On strategies towards the Riemann Hypothesis : Fractal Supersymmetric QM and a Trace Formula

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Abstract

The Riemann's hypothesis (RH) states that the nontrivial zeros of the Riemann zeta-function are of the form $s_n = 1/2 + i\lambda_n$. An improvement of our previous construction to prove the RH is presented by implementing the Hilbert-Polya proposal and furnishing the Fractal Supersymmetric Quantum Mechanical (SUSY-QM) model whose spectrum reproduces the imaginary parts of the zeta zeros. We model the fractal fluctuations of the smooth Wu-Sprung potential (that capture the average level density of zeros) by recurring to a *weighted superposition* of Weierstrass functions $\sum_p W(x, p, D)$ and where the summation has to be performed over *all* primes p in order to recapture the connection between the distribution of zeta zeros and prime numbers. We proceed next with the construction of a smooth version of the fractal QM wave equation by writing an ordinary Schroedinger equation whose fluctuating potential (relative to the smooth Wu-Sprung potential) has the same functional form as the fluctuating part of the level density of zeros. The second approach to prove the RH relies on the existence of a continuous family of scaling-like operators involving the Gauss-Jacobi theta series. An explicit *completion relation* ("trace formula") related to a superposition of eigenfunctions of these scaling-like operators is defined. If the completion relation is satisfied this could be another test of the Riemann Hypothesis. In an appendix we briefly describe our recent findings showing why the Riemann Hypothesis is a consequence of $\mathcal{CT}$-invariant Quantum Mechanics, because $< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \neq 0$ where s are the complex eigenvalues of the scaling-like operators.

1 Introduction

Riemann's outstanding hypothesis that the non-trivial complex zeros of the zeta-function $\zeta(s)$ must be of the form $s_n = 1/2 \pm i\lambda_n$, is one of most important open problems in pure mathematics. The zeta-function has a relation with the number of prime numbers less than a given quantity and the zeros of zeta are deeply connected with the distribution of primes [1]. References [2] are devoted to the mathematical properties of the zeta-function.

The RH has also been studied from the point of view of mathematics and physics [22], [4], [5], [6] among many others. We found recently a novel physical interpretation of the location of the nontrivial Riemann zeta zeros which

corresponds to the presence of tachyonic-resonances/tachyonic-condensates in bosonic string theory. If there were zeros outside the critical line violating the RH these zeros do not correspond to poles of the string scattering amplitude [8]. The spectral properties of the λ_n's are associated with the random statistical fluctuations of the energy levels (quantum chaos) of a classical chaotic system [25]. Montgomery [9] has shown that the two-level correlation function of the distribution of the λ_n's coincides with the expression obtained by Dyson with the help of random matrices corresponding to a Gaussian unitary ensemble.

Wu and Sprung [10] have numerically shown that the lower lying non-trivial zeros can be related to the eigenvalues of a Hamiltonian whose potential has a $fractal$ shape and fractal dimension equal to $D = 1.5$. Wu and Sprung have made a very insightful and key remark pertaining the conundrum of constructing a one-dimensional integrable and time-reversal quantum Hamiltonian to model the imaginary parts of the zeros of zeta as an eigenvalue problem. This riddle of merging chaos with integrability is solved by choosing a fractal local potential that captures the chaotic dynamics inherent with the zeta zeros.

In section **2** we will *generalize* our previous strategy [3] to prove the RH based on extending the Wu and Sprung QM problem by invoking a judicious *superposition* of an infinite family of fractal Weierstrass functions parametrized by the prime numbers p in order to *improve* the expression for the fractal potential. By a fractal SUSY QM model studied here we do not mean systems with fractional supersymmetries which are common in the string and M-theory literature, but a Hamiltonian operator that admits a factorization into two factors involving fractional derivative operators whose fractional (irrational) order is one-half of the fractal dimension of the fractal potential. A model of fractal spin has been constructed by Wellington da Cruz [21] in connection to the fractional quantum Hall effect based on the filling factors associated with the Farey fractions. The self-similarity properties of the Farey fractions are widely known to posses remarkable fractal properties [23]. For further details of the validity of the RH based on the Farey fractions and the Franel-Landau [24] shifts we refer to the literature on the zeta function.

In section **3** we start by reviewing our previous work [7] based on a family of scaling-like operators in one dimension involving the Gauss-Jacobi theta series before introducing the *novel completion relations* ("trace formulae") that have *not* been discussed before (to our knowledge). The last part of section **2** is new material which did not appear in [7]. The inner product of the eigenfunctions $\Psi_s(t;l)$ of these scaling-like operators is given by $(2/l)Z\left[\frac{2}{l}(2k-s^*-s)\right]$; where $Z(s)$ is the fundamental Riemann completed zeta function and $(l+4)/8 = k$. There is a one-to-one correspondence among the zeta zeros s_m (such that $Z[s_m] = 0$) with the eigenfunctions Ψ_{s_m} of the latter scaling-like operators that permits to implement a resolution of the identity via the following completion relation ("trace formula") in QM and given by $\sum_n \varphi_n^*(t';l')\,\varphi_n(t;l) \sim C_{l,l'}\delta(t-t')$. The summation must be taken over *all* the ortho-normal basis elements φ_n which are given by a superposition of the eigen-functions Ψ_{s_m} of the scaling-like operators. If the completion relation is satisfied for those states φ_n given by a superposition of Ψ_{s_m}, where s_m belong to the zeros living in the critical line

$s_m = \frac{1}{2} + i\lambda_m$, this would imply that there are no missing nontrivial zeros. However, if the completion relation is not satisfied, there must be other missing zeros that were omitted in the construction of φ_n and, consequently, there must be zeros s_m *outside* the critical line (and inside the critical strip) that should have been included in the summation. This would be another test of the Riemann Hypothesis.

In an appendix we briefly describe our recent findings [37] showing why the Riemann Hypothesis is a consequence of $\mathcal{CT}$-invariant Quantum Mechanics, because $< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \neq 0$ where s are the complex eigenvalues of the scaling-like operators.

2 The Operator that yields the Riemann zeros

Now let us turn to the fractal SUSY QM problem associated to the Riemann Hypothesis. Armitage [14], considered that the RH can be expressed in terms of diffusion processes with an imaginary time. In this way the Hamiltonian of some QM system could be constructed, which in turn implements the Hilbert-Polya's original program.

