $(-i/\hbar)$ times the commutator bracket is the quantum analog of the classical Poisson bracket, and together serve to incorporate both the correspondence principle and the uncertainty principle into orthodox operator quantum dynamics. Dirac's 1925 version of the canonical commutation rule is well-known, however, to be too weak to determine the ordering of noncommuting factors that in principle can occur in the quantization of an arbitrary classical dynamical variable—albeit this has never been a significant issue in practice because such factors rarely feature in the classical Hamiltonians that are hypothesized for those physical systems for which quantum dynamics appears to be useful. As a matter of principle, however, this ordering ambiguity in the quantization of general classical dynamical variables can obviously be viewed as an annoying gap in the theoretical completeness of orthodox quantum dynamics. Dirac in 1925 was under the impression that he had little choice but to opt for his weak version of the canonical commutation rule because the most obvious stronger alternative turns out to self-inconsistently *overdetermine* the quantization of general classical dynamical variables, and obviously an annoying apparent gap in theoretical completeness is a lesser evil than outright self-inconsistency. The fleshing out of the alternative Hamiltonian phase-space path integral approach to quantum dynamics in the late 1960's, however, yielded an unambiguous quantization of *all* classical dynamical variables—a groundbreaking result which unfortunately was soon mistakenly disputed. That result motivated reexamination of the range of possible canonical commutation rules, which led to the realization that a *slightly stronger one* than Dirac's weak 1925 version *still retains the latter's self-consistency* but nevertheless *completely resolves its ordering ambiguity*—this slightly stronger canonical commutation rule in fact yields *exactly the same* unambiguous quantization of all classical dynamical variables as that which is implied by the Hamiltonian phase-space path integral. Lamentably, as a psychologically freighted consequence of the erroneous disputation of the unambiguous quantization result for the Hamiltonian phase-space path integral, this slightly stronger canonical commutation rule was never published nor publicly disclosed by its original discoverer. That notwithstanding, the fact is that Dirac in 1925 *was intimately familiar with all the knowledge and tools that are needed for its discovery*; it is a matter of mere *historical happenstance* that he failed to light on it *at that time.* Thus it was *very far from inevitable* that the ordering ambiguity gap in the theoretical completeness of quantum dynamics *should even have occurred.*

## Dirac's quantum analog of the classical Poisson bracket

By way of placing Heisenberg's matrix quantum mechanics on a more general footing, Dirac [1] abstracted Heisenberg's *matrix* quantum dynamical variables as simply *noncommuting* quantum dynamical variables of the form $\widehat{F}(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$, whose underlying phase space vector $(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$ *also* consists of mutually noncommuting components. In step with Heisenberg's practice, Dirac required these quantized dynamical variables to obey equations of motion which correspond as *closely* to the *classical Hamiltonian* equations of motion that are satisfied by their *unquantized predecessors* as their *noncommuting character* can accommodate. It was therefore envisaged that the classical equation of motion,

$$dF/dt = \{F, H\}, \tag{1a}$$

where $\{\,,\}$ is the Poisson bracket, has the quantized counterpart,

$$d\widehat{F}/dt = \{\widehat{F}, \widehat{H}\}_Q, \tag{1b}$$

where $\{\,,\}_Q$ is Dirac's *quantum analog* of the Poisson bracket. In parallel with $\{\,,\}$, $\{\,,\}_Q$ is assumed to be *bilinear* in its two arguments (its linearity in its *first* argument is already implied by the linearity of the time derivative operation on the left hand side of Eq. (1b)). Since the classical Hamiltonian $H$ is a constant of motion, its quantization $\widehat{H}$ is postulated to be so as well, i.e., $d\widehat{H}/dt = 0$, which, from Eq. (1b), implies that $\{\widehat{H}, \widehat{H}\}_Q = 0$. Given two quantized Hamiltonians, $\widehat{H_1}$ and $\widehat{H_2}$, their sum $\widehat{H} = \widehat{H_1} + \widehat{H_2}$ is also a quantized Hamiltonian. This, together with the *bilinearity* of $\{\,,\}_Q$ and the *vanishing* of $\{\widehat{H}, \widehat{H}\}_Q$, $\{\widehat{H_1}, \widehat{H_1}\}_Q$, and $\{\widehat{H_2}, \widehat{H_2}\}_Q$, obviously implies that,

$$\{\widehat{H_1}, \widehat{H_2}\}_Q + \{\widehat{H_2}, \widehat{H_1}\}_Q = 0. \tag{2a}$$

Now the only evident distinction between a quantized Hamiltonian $\widehat{H}(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$ and a general quantized dynamical variable $\widehat{F}(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$ is that $\widehat{H}(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$ has dimensions of energy; therefore multiplying the arbitrary $\widehat{F}(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$ by a nonzero *constant* (in the quantized phase space variariables $(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$) that has the appropriate *dimensions* will *change* it to such a quantized $\widehat{H}(\widehat{\mathbf{q}},\widehat{\mathbf{p}})$. This fact, together with the *bilinearity* of $\{\,,\}_Q$ and Eq. (2a), permits us to conclude that $\{\,,\}_Q$, like $\{\,,\}$, is always *antisymmetric* in its two arguments, i.e.,

$$\{\widehat{F_1}, \widehat{F_2}\}_Q = -\{\widehat{F_2}, \widehat{F_1}\}_Q. \tag{2b}$$

The time derivative of the *product* of two noncommuting quantized dynamical variables $\widehat{F}$ and $\widehat{G}$ (e.g., Heisenberg's matrices) is well-known to be given by the familiar *product differentiation rule*, but with the *order* of the two factors always *strictly maintained*,

$$d(\widehat{F}\widehat{G})/dt = (d\widehat{F}/dt)\widehat{G} + \widehat{F}(d\widehat{G}/dt). \tag{3a}$$

