Polyvector-valued Gauge Field Theories and Quantum Mechanics in Noncommutative Clifford Spaces

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August 2009

Abstract

The basic ideas and results behind polyvector-valued gauge field theories and Quantum Mechanics in Noncommutative Clifford spaces are presented. The star products are noncommutative and associative and require the use of the Baker-Campbell-Hausdorff formula. The construction of Noncommutative Clifford-space gravity as polyvector-valued gauge theories of twisted diffeomorphisms in Clifford-spaces would require quantum Hopf algebraic deformations of Clifford algebras.

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [3] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds.

It was recently shown [1] how an unification of Conformal Gravity and a $U(4) \times U(4)$ Yang-Mills theory in four dimensions could be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 TeV regime [2]. Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have been studied in [2], [3] where $\mathbf{X} = X_M \Gamma^M$ is a C-space polyvector valued coordinate

$$\mathbf{X} \ = \ \sigma \, \mathbf{1} \ + \ x_\mu \ \gamma^\mu \ + \ x_{\mu_1\mu_2} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \ + \ x_{\mu_1\mu_2\mu_3} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \ + \ ...... \ +$$

$$x_{\mu_1\mu_2\mu_3......\mu_d}\ \gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\ \gamma^{\mu_3}.......\ \wedge\gamma^{\mu_d} \qquad (1)$$

In order to match dimensions in each term of (1) a length scale parameter must be suitably introduced. In [3] we introduced the Planck scale as the expansion parameter in (1). The scalar component σ of the C-space poly-vector valued coordinate $\mathbf{X}$ was interpreted by [4] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

A Clifford gauge field theory in the C-space associated with the ordinary $4D$ spacetime requires $\mathcal{A}_M(\mathbf{X}) = \mathcal{A}^A_M(\mathbf{X})\ \Gamma_A$ that is a poly-vector valued gauge field where M represents the poly-vector index associated with the C-space, and whose gauge group $\mathcal{G}$ is itself based on the Clifford algebra $Cl(3,1)$ of the tangent space spanned by 16 generators Γ_A. The expansion of the poly-vector Clifford-algebra-valued gauge field $\mathcal{A}^A_M$, for $fixed$ values of A, is of the form

$$\mathcal{A}^A_M\,\Gamma^M\ =\ \Phi^A+\mathcal{A}^A_\mu\,\gamma^\mu+\mathcal{A}^A_{\mu_1\mu_2}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}+\mathcal{A}^A_{\mu_1\mu_2\mu_3}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3}+....... \qquad (2)$$

The index A spans the 16-dim Clifford algebra $Cl(3,1)$ of the tangent space such as

$$\Phi^A\ \Gamma_A\ =\ \Phi\ +\ \Phi^a\ \Gamma_a\ +\ \Phi^{ab}\ \Gamma_{ab}\ +\ \Phi^{abc}\ \Gamma_{abc}\ +\ \Phi^{abcd}\ \Gamma_{abcd}. \qquad (3a)$$

$$\mathcal{A}^A_\mu\ \Gamma_A\ =\ \mathcal{A}_\mu\ +\ \mathcal{A}^a_\mu\ \Gamma_a\ +\ \mathcal{A}^{ab}_\mu\ \Gamma_{ab}\ +\ \mathcal{A}^{abc}_\mu\ \Gamma_{abc}\ +\ \mathcal{A}^{abcd}_\mu\ \Gamma_{abcd}. \qquad (3b)$$

$$\mathcal{A}^A_{\mu\nu}\ \Gamma_A\ =\ \mathcal{A}_{\mu\nu}\ +\ \mathcal{A}^a_{\mu\nu}\ \Gamma_a\ +\ \mathcal{A}^{ab}_{\mu\nu}\ \Gamma_{ab}\ +\ \mathcal{A}^{abc}_{\mu\nu}\ \Gamma_{abc}\ +\ \mathcal{A}^{abcd}_{\mu\nu}\ \Gamma_{abcd}. \qquad (3c)$$

etc......