A numerical exploration of the Hilbert-Polya idea was done by Wu and Sprung [10]. The potential found in [10] has random oscillations around an average value, the average potential allowed them to construct a conventional Hamiltonian whose density of states coincides with the average distribution of the imaginary parts of the Riemann's zeta non trivial zeros. The fluctuations are necessary in order to make the individual eingenvalues fit a set of such zeros within a prescribed error bound. They found that the imaginary parts of the 500 lower lying nontrivial Riemann zeros can be reproduced by a one-dimensional local-potential model, and that a close look at the potential suggests that it has a fractal structure of dimension $D = 1.5$. The references [15],[16] deal in particular with the spectrum of fractal strings and the zeros of the Riemann zeta function.

Here, we construct an operator based on a Fractal SUSY-QM model. In ordinary SUSY-QM two isospectral operators $H^{(+)}$ and $H^{(-)}$ are defined in terms of the so called SUSY-QM potential. A SUSY-QM model was proposed originally in [26] based on the pioneering work of B. Julia [17], where the zeta-function and its fermionic version were related to the partition function of a system of p-adic oscillators in thermal equilibrium at a temperature T. The fermionic zeta-function has zeros at the same positions of the ordinary Riemann function plus a zero at $1/2+0i$, this zero is associated to the SUSY ground state (see also the reference [29]). The fermionic version of the zeta functions is defined by

$$Z_f = \frac{\zeta(s)}{\zeta(2s)} = \sum_n \frac{|\mu(n)|}{n^s}, \tag{1}$$

where $\mu(n)$ is the Mobius function. Spector [31] has shown why the Mobius inversion function of number theory can be interpreted as the operator $(-1)^F$ (

Witten index where F is the fermion number) in QFT. Physical interpretations of various properties of the Mobius function are provided due to the central role played by Supersymmetry and the Witten index. Supersymmetry does have an important place in our construction below.

Here we consider a fractal potential defined by a set unknown phases, to be determined a posteriori after using the CBC formula (Comtet-Bandrauk-Campbell) [13], associated with a Weierstrass function, continuous but nowhere differentiable. A fractal SUSY-QM Hamiltonian, using fractional derivatives, can be constructed in principe, whose eigenvalues coincide with the imaginary parts of the nontrivial zeros of the zeta, λ_n. The fractal dimension of the potential is $D = 1.5$ and the sought-after phases will be determined by solving the inverse eigenvalue problem via the CBC formula..

Our previous ansatz for our fractal SUSY-QM potential was based on the Weierstrass-Madelbrot fractal function, continuous and nowhere differentiable functions

$$W(x, \gamma, D, \alpha_n) = \sum_{n=0}^{\infty} \frac{1 - e^{ix\gamma^n}}{\gamma^{n(2-D)}} e^{i\alpha_n}, \tag{2}$$

n are integers, the powers γ^n are the corresponding set of frequencies and the α_n are the sought-after phases. The expansion (2) is convergent if $1 < D < 2$ and $\gamma > 1$. For these values of the parameters the function W is continuous but nowhere differentiable and has D for fractal dimension [18],[19]. In this work we shall use for the frequencies suitable powers p^n of a given prime p number and perform a superposition of all the Weiertrass functions by summing over all the primes p.

The aim is to relate the SUSY potential-squared Φ^2 to the fractal potential. The choice for the $\Phi^2(x)$ expression that appears in the fractal version of the CBC formula will be comprised of a smooth part given by the Wu-Sprung potential $V_{WS}(x)$ plus an oscillatory fluctuating Weierstrass part of the form

$$W(x, D, \alpha_m) = W_0 \sum_p \frac{1}{p^{x/2}} \sum_m \frac{1}{p^{(2-D)m}} \, sin \, (p^m x \, + \alpha_m). \tag{3}$$

where we must symmetrize the function $W(x, D, \alpha_m)$ with respect to the x variables and an additive constant ϕ_o has to be included also in order to have a vanishing Φ^2 at the origin $x = 0$. Supersymmetry requires that the Φ^2 is symmetric and vanishes at the origin.

In [10] it was shown that the smooth value of the potential V_{WS} that reproduces the average level density of zeta zeros can be obtained as solution of the Abel integral equation. The Wu-Sprung potential $V_{WS}(x)$ is given implicitly as:

$$x = x(V) = \frac{V_o^{1/2}}{\pi} \left[(y-1)^{1/2} \ln \frac{V_o}{2\pi e^2} + y^{1/2} \ln \frac{y^{1/2} + (y-1)^{1/2}}{y^{1/2} - (y-1)^{1/2}} \right]. \tag{4}$$

where the rescaled variable is $y = V/V_o$, and $V_o = 3.10073 \, \pi$.

The inversion $V = V(x)$ of the Wu-Sprung expression $x = x(V)$ for the potential can be attained upon using the Interpolation command of Mathematica

as show by [33] or by recurring to the series expansion [34]

$$V_{WS}(x) = V_o + \sum_{k=1}^{\infty} a_k \ (\pi x)^{2k} \ \omega^{2k-1} \ (-V_o)^{1-k} \ ; \quad \omega = [\ ln \ (\frac{V_o}{2\pi}) \ ]^{-1} \tag{5}$$

The asymptotic behaviour $|x| \to \infty$ of the series expansion behaves as [34]

$$V(x) = \frac{\pi^2 x^2}{4} \ [\ LW(\sqrt{\frac{\pi}{2}} \frac{|x|}{e}) \ ]^{-2}. \tag{6}$$

where LW is the celebrated Lambert-W function, also called the omega function, that is the inverse function of $f(W) = We^W$.

With the SUSY potential Φ at hand one may construct the following fractal SUSY Schrödinger equation associated with the Hamiltonian

$$[\ (\mathcal{D}^\beta + \Phi) \ (-\mathcal{D}^\beta + \Phi) \ ] \ \Psi_j(x) \ = \lambda_j \ \Psi_j(x). \quad and \quad \beta = \frac{D}{2} = \frac{3}{4}. \tag{7}$$

where one sets $\hbar = 2m = 1 \Rightarrow \hbar/mc = 2$, a Compton wavelength of 2 units means that excluding the primes $p = 1, 2$ the minimal spacing among primes is 2 units. The superpotential squared is finally given by

$$\Phi^2(x) = V_{WS}(x) \ + \ \frac{1}{2}[\ W(x) + W(-x) \ ] \ + \phi_o. \tag{8}$$

such that the turning points are defined by $\Phi^2(x = x_j) = \lambda_j$. ϕ_o is suitable constant to ensure that Φ^2 vanishes at $x = 0$. The leading term $V_{WS}(x)$ is the Wu-Sprung potential after performing the inversion procedure and the fractal fluctuating part of the superpotential-squared $\Phi^2(x)$ is given by a superposition of weighted Weiertrass functions

$$W(x) = W_0 \ \sum_p \ \frac{1}{p^{x/2}} \ \sum_m \ \frac{1}{p^{(2-D)m}} \ sin \ (p^m x \ + \alpha_m). \tag{9}$$

that is based on the Riemann-von Mangoldt expression for the fluctuations

$$\mathcal{N}_{fluctuating}(E) = \frac{1}{\pi} \ lim_{\epsilon \to 0} \ Im \ ln \ \zeta(\frac{1}{2} + iE + \epsilon) =$$

$$-\frac{1}{\pi} \sum_p \ \sum_m \ \frac{1}{m \ p^{m/2}} \ sin \ (\ Em \ log(p) \ ).$$