This, together with Eq. (1b), implies that,

$$\{\widehat{F}\widehat{G}, \widehat{H}\}_Q = \{\widehat{F}, \widehat{H}\}_Q\widehat{G} + \widehat{F}\{\widehat{G}, \widehat{H}\}_Q, \tag{3b}$$

and we can, upon taking the *bilinearity* of $\{\,,\}_Q$ into account along the lines discussed just before Eq. (2b), substitute an *arbitrary* quantized dynamical variable for the $\widehat{H}$ which appears in Eq. (3b). Of course the *order* of the factors that occur in Eq. (3b) must be strictly maintained—the classical Poisson bracket $\{\,,\}$ satisfies a relation that formally parallels Eq. (3b), but there the ordering of the factors that occur makes no difference. Given four arbitrary quantized dynamical variables $\widehat{F_1}$, $\widehat{G_1}$, $\widehat{F_2}$, $\widehat{G_2}$, Dirac started with the relation,

$$\{\widehat{F_1}\widehat{G_1}, \widehat{F_2}\widehat{G_2}\}_Q = -\{\widehat{F_2}\widehat{G_2}, \widehat{F_1}\widehat{G_1}\}_Q,$$

which follows from the antisymmetry of $\{\,,\}_Q$ (given by Eq. (2b)). He then systematically applied Eqs. (3b) and (2b) in repeated succession to both its left and right hand sides, always keeping scrupulous track of the *order* of multiplicative factors. After taking account of the cancellation of identical terms, there results,

$$(\widehat{F_1}\widehat{F_2} - \widehat{F_2}\widehat{F_1})\{\widehat{G_1}, \widehat{G_2}\}_Q = \{\widehat{F_1}, \widehat{F_2}\}_Q(\widehat{G_1}\widehat{G_2} - \widehat{G_2}\widehat{G_1}). \tag{4}$$

If $\widehat{F_1}$ commuted with $\widehat{F_2}$ and $\widehat{G_1}$ with $\widehat{G_2}$, this would be an implication-free *identity*—as its *classical* Poisson bracket analog indeed *is*. For the *arbitrary* noncommuting *quantized* dynamical variables $\widehat{F_1}$, $\widehat{G_1}$, $\widehat{F_2}$, and $\widehat{G_2}$, however, Eq. (4) implies that,

$$\begin{aligned} \{\widehat{G_1}, \widehat{G_2}\}_Q &= K(\widehat{G_1}\widehat{G_2} - \widehat{G_2}\widehat{G_1}) \text{ and} \\ \{\widehat{F_1}, \widehat{F_2}\}_Q &= K(\widehat{F_1}\widehat{F_2} - \widehat{F_2}\widehat{F_1}), \end{aligned} \tag{5a}$$

where $K$ is a *universal* constant. By referring to the equation of motion of Heisenberg's Hermitian matrix quantum dynamical variables, Dirac could see that $K$ was equal to the imaginary universal constant $-i/\hbar$. Therefore, $\{\,,\}_Q$ is given by the *commutator* expression,

$$\{\widehat{F_1}, \widehat{F_2}\}_Q = (-i/\hbar)(\widehat{F_1}\widehat{F_2} - \widehat{F_2}\widehat{F_1}) = (-i/\hbar)[\widehat{F_1}, \widehat{F_2}]. \tag{5b}$$

## Exploring potential canonical commutation rules

With the definitive commutator expression of Eq. (5b) in hand for $\{\,,\}_Q$, Dirac was naturally tempted to postulate the following complete correspondence of the quantum Poisson bracket $\{\,,\}_Q$ to its classical counterpart $\{\,,\}$,

$$\{\widehat{F_1}, \widehat{F_2}\}_Q = \overbrace{\{F_1, F_2\}}\ , \tag{6}$$

where we have used the overbrace symbol synonymously with the hat symbol to denote the quantization of a classical dynamical variable (we use the overbrace mainly where lack of sufficient extensibility of the hat symbol presents a problem). For a dynamical system having one degree of freedom, Eq. (6) implies that $\overbrace{qp} = \frac{1}{2}(\widehat{q}\widehat{p} + \widehat{p}\widehat{q})$, $\overbrace{q^2p} = \widehat{q}\widehat{p}\widehat{q}$, and $\overbrace{qp^2} = \widehat{p}\widehat{q}\widehat{p}$. However, for the still higher order quantization $\overbrace{q^2p^2}$, the *two* relations $\{\ \overbrace{q^2p}\ ,\ \overbrace{qp^2}\ \}_Q = \overbrace{\{q^2p, qp^2\}}$ and $\{\ \overbrace{q^3}\ ,\ \overbrace{p^3}\ \}_Q = \overbrace{\{q^3, p^3\}}$ that are *both* implied by Eq. (6) produce results which unfortunately *differ* from each other by $\hbar^2/3$. Though he did not publish the details (the calculations just sketched were first published by Groenewold [2] many years later), Dirac was acutely aware that the straightforward postulate of Eq. (6) self-inconsistently *overdetermines* the quantization of classical dynamical variables. Casting about for a way to avoid such overdetermination, Dirac *restricted* the classical dynamical variables $F_1$ and $F_2$ that appear in Eq. (6) to be *only* the Cartesian components of the phase space vector $(\mathbf{q}, \mathbf{p})$. For these, the right hand side of Eq. (6) is invariably the quantization of either zero or unity, which are naturally taken to be, respectively, zero and the quantum identity I (which is, for example, the identity matrix in the Heisenberg matrix quantum mechanics). Therefore, Dirac's restriction *prevents* Eq. (6) from conclusively *determining* the quantization of *any* nonconstant classical dynamical variable, which obviously (if hardly elegantly!) *eliminates* the issue of its possible *overdetermination*. The obvious downside of Dirac's drastic restriction on Eq. (6) is the resulting *underdetermination* of the quantizations of a vast class of classical dynamical variables. Even the *additional* requirements that such quantizations be *Hermitian* matrices in the Heisenberg matrix quantum mechanics (for *real* classical dynamical variables), and that they reduce properly in the limit that the Cartesian components of $(\widehat{\mathbf{q}}, \widehat{\mathbf{p}})$ all *commute* with each other (i.e., when $\hbar \to 0$), still leaves *major* ambiguities: such a simple entity as $\overbrace{qp}$ cannot be determined beyond $(\frac{1}{2} - ik)\widehat{q}\widehat{p} + (\frac{1}{2} + ik)\widehat{p}\widehat{q} = \frac{1}{2}(\widehat{q}\widehat{p} + \widehat{p}\widehat{q}) + k\hbar$, where k is an *arbitrary* real constant. The main reason that Dirac's drastic restriction on Eq. (6) has been tolerated for so long is the prevalence in practice of classical Hamiltonians of the special form $|\mathbf{p}|^2/(2m) + V(\mathbf{q})$, whose standard quantization as $|\widehat{\mathbf{p}}|^2/(2m) + V(\widehat{\mathbf{q}})$ follows from supplementing Dirac's restricted form of Eq. (6) with the compatible further implicit natural assumptions that $\overbrace{F(\mathbf{p})} = F(\widehat{\mathbf{p}})$, that $\overbrace{G(\mathbf{q})} = G(\widehat{\mathbf{q}})$, and that quantization is a *linear* process. These implicit assumptions, plus the additional one that $\widehat{c} = c\,\mathrm{I}$ for any $c$ that is *constant* in $(\mathbf{q}, \mathbf{p})$, are needed auxiliaries to postulation of any restricted form of Eq. (6). Taking account of these implicit auxiliary assumptions and of the bilinearity of $\{\,,\}_Q$ and $\{\,,\}$, Dirac's restricted form of Eq. (6) can be extended and reexpressed as,