In order to match dimensions in each term of (2) another length scale parameter must be suitably introduced. For example, since $\mathcal{A}^A_{\mu\nu\rho}$ has dimensions of $(length)^{-3}$ and $\mathcal{A}^A_\mu$ has dimensions of $(length)^{-1}$ one needs to introduce another length parameter in order to match dimensions. This length parameter does not need to coincide with the Planck scale. The Clifford-algebra-valued gauge field $\mathcal{A}^A_\mu(x^\mu)\Gamma_A$ in ordinary spacetime is naturally embedded into a far richer object $\mathcal{A}^A_M(\mathbf{X})\Gamma_A$ in C-spaces. The advantage of recurring to C-spaces associated with the $4D$ spacetime manifold is that one can have a (complex) Conformal Gravity, Maxwell and $U(4)\times U(4)$ Yang-Mills unification in a very geometric fashion as provided by [1]

Field theories in Noncommutative spacetimes have been the subject of intense investigation in recent years, see [8] and references therein. Star Product deformations of Clifford Gauge Field Theories based on ordinary Noncommutative spacetimes are straightforward generalizations of the work by [5]. The wedge star product of two Clifford-valued one-forms is defined as

$$\mathbf{A}\ \wedge_*\ \mathbf{A}\ =\ \big(\ (\mathcal{A}^A_\mu\ *\ \mathcal{A}^B_\nu)\ \Gamma_A\ \Gamma_B\ \big)\ dx^\mu\wedge dx^\nu\ =$$

$$\frac{1}{2}\ \Big(\ (\mathcal{A}^A_\mu *_s \mathcal{A}^B_\nu)\ [\Gamma_A,\ \Gamma_B]\ +\ (\mathcal{A}^A_\mu *_a \mathcal{A}^B_\nu)\ \{\Gamma_A,\ \Gamma_B\}\ \Big)\ dx^\mu\ \wedge\ dx^\nu. \tag{4}$$

In the case when the coordinates don't commute $[x^\mu, x^\nu] = \theta^{\mu\nu}$ (constants), the cosine (symmetric) star product is defined by [5]

$$f *_s g\ \equiv\ \frac{1}{2}(f * g + g * f)\ =\ f\ g + \left(\frac{i}{2}\right)^2 \theta^{\mu\nu}\ \theta^{\kappa\lambda}\ (\partial_\mu\ \partial_\kappa f)\ (\partial_\nu\ \partial_\lambda g) + O(\theta^4). \tag{5}$$

and the sine (anti-symmetric Moyal bracket) star product is

$$f *_a g\ \equiv\ \frac{1}{2}(f * g - g * f)\ =\ \left(\frac{i}{2}\right)\ \theta^{\mu\nu}\ (\partial_\mu f)\ (\partial_\nu g)\ +$$

$$\left(\frac{i}{2}\right)^3\ \theta^{\mu\nu}\ \theta^{\kappa\lambda}\ \theta^{\alpha\beta}\ (\partial_\mu\ \partial_\kappa\ \partial_\alpha f)\ (\partial_\nu\ \partial_\lambda\ \partial_\beta g)\ +\ O(\theta^5). \tag{6}$$

Notice that both commutators *and* anticommutators of the gammas appear in the star deformed products in (4). The star product deformations of the gauge field strengths in the case of the $U(2,2)$ gauge group were given by [5] and the expressions for the star product deformed action are very cumbersome .

In this letter we proceed with the construction of Polyvector-valued Gauge Field Theories in *noncommutative* Clifford Spaces (C-spaces) which are polyvector-valued *extensions* and *generalizations* of the ordinary *noncommutative* spacetimes. We begin firstly by writing the commutators $[\Gamma_A, \Gamma_B]$. For $pq = odd$ one has [7]

$$[\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ ]\ =\ 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p}\ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]}\ -\ ...... \tag{7}$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ ]\ =\ -\ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]}\ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1 ....a_3}_{[b_1 ....b_3}\ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]}\ +\ ...... \tag{8}$$

The anti-commutators for $pq = even$ are

$$\{\ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\ \}\ =\ 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p}\ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]}\ -\ ...... \tag{9}$$

and the anti-commutators for $pq = odd$ are

$$\{ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q} \} = - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} + ... \quad (10)$$

For instance,

$$\mathcal{J}^a_b = [\gamma_b, \gamma^a] = 2\gamma^a_b; \quad \mathcal{J}^{a_1 a_2}_{b_1 b_2}, = [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] = -8 \delta^{[a_1}_{[b_1} \gamma^{a_2]}_{b_2]}. \quad (11)$$

$$\mathcal{J}^{a_1 a_2 a_3}_{b_1 b_2 b_3} = [\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] = 2 \gamma^{a_1 a_2 a_3}_{b_1 b_2 b_3} - 36 \delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3]}_{b_3]}. \quad (12)$$

$$\mathcal{J}^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} = [\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] = -32 \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 a_4]}_{b_2 b_3 b_4]} + 192 \delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3} \gamma^{a_4]}_{b_4]}. \quad (13)$$

etc...