The expression above is valid if the zeta zeros are simple. A standard form of the Weierstrass function is

$$W(x, D, \gamma) = \sum_m \ \frac{1}{\gamma^{(2-D)m}} \ sin \ (\gamma^m x \ + \alpha_m) \tag{10}$$

where $\gamma > 1$ is an arbitrary parameter. Our ansatz for the fluctuating part of the potential is based on replacing γ by a prime p and summing over all primes

p. One can notice that the fractal dimension of $D = 1.5$ is perfectly consistent with the behavior of the powers of $p^{m/2}$ of the denominators in the fluctuating part $\mathcal{N}_{fluctuating}(E)$ because when $D = 1.5$, the Weiertrass function has the correct matching powers $p^{(2-D)m} = p^{\frac{m}{2}}$ as well in the denominator.

The reason we performed a weighted superposition of $W(x, D, p)$ with prefactors of $p^{-x/2}$ as explained by Tricot [32] was to modulate the amplitudes of the fractal fluctuations (w.r.t the smooth part of the potential) in such a way that these fluctuations will fit precisely the numerical behavior of the graphs associated with the fitted zeta zeros by Wu and Sprung [1]

The Riemann-Liouville fractal left and right derivatives are defined respectively

$$D_L^{\beta} F(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{-\infty}^{x} \frac{F(x')}{(x-x')^{\beta}} dx'. \tag{11}$$

$$D_R^{\beta} F(x) = \frac{1}{\Gamma(1-\beta)} (-\frac{d}{dx}) \int_{x}^{\infty} \frac{F(x')}{(x'-x)^{\beta}} dx'. \tag{12}$$

For consistency purposes, the kinetic terms must be given by fractal-like operators since the shape of the potential was fractal . This is the reason why fractal derivatives were introduced to reconcile the fact that the potential was a fractal with dimension $D = 1.5$. The fractal derivative operators are *not* Hermitian. Thus, strictly speaking, we are deviating from the Hilbert-Polya proposal which assumed a Hermitian operator. The closest analog of Hermitian operators involve a combination of both right and left derivatives.

$$[\ (\mathcal{D}_L^{\beta} + \Phi)\ (-\mathcal{D}_L^{\beta} + \Phi)\ +\ L \leftrightarrow R\ ]\ \Psi_j(x)\ = \lambda_j\ \Psi_j(x). \quad \beta = \frac{D}{2} = \frac{3}{4}. \tag{13}$$

The fractal analog of the CBC formula associated with the fractal SUSY-QM equation

$$[\ (\mathcal{D}_L^{\beta} + \Phi)\ (-\mathcal{D}_L^{\beta} + \Phi)\ ]\ \Psi_j(x)\ = \lambda_j\ \Psi_j(x). \quad \beta = \frac{D}{2} = \frac{3}{4}. \tag{14}$$

is given by the integrals

$$I_j(x_j, \lambda_j)\ = \frac{2}{\Gamma(\beta)} \int_{-x_j}^{x_j} \frac{[\ \lambda_j\ - \Phi^2(x')\ ]^{1/2}}{(x_j - x')^{1-\beta}} dx' = \ j\pi. \quad j = 1, 2, 3, ... \infty \tag{15}$$

If we had used right fractal derivatives instead of left fractal derivatives, one must use $(x' - x_j)^{1-\beta}$ in the denominator. The turning points x_j are defined by $\Phi^2(x = x_j) = \lambda_j$. Due to the oscillatory behavior of the $\sum_p W(x, p, D)$ terms, defining the fractal fluctuations of the potential, there are multiple turning points [2] that solve the condition $\Phi^2(x = x_j) = \lambda_j$. One can remove this

[1] We thank Paul Slater for referring us to Tricot's observation

[2] We thank Michael Trott for pointing this out

degeneracy in the number of turning points by selecting those particular values which optimize the fractal analog of the CBC formula.

The unknown parameters to solve for from the coupled set of equations, given by the CBC integrals and the turning points $\Phi^2(x = x_j) = \lambda_j$, are the phases α_m, and the overall global scaling parameter W_o [3] . The additive constant ϕ_o that ensures that $\Phi^2(x = 0) = 0$ is not an independent parameter since it is given in terms of the α_m and W_o. There is a one-to-one correspondence among the (imaginary parts of) zeta zeros λ's and the phases α's. From the well known relationship between the Spectral Statistics of Random Matrix Models based on the Gaussian Unitary Ensembles (GUE) and the Montgomery-Dyson nearest-neighbour-spacing correlations of the imaginary parts of the zeta zeros, one would expect that the phases α_m should obey similar correlation functions associated (for instance) with Dyson's Circular Unitary Ensemble.

It would be astonishing if the phases turned out to deterministic, namely if they obeyed a relation like $\alpha_m = F(m)$; in particular, a linear relationship $\alpha_m = m\mu$ studied by Berry and Lewis [18] in their analysis of the Mandelbrot-Weierstrass function . In this very unlikely case when $\phi_m = m\mu$ one would need to fit only *two* parameters μ and W_o in order to generate *all* the zeta zeros for the spectrum. If this turns out to be the case it certainly would be a truly remarkable finding. Whether or not a preferred set of values for the phases, like $\alpha_m = 0, \frac{\pi}{2}$ occurs is unknown at the moment. In the analysis by Slater [33] of our initial model, he found that the zero phases occurred with a high incidence number that could have been due to the boundary chosen for the domain of the values of the angles; i.e $0, 2\pi$. Nevertheless, after a closer study, it is our belief that the phases should be randomly distributed according to Dyson's Circular Unitary Ensemble, in the same fashion that the spectral statistics of the zeta zeros obeyed the Montgomery-Dyson nearest-neighbour-spacing correlation function associated with a GUE and given by the celebrated expression

$$1 - \frac{sin^2(\pi x)}{(\pi x)^2} \tag{16}$$

The smooth version of the Fractal QM wave equation is an ordinary Schroedinger equation involving a truly Hermitian operator

$$[\ -\frac{d^2}{dx^2}\ + V_{WS}\ + V_{fluctuating}\ ]\ \Psi_n(x) = E_n\ \Psi_n(x) \tag{17}$$

where the superposition of fractal Weiertrass functions is replaced now by a non-fractal function $V_{fluctuating}$ that is chosen to have the same functional form as $\mathcal{N}_{fluctuating}(E)$ but with the inclusion of phases

$$V_{fluctuating}(x) = \frac{V_o}{2} \sum_p \sum_m \frac{1}{m\ p^{m/2}}\ sin\ (\ m\ x\ log(p) + \alpha_m\ ) + x \rightarrow -x. \tag{18}$$

[3] Paul Slater suggested to us to introduce this global scaling factor

The open problem remains to verify whether or not we are able to reproduce the λ_n for the eigenvalues without the need to solve the Fractal SUSY-QM wave equation comprised of fractal derivatives and a fractal potential term.