$$\begin{aligned}\{\,\overbrace{c_1 + \mathbf{k}_1\cdot\mathbf{q} + \mathbf{l}_1\cdot\mathbf{p}}\,,\,\overbrace{c_2 + \mathbf{k}_2\cdot\mathbf{q} + \mathbf{l}_2\cdot\mathbf{p}}\,\}_Q &= \overbrace{\{c_1 + \mathbf{k}_1\cdot\mathbf{q} + \mathbf{l}_1\cdot\mathbf{p}, c_2 + \mathbf{k}_2\cdot\mathbf{q} + \mathbf{l}_2\cdot\mathbf{p}\}} \\ &= (\mathbf{k}_1\cdot\mathbf{l}_2 - \mathbf{l}_1\cdot\mathbf{k}_2)\,\mathrm{I},\end{aligned} \tag{7}$$

where $c_1$, $c_2$, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{l}_1$, and $\mathbf{l}_2$ are all arbitrary *constants* as functions of $(\mathbf{q}, \mathbf{p})$, and I is, of course, the quantum identity. Eq. (7) is equivalent to Eq. (6) with the *restriction* that *all* the phase-space second partial derivatives of $F_1$ and $F_2$ *vanish*.

Dirac pointed out that with the aid of Eq. (3b) and any Taylor series representation of $F(\mathbf{q})$, an arbitrary function of $\mathbf{q}$, his commutation postulate (e.g., Eq. (7)) can be shown to imply,

$$\{\,\overbrace{\mathbf{l}\cdot\mathbf{p}}\,,\,\overbrace{F(\mathbf{q})}\,\}_Q = \overbrace{\{\mathbf{l}\cdot\mathbf{p}, F(\mathbf{q})\}} = -(\mathbf{l}\cdot\nabla_{\widehat{\mathbf{q}}}F(\widehat{\mathbf{q}})). \tag{8a}$$

Indeed, such consequences of Dirac's commutation rule can themselves be explicitly incorporated into a further extension of Eq. (7), which thereupon reads,

$$\{\,\overbrace{c + \mathbf{k}\cdot\mathbf{q} + \mathbf{l}\cdot\mathbf{p}}\,,\,\overbrace{F(\mathbf{q}) + G(\mathbf{p})}\,\}_Q = \overbrace{\{c + \mathbf{k}\cdot\mathbf{q} + \mathbf{l}\cdot\mathbf{p}, F(\mathbf{q}) + G(\mathbf{p})\}} = \mathbf{k}\cdot\nabla_{\widehat{\mathbf{p}}}G(\widehat{\mathbf{p}}) - \mathbf{l}\cdot\nabla_{\widehat{\mathbf{q}}}F(\widehat{\mathbf{q}}). \tag{8b}$$

Eq. (8b) is equivalent to Eq. (6) with the *restriction* that *all* the phase-space second partial derivatives of $F_1$ *vanish*, but that *only* the $(\mathbf{q}, \mathbf{p})$–*mixed gradients* of $F_2$, i.e., those of the form $(\mathbf{l}\cdot\nabla_{\mathbf{q}})(\mathbf{k}\cdot\nabla_{\mathbf{p}})F_2$, need vanish (or, alternatively, the same with $F_1$ and $F_2$ interchanged). It is a great pity that, notwithstanding that he possessed all the tools needed for this, Dirac apparently never wrote down and pondered his postulate in the extended form given by Eq. (8b). Had he done so, there can be no doubt that he would have been struck by the peculiar juxtaposition of the general expression $F(\mathbf{q}) + G(\mathbf{p})$ with its mere Taylor expansion through *linear* terms *only*, i.e., $c + \mathbf{k}\cdot\mathbf{q} + \mathbf{l}\cdot\mathbf{p}$. As no quantum physical argument requiring this decidedly unaesthetic *dichotomy* suggests itself, Dirac would unquestionably have been anxious to explore the *implications of removing it*—in particular whether these implications are, *unlike* those of the *unrestricted* Eq. (6), *self-consistent*. The resulting slightly stronger canonical commutation rule,

$$\{\ \overbrace{F_1(\mathbf{q})+G_1(\mathbf{p})}\ ,\ \overbrace{F_2(\mathbf{q})+G_2(\mathbf{p})}\ \}_Q = \overbrace{\{F_1(\mathbf{q})+G_1(\mathbf{p}), F_2(\mathbf{q})+G_2(\mathbf{p})\}}$$
$$= \overbrace{\nabla_{\mathbf{q}}F_1(\mathbf{q})\cdot\nabla_{\mathbf{p}}G_2(\mathbf{p})} - \overbrace{\nabla_{\mathbf{p}}G_1(\mathbf{p})\cdot\nabla_{\mathbf{q}}F_2(\mathbf{q})}\ , \tag{9a}$$