We must reiterate once again that if one introduces a fractal potential based on the (superposition of) Weiertrass functions one should also introduce fractal derivatives corresponding to the fractal kinetic terms associated with the fractal QM wave equation. This procedure must be clearly *distinguished* from the procedure used by Wu and Sprung [10] in which they fitted the first 500 zeros using an ordinary Schroedinger equation via a numerical Numerov method on a grid. Zyl and Hutchinson [20] used the dressing transformation method (well known in the study of nonlinear solitons) involving an ordinary first order nonlinear differential equation that allows to fit a larger number of zeros $N = 40000$ and all of these authors concluded that the fitted potential was a fractal of dimension $D = 1.5$.

3 The Scaling Operators related to the Gauss-Jacobi Theta series and the Riemann zeros

Our second proposal towards a proof of the RH begins with the introduction of the appropriate generalized scaling operator D_1 [7]

$$D_1 = -\frac{d}{d\ln t} + \frac{dV}{d\ln t} + k, \tag{19}$$

such that its eigenvalues s are complex-valued, and its eigenfunctions are given by

$$\psi_s(t) = t^{-s+k} e^{V(t)}. \tag{20}$$

D_1 is not self-adjoint since it is an operator that does not admit an adjoint extension to the whole real line and its eigenvalues are complex valued numbers s. We also define the operator dual to D_1 as follows,

$$D_2 = \frac{d}{d\ln t} + \frac{dV}{d\ln t} + k, \tag{21}$$

that is related to D_1 by the substitution $t \to 1/t$ and by noticing that

$$\frac{dV(1/t)}{d\ln(1/t)} = -\frac{dV(1/t)}{d\ln t},$$

where $V(1/t)$ is not equal to $V(t)$.

Since $V(t)$ can be chosen arbitrarily, we choose it to be related to the Bernoulli string spectral counting function, given by the Jacobi theta series,

$$e^{2V(t)} = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t^l} = 2\omega(t^l) + 1. \tag{22}$$

This choice is justified in part by the fact that Jacobi's theta series ω has a deep connection to the integral representations of the Riemann zeta-function [27].

Latter arguments will rely also on the following related function defined by Gauss,

$$G(1/x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} = 2\omega(1/x) + 1, \tag{23}$$

where $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Then, our V is such that $e^{2V(t)} = G(t^l)$. We defined x as t^l. We call $G(x)$ the Gauss-Jacobi theta series (GJ).

Defining $H_A = D_2 D_1$ and $H_B = D_1 D_2$ we were able to show in [7], due to the relation $\Psi_s(1/t) = \Psi_{1-s}(t)$ based on the properties of the Gauss-Jacobi series, that

$$H_A\ \Psi_s(t) = s(1-s)\Psi_s(t). \quad H_B\ \Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t}). \tag{24}$$

Therefore, despite that H_A, H_B are not Hermitian they have the same spectrum $s(1-s)$ which is real-valued only in the critical line and in the real line.

We have to consider a family of D_1 operators, each characterized by two real numbers k and l which can be chosen arbitrarily. The measure of integration $d\ln t$ is scale invariant. Let us mention that D_1 is also invariant under scale transformations of t and $F = e^V$ since $dV/(d\ln t) = d\ln F/(d\ln t)$. In [30] only one operator D_1 is introduced with the number $k = 0$ and a different (from ours) definition of F.

We define the inner product as follows,

$$\langle f|g\rangle = \int_0^{\infty} f^* g \frac{dt}{t}. \tag{25}$$

Based on this definition the inner product of two eigenfunctions of D_1 is

$$\begin{aligned} \langle \psi_{s_1}|\psi_{s_2}\rangle &= \int_0^{\infty} e^{2V} t^{-s_{12}+2k-1} dt \\ &= \frac{2}{l} Z\left[\frac{2}{l}(2k - s_{12})\right], \end{aligned} \tag{26}$$

where we have denoted $s_{12} = s_1^* + s_2 = x_1 + x_2 + i(y_2 - y_1)$ and used the expressions (22) and (28).

We notice that

$$\langle \psi_{s_1}|\psi_{s_2}\rangle = \langle \psi_{s_o}|\psi_s\rangle,$$

thus, the inner product of ψ_{s_1} and ψ_{s_2} is equivalent to the inner product of ψ_{s_o} and ψ_s, where $s_o = 1/2 + i0$ and $s = s_{12} - 1/2$. The integral is evaluated by introducing a change of variables $t^l = x$ (which gives $dt/t = (1/l)dx/x$) and using the result provided by the equation (23), given in Karatsuba and Voronin's

book [2]. The fundamental Riemann function Z in (28) can be expressed in terms of the Jacobi theta series, $\omega(x)$ defined by (22)

$$\begin{aligned}\int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{s/2-1} dx &= \\ &= \int_0^\infty x^{s/2-1}\omega(x)dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty [x^{s/2-1} + x^{(1-s)/2-1}]\omega(x)dx \\ &= Z(s) = Z(1-s),\end{aligned} \tag{27}$$

where the fundamental Riemann (completed zeta) function is

$$Z(s) \equiv \pi^{-s/2}\, \Gamma(\frac{s}{2})\, \zeta(s), \tag{28}$$

which obeys the functional relation $Z(s) = Z(1-s)$.

Since the right-hand side of (27) is defined for all s this expression gives the analytic continuation of the function $Z(s)$ to the entire complex s-plane [2]. In this sense the fourth "=" in (27) is not a genuine equality. Such an analytic continuation transforms this expression into the inner product, defined by (26).

A recently published report by Elizalde, Moretti and Zerbini [11] (containing comments about the first version of our paper [7]) considers in detail the consequences of the analytic continuation implied by equation (27). One of the consequences is that equation (26) loses the meaning of being a scalar product. Arguments by Elizalde *et al.* [11] show that the construction of a genuine inner product is impossible.