is equivalent to Eq. (6) with the *restriction* that *only* the $(\mathbf{q},\mathbf{p})$–*mixed gradients* of $F_1$ and $F_2$, i.e., those of the form $(\mathbf{l}\cdot\nabla_{\mathbf{q}})(\mathbf{k}\cdot\nabla_{\mathbf{p}})F_i$, $i=1,2$, need vanish. This slightly stronger canonical commutation rule certainly shows promise for a self-consistent determination of the quantization of $q^2p^2$—unlike Dirac's canonical commutation rule (given, for example, by Eq. (8b)), Eq. (9a) actually implies a relation, namely,

$$\{\ \overbrace{q^3}\ ,\ \overbrace{p^3}\ \}_Q = \overbrace{\{q^3, p^3\}},$$

that *determines* the quantization of $q^2p^2$, while, *unlike* the *unrestricted* Eq. (6), it apparently does *not* permit inference of the *conflicting* relation,

$$\{\ \overbrace{q^2p}\ ,\ \overbrace{qp^2}\ \}_Q = \overbrace{\{q^2p, qp^2\}}\,.$$

## Unambiguous quantization of classical dynamical variables

More generally, the slightly stronger canonical commutation rule of Eq. (9a) also implies apparently unambiguous quantization of the crucially important Fourier expansion components $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ for classical dynamical variables $F(\mathbf{q},\mathbf{p})$. It does so via the relation,

$$\{e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\}_Q = \ \overbrace{\{e^{i\mathbf{k}\cdot\mathbf{q}}, e^{i\mathbf{l}\cdot\mathbf{p}}\}}\ = -(\mathbf{k}\cdot\mathbf{l})\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,, \tag{9b}$$

which, together with Eq. (5b), yields,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = (i/(\hbar\mathbf{k}\cdot\mathbf{l}))(e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}\,e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} - e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\,e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}})\,. \tag{10a}$$

The right hand side of Eq. (10a) is a bit awkward in that its limit as $\mathbf{k}\to\mathbf{0}$ or $\mathbf{l}\to\mathbf{0}$ or even $\hbar\to 0$ fails to be manifestly apparent. These obscurities can be resolved after first formally reexpressing the commutator on the right side as the integral of a perfect differential,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = (i/(\hbar\mathbf{k}\cdot\mathbf{l}))\int_0^1 d\alpha\,\frac{d}{d\alpha}\left(e^{i\alpha\mathbf{k}\cdot\widehat{\mathbf{q}}}\,e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\,e^{i(1-\alpha)\mathbf{k}\cdot\widehat{\mathbf{q}}}\right), \tag{10b}$$

where, upon carrying out the differentiation with respect to $\alpha$, there results,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = (-1/(\hbar\mathbf{k}\cdot\mathbf{l}))\int_0^1 d\alpha\,e^{i\alpha\mathbf{k}\cdot\widehat{\mathbf{q}}}\left[\mathbf{k}\cdot\widehat{\mathbf{q}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\right]e^{i(1-\alpha)\mathbf{k}\cdot\widehat{\mathbf{q}}}\,. \tag{10c}$$

The $[\,,]$ commutator bracket which occurs on the right hand side of Eq. (10c) can, using Eq. (5b), be reexpressed as a $\{\,,\}_Q$ quantum analog of the Poisson bracket to yield,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = (-i/(\mathbf{k}\cdot\mathbf{l}))\int_0^1 d\alpha\,e^{i\alpha\mathbf{k}\cdot\widehat{\mathbf{q}}}\left\{\mathbf{k}\cdot\widehat{\mathbf{q}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\right\}_Q e^{i(1-\alpha)\mathbf{k}\cdot\widehat{\mathbf{q}}}\,. \tag{10d}$$

To the $\{\,,\}_Q$ on the right hand side of Eq. (10d) we apply the Dirac commutation rule of Eq. (8b) to obtain,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = \int_0^1 d\alpha\,e^{i\alpha\mathbf{k}\cdot\widehat{\mathbf{q}}}\,e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\,e^{i(1-\alpha)\mathbf{k}\cdot\widehat{\mathbf{q}}}\,. \tag{10e}$$

It can in very similar fashion as well be shown that,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = \int_0^1 d\alpha\,e^{i\alpha\mathbf{l}\cdot\widehat{\mathbf{p}}}\,e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}\,e^{i(1-\alpha)\mathbf{l}\cdot\widehat{\mathbf{p}}}\,. \tag{10f}$$

Now since the arbitrary classical dynamical variable $F(\mathbf{q},\mathbf{p})$ can be *linearly* expanded in the Fourier components $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ as,

$$F(\mathbf{q},\mathbf{p}) = (2\pi)^{-2n}\int d^n\mathbf{q}'\, d^n\mathbf{p}'\, d^n\mathbf{k}\, d^n\mathbf{l}\, F(\mathbf{q}',\mathbf{p}')\, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')}\, e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}\,, \tag{11a}$$

the assumed linearity of quantization implies that its quantization $\widehat{F(\mathbf{q},\mathbf{p})}$ is given by,

$$\widehat{F(\mathbf{q},\mathbf{p})} = (2\pi)^{-2n}\int d^n\mathbf{q}'\, d^n\mathbf{p}'\, d^n\mathbf{k}\, d^n\mathbf{l}\, F(\mathbf{q}',\mathbf{p}')\, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')}\, \widehat{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,, \tag{11b}$$

with the indicated quantization of $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ being supplied by Eq. (10e) or (10f). The closed integral quantization formula of Eq. (11b) conversely *implies* the linearity of quantization. Because Eq. (10e) or (10f) implies that,

$$\left.\widehat{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\right|_{\mathbf{l}=\mathbf{0}} = e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}} \;\text{ and }\; \left.\widehat{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\right|_{\mathbf{k}=\mathbf{0}} = e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\,, \tag{12a}$$

we have as special cases of Eq. (11b) that,

$$\begin{aligned} \widehat{F(\mathbf{q})} &= (2\pi)^{-n}\textstyle\int d^n\mathbf{q}'\, d^n\mathbf{k}\, F(\mathbf{q}')\, e^{-i\mathbf{k}\cdot\mathbf{q}'}\, e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}} \;=\; F(\widehat{\mathbf{q}}) \;\text{ and} \\ \widehat{G(\mathbf{p})} &= (2\pi)^{-n}\textstyle\int d^n\mathbf{p}'\, d^n\mathbf{l}\, G(\mathbf{p}')\, e^{-i\mathbf{l}\cdot\mathbf{p}'}\, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} \;=\; G(\widehat{\mathbf{p}}). \end{aligned} \tag{12b}$$