Therefore from now on we will loosely speak of a "scalar product" realizing that we do not have a scalar product as such. The crucial problem is whether there are zeros outside the critical line (but still inside the critical strip) and not the interpretation of equation (26) as a genuine inner product. Despite this, we still rather loosely refer to this mapping as a scalar product. The states still have a real norm squared, which however need not to be positive-definite.

Here we must emphasize that our arguments do not rely on the validity of the zeta-function regularization procedure [12], which precludes a rigorous interpretation of the right hand side of (27) as a scalar product. Instead, we can simply replace the expression "scalar product of ψ_{s_1} and ψ_{s_2}" by the map S of complex numbers defined as

$$\begin{aligned} S: \quad \mathcal{C}\otimes\mathcal{C} &\rightarrow \mathcal{C} \\ (s_1, s_2) &\mapsto S(s_1, s_2) = \frac{2}{l} Z(as+b), \end{aligned} \tag{29}$$

where $s = s_1^* + s_2 - 1/2$ and $a = -2/l; b = (4k-1)/l$. In other words, our arguments do not rely on an evaluation of the integral $\langle \psi_{s_1} | \psi_{s_2} \rangle$, but only on the mapping $S(s_1, s_2)$, defined as the finite part of the integral (26). The kernel of the map $S(s_1, s_2) = \frac{2}{l} Z(as+b)$ is given by the values of s such that $Z(as+b) = 0$, where $\langle \psi_{s_1} | \psi_{s_2} \rangle = \langle \psi_{s_o} | \psi_s \rangle$ and $s_o = 1/2 + i0$. Notice that $2b + a = 4(2k-1)/l$. We only need to study the "orthogonality" (and symmetry) conditions with respect to the "vacuum" state s_o to prove why $a + 2b = 1$. By symmetries of the "orthogonal" states to the "vacuum" we mean always the symmetries of the kernel of the S map.

The "inner" products are trivially divergent due to the contribution of the $n = 0$ term of the GJ theta series in the integral (26). From now on, we denote for "inner" product in (26) and (29) as the finite part of the integrals by simply removing the trivial infinity. We shall see in the next paragraphs, that this "additive" regularization is in fact compatible with the symmetries of the problem.

We can easily show that if a and b are such that $2b + a = 1$, then the symmetries of all the states ψ_s orthogonal to the "vacuum" state are preserved by any map S, equation (29), which leads to $Z(as + b)$. In fact, if the state associated with the complex number $s = x + iy$ is orthogonal to the "vacuum" state and the "scalar product" is given by $Z(as+b) = Z(s')$, then the Riemann zeta-function has zeros at $s' = x' + iy'$, s'^*, $1 - s'$ and $1 - s'^*$. If we equate $as + b = s'$, then $as^* + b = s'^*$. Now, $1 - s'$ will be equal to $a(1 - s) + b$, and $1 - s'^*$ will be equal to $a(1 - s^*) + b$, if, and only if, $2b + a = 1$. Therefore, all the states ψ_s orthogonal to the "vacuum" state, parameterized by the complex number $1/2 + i0$, will then have the same symmetry properties with respect to the critical line as the nontrivial zeros of zeta.

Notice that our choice of $a = -2/l$ and $b = (4k-1)/l$ is compatible with this symmetry if k and l are related by $l = 4(2k-1)$. Conversely, if we assume that the orthogonal states to the "vacuum" state have the same symmetries of $Z(s)$, then a and b must be constrained to obey $2b + a = 1$. It is clear that a map with arbitrary values of a and b does not preserve the above symmetries and for this reason we have now that $s = as + b = a(s - 1/2) + 1/2$

Therefore, concluding, the inner product $\langle \psi_{s_1} | \psi_{s_2} \rangle$ is equal to $\langle \psi_{s_o} | \psi_s \rangle = \frac{2}{l}\ Z[a(s-1/2)+1/2] = \frac{2}{l} Z(s')$ where $s = s_1^* + s_2 - 1/2$. For example, if we set $l = -2$, then $k = 1/4$, $a = 1$, $b = 0$, and consequently $s' = s$ in this case; *i.e.* the position of the zeros $(s_n)' = s_n$ coincide with the location of the orthogonal states ψ_{s_n} to the reference state ψ_{s_o}.

To sum up, for states Ψ_s such that $s = \frac{1}{2} + i\lambda$ lies in the critical line one learns that

$$\langle \psi_{s_1} | \psi_{s_2} \rangle \ = \ (\frac{2}{l})\ Z\ [\ \frac{2}{l}(2k - (s_1^* + s_2))\ ]\ =\ (\frac{2}{l})\ Z\ [\ \frac{1}{2} + \frac{2i}{l}(\lambda_1 - \lambda_2)\ ] \quad (30)$$

and one finds that the states Ψ_s have equal norm

$$< \Psi_s\ |\Psi_s > \ = \ \frac{2}{l}\ Z\ [\frac{1}{2}] \quad (31)$$

Since $Z[\frac{1}{2}] < 0$ in order to have a positive definite norm one requires to choose $l < 0$. $Z(s)$ is real-valued along the critical line because when $s = \frac{1}{2} + i\lambda \Rightarrow 1 - s = s^*$ then as a result of the functional equation one must have $Z(s) = Z(1-s) = Z(s^*) = (Z(s))^*$ which implies that $Z(1/2 + i\lambda) = real$. Let us follow the Gram-Schmidt procedure to construct an orthonormal basis φ_n from the family of functions Ψ_{s_m}. We begin with the first state $\varphi_1 = \Psi_{s_1}$ associated with the first zero $s_1 = \frac{1}{2} + i\lambda_1$. The next orthonormal state is

$$\varphi_2 = a_{21}\ \Psi_{s_1}\ +\ a_{22}\ \Psi_{s_2} \tag{32}$$

such that

$$< \varphi_2 \mid \varphi_2 > \ = \ 1 \ = \ (\ |a_{21}|^2 + |a_{22}|^2\ )\ (\frac{2}{l})\ Z\ [\frac{1}{2}]\ + \tag{33}$$

$$a_{21}^*\ a_{22}\ (\frac{2}{l})\ Z\ [\ \frac{1}{2} + \frac{2i}{l}(\lambda_1 - \lambda_2)\ ] + complex\ conjugate$$

and

$$< \varphi_2 \mid \varphi_1 > \ = \ 0 \ \Rightarrow 0 \ = \ a_{21}\ (\frac{2}{l})\ Z\ [\frac{1}{2}]\ +\ a_{22}\ (\frac{2}{l})\ Z[\ \frac{1}{2} + \frac{2i}{l}(\lambda_1 - \lambda_2)\ ] \tag{34}$$

yield the defining relations for the coefficients a_{21}, a_{22}. One continues with