If we further specialize $F(\mathbf{q},\mathbf{p})$ to just a constant $c$ in the phase space argument $(\mathbf{q},\mathbf{p})$, Eqs. (12b) imply that $\widehat{c} = c\,\mathrm{I}$, where I is the quantum identity. Mindful of the bilinearity of $\{\,,\,\}_Q$, we also see that Eqs. (12b) imply that,

$$\{F(\widehat{\mathbf{q}}), G(\widehat{\mathbf{p}})\}_Q = (2\pi)^{-2n}\int d^n\mathbf{q}'\, d^n\mathbf{p}'\, d^n\mathbf{k}\, d^n\mathbf{l}\, F(\mathbf{q}')G(\mathbf{p}')\, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')}\, \{e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\}_Q. \tag{12c}$$

The prior deduction of Eq. (10e) or (10f) from Eq. (10a) in concert with the Dirac commutation rule of Eq. (8b) is *reversible*: Eq. (10a) follows from either Eq. (10e) or (10f) coupled with Eq. (8b). As Eq. (10a) is just Eq. (9b) rewritten, we now put Eq. (9b) into Eq. (12c) to obtain,

$$\{F(\widehat{\mathbf{q}}), G(\widehat{\mathbf{p}})\}_Q = (2\pi)^{-2n}\int d^n\mathbf{q}'\, d^n\mathbf{p}'\, d^n\mathbf{k}\, d^n\mathbf{l}\, F(\mathbf{q}')G(\mathbf{p}')(-(\mathbf{k}\cdot\mathbf{l}))\, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')}\, \widehat{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,, \tag{12d}$$

which, via integrations by parts, can be reexpressed as,

$$\{F(\widehat{\mathbf{q}}), G(\widehat{\mathbf{p}})\}_Q = (2\pi)^{-2n}\int d^n\mathbf{q}'\, d^n\mathbf{p}'\, d^n\mathbf{k}\, d^n\mathbf{l}\, (\nabla_{\mathbf{q}'}F(\mathbf{q}')\cdot\nabla_{\mathbf{p}'}G(\mathbf{p}'))\, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')}\, \widehat{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,. \tag{12e}$$

If we now refer to Eq. (11b), we see that Eq. (12e) implies that,

$$\{F(\widehat{\mathbf{q}}), G(\widehat{\mathbf{p}})\}_Q = \widehat{\nabla_{\mathbf{q}}F(\mathbf{q})\cdot\nabla_{\mathbf{p}}G(\mathbf{p})} = \widehat{\{F(\mathbf{q}), G(\mathbf{p})\}}\,, \tag{12f}$$

which is the key part of the slightly stronger canonical commutation rule given by Eq. (9a). The *remainder* of Eq. (9a) follows from the properties of the Poisson bracket, the linearity of quantization, and the relations,

$$\{F_1(\widehat{\mathbf{q}}), F_2(\widehat{\mathbf{q}})\}_Q = 0 \;\text{ and }\; \{G_1(\widehat{\mathbf{p}}), G_2(\widehat{\mathbf{p}})\}_Q = 0,$$

which are an obvious consequence of just the Dirac canonical commutation postulate given by Eq. (8b).

Therefore, we have seen *not only* that the slightly stronger canonical commutation rule of Eq. (9a) and its natural auxiliary assumptions (e.g., linearity of quantization, $\widehat{F(\mathbf{q})} = F(\widehat{\mathbf{q}})$, etc.) implies the closed form quantization formula given by Eq. (11b) and either of Eqs. (10e) or (10f), as *well*, of course, as implying the Dirac canonical commutation postulate of Eq. (8b), but that the *converse* is *also* the case. Since that closed form quantization formula is obviously self-consistent, unique, and compatible with the *as well* self-consistent Dirac canonical commutation postulate of Eq. (8b), the *same* self-consistency and uniqueness therefore must *also* hold for the *equivalent* quantization embodied by the slightly stronger canonical commutation rule of Eq. (9a) and its natural auxiliary assumptions. Thus the slightly stronger canonical commutation rule of

Eq. (9a) *navigates to perfection* the tight, perilous channel between the Scylla of self-inconsistently *overdetermining* quantization by placing *insufficient restriction* on the *classical correspondence* embodied by Eq. (6) and the Charybdis of ambiguously *underdetermining* quantization by anxiously *overrestricting* the classical correspondence which flows from Eq. (6), as Dirac did when he limited the inputs of Eq. (6) to *only the Cartesian components* of the phase space vector $(\mathbf{q}, \mathbf{p})$.

In retrospect, it seems truly astonishing that Dirac himself, in the course of his long theoretical physics career, did not eventually hit upon the modest and completely natural upgrade of his inadequate canonical quantization postulate to the vastly more satifactory Eq. (9a)—this is surely a cautionary real life lesson in the fact that even so penetrating and innovative a mind as was Dirac's can still become snagged in a conceptual side stream. Probably the first to discover Eq. (9a) was E.H. Kerner [3], who realized that it confirmed a groundbreaking clarification of Hamiltonian path integral quantization which he had, together with W.G. Sutcliffe, pointed out [4]. Unfortunately, Kerner, who was apparently a very sensitive individual, never published this discovery, possibly because his earlier path integral work with Sutcliffe had met with strong—albeit entirely misconceived—opposition [5] (to which he as well declined to riposte, so as not, in his words, "to pick a fight" [3]). Formal further development and elaboration of the path integral quantization insight of Kerner and Sutcliffe is addressed in another manuscript, but it is interesting to note that the Hamiltonian path integral naturally incorporates a maximally strong type of the classical correspondence: the single *most* important path which enters into the path sum is the one of stationary phase, and that path is also *always* the *classical* phase-space path which obeys Hamilton's equations of motion. It is perhaps not surprising, then, that the results which flow from such a principle of maximum classical correspondence dovetail with those that result from requiring the maximum correspondence between quantum and classical Poisson brackets which is self-consistent. Now because the physically correct Hamiltonian path integral in configuration space directly yields the quantization of its input classical Hamiltonian in *configuration representation*, we shall here work out the quantization of an arbitrary classical dynamical variable in that particular representation. As a preliminary step, we work out the configuration space matrix elements of the quantization of the classical Fourier component $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ that is given by Eq. (10e),

$$\langle\mathbf{q}_2|\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\ |\mathbf{q}_1\rangle = \int_0^1 d\alpha\, e^{i\mathbf{k}\cdot(\alpha\mathbf{q}_2+(1-\alpha)\mathbf{q}_1)}\ \delta^{(n)}(\mathbf{q}_2-\mathbf{q}_1+\hbar\mathbf{l}), \tag{13a}$$

which, together with Eq. (11b), yields these same matrix elements of the quantization of an arbitrary classical dynamical variable $F(\mathbf{q}, \mathbf{p})$,

$$\langle\mathbf{q}_2|\ \overbrace{F(\mathbf{q},\mathbf{p})}\ |\mathbf{q}_1\rangle = (2\pi\hbar)^{-n}\int d^n\mathbf{p}\int_0^1 d\alpha\, F(\alpha\mathbf{q}_2+(1-\alpha)\mathbf{q}_1,\mathbf{p})\, e^{i\mathbf{p}\cdot(\mathbf{q}_2-\mathbf{q}_1)/\hbar}\,. \tag{13b}$$

Though it is not well-known, there is also a Hamiltonian path integral in *momentum space*, which directly yields the quantization of its input classical Hamiltonian in *momentum representation*. For this reason, we use Eqs. (10f) and (11b) to likewise work out the momentum space matrix elements of the quantization of our arbitrary classical dynamical variable $F(\mathbf{q}, \mathbf{p})$,

$$\langle\mathbf{p}_2|\ \overbrace{F(\mathbf{q},\mathbf{p})}\ |\mathbf{p}_1\rangle = (2\pi\hbar)^{-n}\int d^n\mathbf{q}\int_0^1 d\alpha\, F(\mathbf{q},\alpha\mathbf{p}_2+(1-\alpha)\mathbf{p}_1)\, e^{-i\mathbf{q}\cdot(\mathbf{p}_2-\mathbf{p}_1)/\hbar}\,. \tag{13c}$$

The quantization formulas we have derived in Eqs. (13b) and (13c) on the basis of the strengthened canonical commutation postulate of Eq. (9a) arise *entirely naturally* as well from the Hamiltonian path integral *provided* that care is taken to *ensure* that the paths summed over *all actually adhere* to the two imposed endpoint constraints $\mathbf{q}(t_1) = \mathbf{q}_1$ and $\mathbf{q}(t_2) = \mathbf{q}_2$ (or, for the momentum-space path integral, $\mathbf{p}(t_1) = \mathbf{p}_1$ and $\mathbf{p}(t_2) = \mathbf{p}_2$). It was R. P. Feynman himself who introduced a very widely used approximating sequence of path sets which each have paths that *inadvertently deviate* from these physically crucial constraints by *arbitrarily large* amounts, no matter *how far* one goes along that path set sequence [6]. The work of Kerner and Sutcliffe ensures that approximating path sets for the Hamiltonian path integral are free of such egregiously unphysical elements [4].

## The generalized Ehrenfest theorem

Because Dirac's inadequate canonical commutation postulate simply does not determine the quantization of the vast majority of classical dynamical variables, it precludes even *investigation* of the very interesting issue of whether the Ehrenfest theorem relation of the mean dynamical behavior of a quantized system to Hamilton's classical equations of motion for that system is a *universal* fact. The slightly stronger canonical commutation rule of Eq. (9a), however, *always* determines $\overbrace{F(\mathbf{q},\mathbf{p})}$ through Eq. (11b) in conjunction Eq. (10a), (10e), or (10f). From any of these latter three equations, it is readily seen that,

$$\{\widehat{\mathbf{q}},\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\ \}_Q = i\mathbf{l}\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\ \text{ and }\ \{\widehat{\mathbf{p}},\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\ \}_Q = -i\mathbf{k}\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\ , \tag{14a}$$

which, in conjunction with Eq. (11b), implies that,

$$\{\widehat{\mathbf{q}},\ \overbrace{F(\mathbf{q},\mathbf{p})}\ \}_Q = \overbrace{\nabla_{\mathbf{p}}F(\mathbf{q},\mathbf{p})}\ \text{ and }\ \{\widehat{\mathbf{p}},\ \overbrace{F(\mathbf{q},\mathbf{p})}\ \}_Q = -\ \overbrace{\nabla_{\mathbf{q}}F(\mathbf{q},\mathbf{p})}\ . \tag{14b}$$

Eq. (14b), together with the quantum equation of motion (1b) for arbitrary quantized dynamical variables, implies that,

$$d\widehat{\mathbf{q}}/dt = \overbrace{\nabla_{\mathbf{p}}H(\mathbf{q},\mathbf{p})}\ \text{ and }\ d\widehat{\mathbf{p}}/dt = -\ \overbrace{\nabla_{\mathbf{q}}H(\mathbf{q},\mathbf{p})}\ . \tag{15a}$$

Taking arbitrary expectation values of both sides of the two equations in (15a) shows that the Ehrenfest theorem indeed applies *universally* to Hamiltonian dynamical systems,

$$\langle d\widehat{\mathbf{q}}/dt\rangle = \left\langle \overbrace{\nabla_{\mathbf{p}}H(\mathbf{q},\mathbf{p})} \right\rangle\ \text{ and }\ \langle d\widehat{\mathbf{p}}/dt\rangle = -\left\langle \overbrace{\nabla_{\mathbf{q}}H(\mathbf{q},\mathbf{p})} \right\rangle, \tag{15b}$$

which provides an elegant counterpoint to the Correspondence Principle.