$$\varphi_3 = a_{31}\ \Psi_{s_1} + a_{32}\ \Psi_{s_2} + a_{33}\ \Psi_{s_3} \tag{35}$$

obeying

$$< \varphi_3 \mid \varphi_3 > \ = \ 1. \quad < \varphi_3 \mid \varphi_1 > \ = \ 0. \quad < \varphi_3 \mid \varphi_2 > \ = \ 0 \tag{36}$$

that yield the defining relations for the coefficients a_{31}, a_{32}, a_{33}. One continues to follow this procedure

$$\varphi_n \ = \ \sum_{m=1}^{m=n}\ a_{mn}\ \Psi_{s_m} \ = \ a_{1n}\ \Psi_{s_1} + a_{2n}\ \Psi_{s_2} + a_{3n}\ \Psi_{s_3} + ... \ + a_{mn}\ \Psi_{s_m} \tag{37}$$

all the way to $m \to \infty$. Since $Z(1/2 + iy) = real$ this means there could exist real-valued solutions for all the coefficients a_{mn} which simplifies enormously the calculations.

The explicit form of the eigenfunctions, after using the relation $l = 8k - 4$, is

$$\Psi_{s_m}(t;l) = \ t^{-(1/2 + i\lambda_m) + (l+4)/8}\ \sqrt{\sum_{n=-\infty}^{n=\infty}\ e^{-\pi n^2 t^l}} \tag{38}$$

one can notice that the coefficients $a_{mn}(l)$ and the orthonormal basis $\varphi_n(t)$ must depend explicitly on the values of l since the eigenfunctions depend on l. Hence, one should write in general $\varphi_n(t;l)$.

When $l=-2 \Rightarrow a=-2/l=1; b=0$ one has states of *positive* norm

$$< \Psi_s \, |\Psi_s > \; = \; \frac{2}{l} \; Z \, [\frac{1}{2}] \; = \; - \, Z \, [\frac{1}{2}] \; > \; 0 \tag{39}$$

and the orthogonality condition with the reference "ground " state $\Psi_{1/2+i0}$ is

$$< \Psi_{1/2+i0} \; (t; l=-2) \; |\Psi_{s_m} \; (t; l=-2) > \; = \; Z \; [s_m] = 0 \tag{40}$$

Despite the fact that we don't have a true scalar product as such and that $\mathcal{D}_1, \mathcal{D}_2$ are not self-adjoint operators since they don't admit an adjoint extension to the whole real line, we may define a *completion relation* (a "trace formula") based on the resolution of the identity operator in QM and ask if the analog of the following completeness relation holds in the case $l=-2$

$$\sum_n \; \varphi_n^*(t') \; \varphi_n(t) \; = \sum_{n=1}^{n=\infty} \; \sum_{m=1}^{m=n} \; a_{mn}^* \; a_{mn} \; \Psi_{s_m}^*(t') \; \Psi_{s_m}(t) \; \sim \delta(t-t') \tag{41}$$

where

$$\Psi_{s_m}(t; l=-2) = \; t^{-(1/2+i\lambda_m)+(1/4)} \; \sqrt{\sum_{n=-\infty}^{n=\infty} \; e^{-\pi n^2 t^{-2}}} \tag{42}$$

In general for any given value of l one must have

$$\sum_n \; \varphi_n^*(t'; l) \; \varphi_n(t; l) \; = \sum_{n=1}^{n=\infty} \; \sum_{m=1}^{m=n} \; a_{mn}^*(l) \; a_{mn}(l) \; \Psi_{s_m}^*(t'; l) \; \Psi_{s_m}(t; l) = C_l \delta(t-t') \tag{43}$$

where C_l is a constant whose value depends on l.

One could also study the *completion relation* for different values l, l' defined as

$$\sum_n \; \varphi_n^*(t'; l') \; \varphi_n(t; l) \; = \sum_{n=1}^{n=\infty} \; \sum_{m=1}^{m=n} \; a_{mn}^*(l') \; a_{mn}(l) \; \Psi_{s_m}^*(t'; l') \; \Psi_{s_m}(t; l) \tag{44}$$

and verify whether or not it is proportional to $\delta(l-l') \; \delta(t-t')$ or to $C_{ll'} \delta(t-t')$. The *completion relations* are not the same as a trace formulae in number theory, nevertheless it would be instructive to explore the connections to Connes hypothetical trace formula [6] related to the RH; the Deninger's Lefschetz trace formulae; the Selberg trace formulae; the Gutzwiller trace formulae, ... [29].

If the completion relations (41, 43) are not satisfied this could be a signal that there should be missing zeros; i.e. zeros outside the critical line (but inside the critical strip) that were not included in the summation in eqs-(41,43). Naturally, one would need to include all of the infinite number of zeros in the critical line to evaluate explicitly the above summations (41,43). However, despite this technicality it does not preclude us from postulating and writing explicitly such completion relation and performing a numerical analysis over the known zeros.

Appendix : The Riemann Hypothesis is a consequence of $\mathcal{CT}$-invariant Quantum Mechanics

In this appendix we will briefly announce our recent findings [37] showing the equivalence between the RH and $\mathcal{CT}$-invariant Quantum Mechanics that relies also on a family of theta series and their Mellin transforms. Given

$$G(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\omega(x) + 1. \qquad (A.1)$$

where $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. The Gauss-Jacobi series obeys the relation

$$G(\frac{1}{x}) \; = \; \sqrt{x}\; G(x). \qquad (A.2)$$

Then V is chosen such that $e^{2V(t)} = G(t^l)$. One defined x as t^l. Upon defining $H_A = D_2 D_1$ and $H_B = D_1 D_2$ we were able to show in [7], due to the relation $\Psi_s(1/t) = \Psi_{1-s}(t)$ based on the properties of the Gauss-Jacobi series $G(\frac{1}{x}) \; = \; \sqrt{x}\; G(x)$, that

$$H_A \; \Psi_s(t) = s(1-s)\Psi_s(t). \quad H_B \; \Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t}). \qquad (A.3)$$

Therefore, despite that H_A, H_B are not Hermitian they have the same spectrum $s(1-s)$ which is real-valued only in the critical line and in the real line. Had H_A, H_B been Hermitian one would have had an immediate proof of the RH. Hermitian operators have a real spectrum, hence if $s(1-s)$ is real this means that $1 - s = c\; s^*$, for a real valued c, and $1 - s^* = c\; s$. Subtracting :

$$1 - s - (1 - s^*) = -(s - s^*) = -c\; (s - s^*) \; \Rightarrow \; (s - s^*)\; (1 - c) = 0. \qquad (A.4)$$