## Noninvertibility of quantization

It is also quite interesting to note that the quantization given by Eq. (11b) and either Eq. (10e) or (10f), or, equivalently, by Eq. (13b) or Eq. (13c), has the property that every quantized dynamical variable has *more* than one classical precursor (indeed it has an uncountable infinity of them), which implies that quantization is *not* an invertible procedure. We demonstrate this by exhibiting an uncountable number of nontrivial classical precursors to the identically *zero* quantized dynamical variable, from which the more general result follows because of the *linearity* of quantization. To find this plethora of nontrivial classical precursors to the quantized zero, we merely need to look among the members of the now familiar class of nontrivial dynamical variables of the form $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$, where the vector pair $(\mathbf{k},\mathbf{l})$ runs over $R^{2n}$. The members of this class are obviously quantized by Eq. (10e) or Eq. (10f). The exponentiated operators in the right hand sides of these equations can be combined by the repeated application of the simplest special case of the Campbell-Baker-Hausdorff formula, namely,

$$e^{i\widehat{F}}\ e^{i\widehat{G}} = e^{-\frac{1}{2}c}\ e^{i(\widehat{F}+\widehat{G})}\ \text{ when } [\widehat{F},\widehat{G}] = c\,\mathrm{I}, \text{ where c is a constant.} \tag{16}$$

After consolidation of the exponentiated operators in Eq. (10e) or (10f), the integration over the variable $\alpha$ can be straightforwardly carried out to yield,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = \frac{\sin(\frac{1}{2}\hbar(\mathbf{k}\cdot\mathbf{l}))}{(\frac{1}{2}\hbar(\mathbf{k}\cdot\mathbf{l}))}\ e^{i(\mathbf{k}\cdot\widehat{\mathbf{q}}+\mathbf{l}\cdot\widehat{\mathbf{p}})}\,. \tag{17a}$$

Alternatively, the integration over the variable $\alpha$ in Eq. (13a) can also be straightforwardly carried out, and then followed by the application of a trigonometric identity to yield,

$$\langle\mathbf{q}_2|\ \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\ |\mathbf{q}_1\rangle = \frac{\sin(\frac{1}{2}\hbar(\mathbf{k}\cdot\mathbf{l}))}{(\frac{1}{2}\hbar(\mathbf{k}\cdot\mathbf{l}))}\ e^{i(\mathbf{k}\cdot(\mathbf{q}_1+\mathbf{q}_2))/2}\ \delta^{(n)}(\mathbf{q}_2-\mathbf{q}_1+\hbar\mathbf{l}). \tag{17b}$$

It is clear from both Eq. (17a) and (17b) that,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = 0\ \text{ when } (\mathbf{k}\cdot\mathbf{l}) = 2n\pi/\hbar, \text{ where } n \text{ is any } \textit{nonzero} \text{ signed integer.} \tag{17c}$$

Therefore the quantized zero has an uncountable number of nontrivial classical dynamical precursors, and the same obviously also holds for *any* quantized dynamical variable. One might wonder how this state of affairs can be compatible with the Correspondence Principle—the answer involves the highly nonuniform asymptotic "convergence" which is so typical of the correspondence "limit". One sees from Eq. (17c) and the Schwarz inequality that those classical dynamical variables $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ whose quantizations vanish have $\mathbf{k}$ and $\mathbf{l}$ that satisfy $|\mathbf{k}||\mathbf{l}| \geq 2\pi/\hbar$. Thus, in the limit that $\hbar \to 0$, we will have, for these classical dynamical precursors of quantized zero, that $|\mathbf{k}| \to \infty$ or $|\mathbf{l}| \to \infty$ or both, which causes these classical dynamical variables $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ to oscillate arbitrarily rapidly. It is in this highly nonuniform asymptotic sense that these nontrivial classical precursors of *quantized* zero indeed also "wash out" to zero in the correspondence "limit" $\hbar \to 0$. In the *quantum* world, however, all dynamical variables formally correspond to an infinite number of classical precursors, and thus have a kind of "many classical potentialities" aura which is eminently compatible with at least the *spirit* of the Uncertainty Principle.

## The quantization supposition of Born and Jordan

The quantization rule that is given by Eq. (11b) and either Eq. (10e) or (10f) was historically first presented by Born and Jordan in the paper which sets out their intriguing *variational* development of quantum mechanics [7]. That development absorbs Heisenberg's equation of motion into a variational principle which involves a trace rather than an integral, and has as its consequence commutation rules which not only predated those of Dirac, but are also, in principal, stronger than his—indeed Eq. (14b) can be regarded as the central *consequence* of the systematic Born-Jordan development of quantum mechanics. It so happens, however, that the commutation rules of Eq. (14b) are *also* compatible with quantization rules which *differ* from the "Born-Jordan" one embodied by Eq. (11b) and either Eq. (10e) or (10f)—they are, for example, compatible with Weyl's quite different quantization rule, which we shall discuss next. Thus the discovery of "Born-Jordan" quantization by those authors must be regarded as serendipitously premature—the central tenets of the quantum mechanics they developed, while entirely *compatible* with this quantization, do *not* uniquely *imply* it. (Born and Jordan, however, were apparently unaware of that fact.)

## Contrasts with Weyl's maximally austere quantization

The mathematician Weyl, who took an interest in the nascent quantum mechanics, realized that Dirac's canonical commutation rules failed to pin down the quantization of a general classical dynamical variable, and decided to try his hand at rectifying that situation. Weyl immediately grasped the essence of Eq. (11b); i.e., that with the assumption that quantization is a linear process, one need only determine the quantizations of the classical dynamical Fourier component functions $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ [8]. Mathematician that he was, Weyl applied *no* further *physics-related* considerations *whatsoever* to this problem, but in archtypical fashion aimed to construct the *formally* most straightforward, spartan, and elegant quantization of these Fourier component functions possible. Not surprisingly, given this thrust, Weyl arrived at the quantization postulate,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = e^{i(\mathbf{k}\cdot\widehat{\mathbf{q}}+\mathbf{l}\cdot\widehat{\mathbf{p}})}\,, \tag{18a}$$

to be used in conjunction with Eq. (11b). Application of Eq. (16) shows that Eq. (18a) can also be written as,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}/2}\, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}\, e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}/2} = e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}/2}\, e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}\, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}/2}\,, \tag{18b}$$

which permits direct comparison of Weyl's quantization rule with the Born-Jordan quantization Eq. (10e) or (10f). It is thus seen that Weyl's choice is the *single* most *symmetrical* operator ordering of a quite general class, whereas Born-Jordan quantization is the *equally weighted average* of *all* the operator orderings of that class. Notwithstanding that Weyl undoubtedly achieved his goal of spartan elegance, it becomes clear that the character of the quantum physics can involve yet profounder themes—Born-Jordan quantization's even-handed embrace of *all* orderings of the class in question seems to echo the path integral's sum over *all* applicable paths.

In the manner of Eq. (13b) for Born-Jordan quantization of the classical dynamical variable $F(\mathbf{q},\mathbf{p})$ in configuration representation, one has the following Weyl quantization in configuration representation of $F(\mathbf{q},\mathbf{p})$,

$$\langle\mathbf{q}_2|\ \overbrace{F(\mathbf{q},\mathbf{p})}\ |\mathbf{q}_1\rangle = (2\pi\hbar)^{-n}\int d^n\mathbf{p}\, F(\tfrac{1}{2}(\mathbf{q}_1+\mathbf{q}_2),\mathbf{p})\, e^{i\mathbf{p}\cdot(\mathbf{q}_2-\mathbf{q}_1)/\hbar}\,. \tag{19a}$$

Noting that Weyl's Eq. (18a) exhibits none of the mapping into quantized zero of classical dynamical Fourier components which is so apparent for its Born-Jordan counterpart of Eq. (17a), we need to seriously entertain the possibility that Weyl's quantization is one-to-one and *invertible*. This indeed becomes rather apparent upon reexpressing Eq. (19a) in terms of the variables $\mathbf{q}_- = (\mathbf{q}_2 - \mathbf{q}_1)$ and $\mathbf{q}_+ = \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2)$,

$$\langle \mathbf{q}_+ + \tfrac{1}{2}\mathbf{q}_- | \ \widehat{F(\mathbf{q},\mathbf{p})} \ | \mathbf{q}_+ - \tfrac{1}{2}\mathbf{q}_- \rangle = (2\pi\hbar)^{-n} \int d^n\mathbf{p}\, F(\mathbf{q}_+, \mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{q}_-/\hbar}\,, \tag{19b}$$

which reveals a straightforward Fourier transformation of $F(\mathbf{q},\mathbf{p})$ in just its second independent variable $\mathbf{p}$. This transformation is easily inverted to recover the classical $F(\mathbf{q},\mathbf{p})$ itself,

$$F(\mathbf{q},\mathbf{p}) = \int d^n\mathbf{q}_- \, \langle \mathbf{q} + \tfrac{1}{2}\mathbf{q}_- | \ \widehat{F(\mathbf{q},\mathbf{p})} \ | \mathbf{q} - \tfrac{1}{2}\mathbf{q}_- \rangle \, e^{-i\mathbf{p}\cdot\mathbf{q}_-/\hbar}\,. \tag{19c}$$

Eqs. (19b) and (19c) not only show that Weyl's quantization is one-to-one and invertible, but (19c) also shows that inverse to be given by the well-known "classical Wigner representation" for quantized dynamical variables. The *unique* emergence of the Wigner representation at this juncture is something less than a resounding *physical* endorsement of Weyl's quantization—it is, for example, well-known that there exist positive definite quantum operators which have classical Wigner representations that attain negative values over very significant regions of phase space. Even the straightforward apparent elegance of one-to-one invertible quantization *itself* frays a little around the edges on closer physical scrutiny. If one takes the *unique* classical precursor of each quantized dynamical variable seriously, one will need to wrestle philosophically with the *determinism* of the consequent well-defined classical "shadow world"—note that because the quantized *operators* evolve deterministically under the influence of their Heisenberg equations of motion, their *unique* classical "shadows" will do the same. On balance, it would seem wisest to studiously ignore the antics of Weyl's unique classical precursors. This, however, raises the question of the scientific appropriateness of entertaining at all a theoretical hypothesis that gives rise to a prominent mathematical feature (such as invertibility by Wigner representation), whose most *obvious* physical interpretations conflict with *other* tenets of known physical theory—particularly when there exist *alternative* theoretical hypotheses (e.g., Born-Jordan quantization) which simply do *not* give rise to that mathematical feature and which are, in *addition*, far more extensively based on *physics-related* arguments.

Finally, Weyl's quantization *also* turns out to be the result of a sequence of approximating path sets (due to Feynman [6]) for the Hamiltonian path integral which *inadvertently* has the property that *all* of those sets contain paths which have *arbitrarily large deviations* from the two (physically crucial!) imposed endpoint constraints. Born-Jordan quantization, in utter contrast, is the result of a sequence of approximating path sets (due to Kerner and Sutcliffe [4]) for the Hamiltonian path integral which has the property that *each and every path* scrupulously *conforms* to those two imposed endpoint constraints. Ironically, routine *unsuspecting* use of the endpoint constraint *breaching* Feynman sequence of approximating path sets for Hamiltonian path integrals has long been a mainstay of support for a widely shared assumption that Weyl quantization is probably physically *correct* [6], *notwithstanding* the well-known woes of the thereupon *unavoidable* Wigner representation, which Feynman in particular was even moved to desperately try to make sense of through an attempt to "interpret" negative probabilities!

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