If $c \neq 1$ then one has $s - s^* = 0 \Rightarrow s = real$. And if $c = 1$ then $s - s^* \neq 0$ such that the Imaginary part of s is not zero. Therefore, the condition $1 - s = c\; s^*$ for $c = 1$ leads immediately to $s = \frac{1}{2} + i\lambda$. From eq-(A.3) resulting from properties of the Gauss-Jacobi series $G(\frac{1}{x}) \; = \; \sqrt{x}\; G(x)$ it follows that under the "time reversal " $\mathcal{T}$ operation $t \to \frac{1}{t}$ the eigenfunctions $\Psi_s(t)$ behave as

$$\mathcal{T} \; \Psi_s(t) \; = \; \Psi_s(\frac{1}{t}) \; = \; \Psi_{1-s}(t). \qquad (A.5)$$

and the Hamiltonian operators $H_A = D_2 D_1, \; H_B = D_1 D_2$ transform as

$$\mathcal{T} \; H_B \; \mathcal{T}^{-1} \; = \; H_A, \quad \mathcal{T} \; H_A \; \mathcal{T}^{-1} \; = \; H_B. \qquad (A.6)$$

If the Hamiltonians H_A, H_B are *invariant* under the $\mathcal{CT}$ operation, where under the $\mathcal{C}$ "charge conjugation" operation the eigenfunctions transform $\mathcal{C} \; \Psi_s(t) \; = \; \Psi_{s^*}(t)$, one may prove the RH resulting from the condition $< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; \neq \; 0$. The invariance of the H_A, H_B operators under $\mathcal{CT}$ implies the vanishing of the

commutators $[H_A, \mathcal{CT}] = [H_B, \mathcal{CT}] = 0$. When the operators H_A, H_B commute with $\mathcal{CT}$, there exits new eigenfunctions $\Psi_s^{\mathcal{CT}}(t)$ of the H_A operator with eigenvalues $s^*(1-s^*)$. Let us focus only in the H_A operator since similar results follow for the H_B operator. Defining

$$\mid \Psi_s^{\mathcal{CT}}(t) \; > \; \equiv \; \mathcal{CT} \mid \Psi_s(t) \; > . \tag{A.7}$$

one can see that it is also an eigenfunction of H_A with eigenvalue $s^*(1-s^*)$:

$$H_A \mid \Psi_s^{\mathcal{CT}}(t) \; > \; = \; H_A \, \mathcal{CT} \mid \Psi_s(t) \; > \; = \; H_A \mid \Psi_{1-s^*}(t) \; > \; =$$

$$s^*(1-s^*) \mid \Psi_{1-s^*}(t) \; > \; = \; s^*(1-s^*) \, \mathcal{CT} \mid \Psi_s(t) \; > \; = \; (E_s)^* \mid \Psi_s^{\mathcal{CT}}(t) \; > . \tag{A.8}$$

where we have defined $(E_s)^* = s^*(1-s^*)$. Given

$$[H_A, \mathcal{CT}] \; = \; 0 \; \Rightarrow \; < \Psi_s \mid [H_A, \mathcal{CT}] \mid \Psi_s > \; = \; 0 \; \Rightarrow$$

$$< \Psi_s \mid H_A \, \mathcal{CT} \mid \Psi_s > \; - \; < \Psi_s \mid \mathcal{CT} \, H_A \mid \Psi_s > \; =$$

$$(E_s)^* \; < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; - \; E_s \; < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; =$$

$$(E_s^* \; - \; E_s) \; < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; = \; 0. \tag{A.9}$$

From (A.9) one has two cases to consider.

• Case A : If the pseudo-norm is null

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; = \; 0 \; \Rightarrow \; (E_s \; - \; E_s^*) \; \neq \; 0 \tag{A.10a}$$

then the *complex* eigenvalues $E_s = s(1-s)$ and $E_s^* = s^*(1-s^*)$ are *complex* conjugates of each other. In this case the RH would be false and there are quartets of non-trivial Riemann zeta zeros given by $s_n, \; 1-s_n, \; s_n^*, \; 1-s_n^*$.

• Case B : If the pseudo-norm is *not* null :

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; \neq \; 0 \; \Rightarrow \; (E_s \; - \; E_s^*) \; = \; 0 \tag{A.10b}$$

then the eigenvalues are *real* given by $E_s = s(1-s) = E_s^* = s^*(1-s^*)$ and which implies that $s = real$ (location of the trivial zeta zeros) and/or $s = \frac{1}{2} + i\lambda$ (location of the non-trivial zeta zeros). In this case the RH would be true and the non-trivial Riemann zeta zeros are given by $s_n = \frac{1}{2} + i\lambda_n$ and $1 - s_n = s_n^* = \frac{1}{2} - i\lambda_n$. We are going to prove next why Case A does and cannot occur, therefore the RH is true because we are left with case B.

In section **3** we established a one to one correspondence among the zeta zeros s_n with the states $\Psi_{s_n}(t)$ *orthogonal* to the *ground* (vacuum state) $\Psi_{s_o}(t)$ associated with the center of symmetry $s_o = \frac{1}{2} + i0$ of the non-trivial zeta zeros and corresponding to the fundamental Riemann function obeying the "duality" condition $Z(s) = Z(1-s)$. The inner products $< \Psi_{s_o}(t) \mid \Psi_{s_n}(t) > \; \sim \; Z[s_n] = 0$ fix the location of the nontrivial zeta zeros s_n since $Z[s]$ is proportional to $\zeta(s)$.

To sum up, for states Ψ_s such that $s = \frac{1}{2} + i\lambda$ lies in the critical line one learns that

$$\langle \psi_{s_1} | \psi_{s_2} \rangle \; = \; (\frac{2}{l}) \; Z \; [\; \frac{2}{l}(2k - (s_1^* + s_2)) \;] \; = \; (\frac{2}{l}) \; Z \; [\; \frac{1}{2} + \frac{2i}{l}(\lambda_1 - \lambda_2) \;]. \; (A.11)$$

and one finds that the states Ψ_s have equal norm

$$< \Psi_s \; | \Psi_s > \; = \; \frac{2}{l} \; Z \; [\frac{1}{2}] \; \neq \; 0, \quad (l \neq \pm\infty). \qquad (A.12)$$

After this discussion, we are ready now to study cases A and B in eqs-(A.10a, A.10b) respectively. From the explicit form of eq-(A.11) depicting the inner product of two arbitrary states , by choosing for example that $l = -2 \Rightarrow k = \frac{1}{4}$, one concludes that the pseudo-norm is not null

$$< \Psi_s \; | \; \mathcal{CT} \; | \; \Psi_s > \; = \; < \Psi_s \; || \; \Psi_{1-s^*} > \; = \; - \; Z[\; -(\frac{1}{2} - (s^* + 1 - s^*)) \;] \; =$$

$$- \; Z[\frac{1}{2}] \; \neq \; 0. \qquad (A.13)$$

and consequently case **A** of eq-(A.10a) is ruled out and case **B** of eq-(A.10b) stands. Concluding, since the pseudo-norm (A.13) is *not null* this implies that the eigenvalues E_s, E_s^* obey eq-(A.10b) and are *real*-valued $E_s = s(1-s) = E_s^* = s^*(1-s^*)$ which means that the Riemann Hypothesis *is true*.

The results of eq-(A.13) and conclusions remain the same for other choices of the parameters l, k so far as l, k are constrained to obey the condition $l = 4(2k-1) \Leftrightarrow a + 2b = 1$ imposed from the symmetry considerations since the orthogonal states $\Psi_{s_n}(t)$ to the reference state $\Psi_{s_o}(t)$ must obey the same symmetry conditions with respect to the critical line and real line as the non-trivial zeta zeros :

$$< \Psi_s \; | \; \mathcal{CT} \; | \; \Psi_s > \; = \; < \Psi_s \; || \; \Psi_{1-s^*} > \; = \; \frac{2}{l} \; Z[\; \frac{2}{l}(2k - (s^* + 1 - s^*)) \;] \; =$$

$$\frac{2}{l} \; Z[\frac{2}{l}(2k-1)] \; = \frac{2}{l} \; Z[\frac{1}{2}] \; \neq 0, \quad (l \neq \pm\infty). \qquad (A.14)$$

as a result of $l = 4(2k-1)$.

Finally, we show that there exists an infinity family of potentials $V_{2j,2m}(t)$ related to an infinite family of theta series [39], [38] where *no* regularization is needed in the construction of the inner products and *without* the need to extract the zero mode $n = 0$ divergent contribution. The infinite family of potentials is defined by

$$e^{2V_{2j,2m}(t)} \; = \; \Theta_{2j,2m}(t) \; \equiv \; \sum_{n=-\infty}^{n=\infty} \; n^{2m} \; H_{2j}(n\sqrt{2\pi t}) \; e^{-\pi n^2 t}. \qquad (A.15)$$

when $m \neq 0$, the zero mode $n = 0$ does *not* contribute to the sum and the Mellin transform of $\Theta_{2j,2m}(t)$, after exploiting the symmetry of the even-degree Hermite polynomials, is given by [38], [39]

$$\int_0^\infty \; [\; 2 \; \sum_{n=1}^{n=\infty} \; n^{2m} \; H_{2j}(n\sqrt{2\pi t}) \; e^{-\pi n^2 t} \;] \; t^{s/2-1} \; dt \; =$$

$$2 \; (8\pi)^j \; P_j(s) \; \pi^{-s/2} \; \Gamma(\frac{s}{2}) \; \zeta(s-2m); \quad m = 1, 2, ... \qquad (A.16)$$

The polynomial $P_j(s)$ has simple zeros on the critical line $Re\; s = \frac{1}{2}$, obeys the functional relation $P_j(s) = (-1)^j P_j(1-s)$ and in particular $P_j(s = \frac{1}{2}) = 0$ when $j = odd$. It is only when $j = even$ that $P_j(s = \frac{1}{2}) \neq 0$ and when we can implement $\mathcal{CT}$ invariance. In the $j = odd$ case, one cannot implement $\mathcal{CT}$ invariance and $P_j(s = \frac{1}{2}) = 0$. A Poisson re-summation formula leads to a similar relation under $t \to \frac{1}{t}$

$$\frac{(-1)^j}{\sqrt{t}} \; \Theta_{2j,2m}(\frac{1}{t}) \; = \; \Theta_{2j,2m}(t). \qquad (A.17)$$

therefore, only when $j = even$ in (A.17) one can implement $\mathcal{CT}$ invariance to the new family of Hamiltonians H_A, H_B associated with the potentials $V_{2j,2m}(t)$ of (A.15) because $H_A \Psi_s(t) = s(1-s)\Psi(t)$ and $H_B \Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t})$ would only be valid when $j = even$. The inner products with respect to the shifted state $\Psi_{\frac{1}{2}+2m}(t)$ are

$$< \; \Psi_{\frac{1}{2}+2m}(t) \mid \Psi_s(t) \; > \; = \; - \; 2 \; (8\pi)^j \; P_j(s+2m) \; \pi^{-(s+2m)/2} \; \Gamma(\frac{s+2m}{2}) \; \zeta(s). \qquad (A.18)$$

this result requires *fixing* uniquely the values $l = -2; k = \frac{1}{4}$. Thus, the non-trivial zeta zeros s_n would correspond to the states $\Psi_{s_n}(t)$ orthogonal to the shifted "ground" state $\Psi_{\frac{1}{2}+2m}(t)$. It remains to prove when $l = -2, k = \frac{1}{4}$ and $s_{12} = s_1^* + s_2 = s_1^* + (1 - s_1^*) = 1$ that

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; = \; < \Psi_s \mid\mid \Psi_{1-s^*} > \; =$$

$$\int_0^\infty \; [\; 2 \; \sum_{n=1}^{n=\infty} \; n^{2m} \; H_{2j}(n\sqrt{2\pi t}) \; e^{-\pi n^2 t} \;] \; t^{\frac{2(-s_{12}+2k)}{2l}-1} \; dt \; =$$

$$- \; 2 \; (8\pi)^j \; P_j(s = \frac{1}{2}) \; \pi^{-1/4} \; \Gamma(\frac{1}{4}) \; \zeta(\frac{1}{2} - 2m) \; \neq 0; \quad j = even, \; m = 1, 2, 3, ... \qquad (A.19)$$

Hence, one would arrive at a definite solid conclusion based on a well defined inner product : the RH is true because $\zeta(\frac{1}{2} - 2m) \neq 0$ when $m = 1, 2, ...$ and $P_j(\frac{1}{2}) \neq 0$ when $j = even$. This finding can be inferred from the nonzero pseudo-norm $< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \; \neq 0$ in (A.19) and upon following our previous arguments as in (A.10a, A.10b) that rule out case **A** , single out case **B**, and that leads to $E_s = s(1-s) = real \; \Rightarrow s = \frac{1}{2} + i\lambda$ (and/or s = real). The

key reason why the Riemann hypohesis is true is due to the $\mathcal{CT}$ invariance and the fact that the pseudo-norm $< \Psi_s|\mathcal{CT}|\Psi_s >$ *is not null*. Had the pseudo-norm $< \Psi_s|\mathcal{CT}|\Psi_s >$ been *null*, the RH would have been false. It remains to be seen whether our procedure is valid to prove the grand-Riemann Hypothesis associated to the L-functions.

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