The second step is to write down the *noncommutative* algebra associated with the noncommuting poly-vector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra (7-10) by performing the following replacements (and relabeling indices)

$$\gamma^\mu \leftrightarrow X^\mu, \quad \gamma^{\mu_1 \mu_2} \leftrightarrow X^{\mu_1 \mu_2}, \quad .... \gamma^{\mu_1 \mu_2 .....\mu_n} \leftrightarrow X^{\mu_1 \mu_2 ....\mu_n}. \quad (14)$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (14) and the Clifford algebra (7-10) (after one relabels indices), one can then construct the following *noncommutative* algebra among the poly-vector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations

$$[X^{\mu_1}, X^{\mu_2}]_* = X^{\mu_1} * X^{\mu_2} - X^{\mu_2} * X^{\mu_1} = 2 X^{\mu_1 \mu_2}. \quad (15)$$

$$[X^{\mu_1 \mu_2}, X^\nu]_* = 4 (g^{\mu_2 \nu} X^{\mu_1} - g^{\mu_1 \nu} X^{\mu_2}). \quad (16)$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^\nu]_* = 2 X^{\mu_1 \mu_2 \mu_3 \nu}, \quad [X^{\mu_1 \mu_2 \mu_3 \mu_4}, X^\nu]_* = -8 g^{\mu_1 \nu} X^{\mu_2 \mu_3 \mu_4} \pm ...... \quad (17)$$

$$[X^{\mu_1 \mu_2}, X^{\nu_1 \nu_2}]_* = -8 g^{\mu_1 \nu_1} X^{\mu_2 \nu_2} + 8 g^{\mu_1 \nu_2} X^{\mu_2 \nu_1} +$$

$$8 g^{\mu_2 \nu_1} X^{\mu_1 \nu_2} - 8 g^{\mu_2 \nu_2} X^{\mu_1 \nu_1}. \quad (18)$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^{\nu_1 \nu_2}]_* = 12 g^{\mu_1 \nu_1} X^{\mu_2 \mu_3 \nu_2} \pm .... \quad (19)$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^{\nu_1 \nu_2 \nu_3}]_* = -36 G^{\mu_1 \mu_2 \ \nu_1 \nu_2} X^{\mu_3 \nu_3} \pm ...... \quad (20)$$

$$[\, X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\,]_* \ = \ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \tag{21}$$

$$[\, X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\,]_* \ = \ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \tag{22}$$

$$[\, X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}\,]_* \ = \ 48\, G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4} - 48\, G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3} + ... \tag{23}$$

$$[\, X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3\nu_4}\,]_* \ = \ 192\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ -\ .... \tag{24}$$

etc...... where

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n} \ = \ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n}\ +\ signed\ permutations \tag{25}$$

The metric components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n \times n$ matrix $\mathbf{G}$ whose entries are $g^{\mu_I\nu_J}$

$$det\ \mathbf{G}_{n\times n} \ = \ \frac{1}{n!}\ \epsilon_{i_1 i_2.....i_n}\ \epsilon_{j_1 j_2....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \tag{26}$$

$i_1, i_2, ..., i_n \subset I = 1, 2, ..., D$ and $j_1, j_2, ..., j_n \subset J = 1, 2, ..., D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field).

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the poly-vector-valued coordinates in $D = 4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$.

The noncommutative conditions on the polyvector coordinates in condensed notation can be written as

$$[\, X^M,\ X^N\,]_* \ = \ X^M * X^N - X^N * X^M \ = \ \Omega^{MN}(X) \ = \ f^{MN}{}_L\, X^L \ = \ f^{MNL}\, X_L \tag{27}$$

the structure constants f^{MNL} are antisymmetric under the exchange of polyvector valued indices. An immediate consequence of the noncommutativity of coordinates is

$$[\ \hat{X}^{\mu_1},\ \hat{X}^{\mu_2}\ ] \ = \ 2\ \hat{X}^{\mu_1\mu_2} \ \Rightarrow \ \Delta X^\mu\ \Delta X^\nu \ \geq \ \frac{1}{2}\ |\ <\ \hat{X}^{\mu\nu}\ >\ | = \ X^{\mu\nu} \tag{28}$$

Hence, the bivector area coordinates $X^{\mu\nu}$ in C-space can be seen as a measure of the noncommutative nature of the "quantized" spacetime coordinates $\hat{X}^\mu$.

The third step is to define the noncommutative star product of functions of X. The following naive noncommutative star product is *not* associative

$$(A_1 \ * \ A_2)(Z) \ = \ exp \left(\ \frac{1}{2} \ \Omega^{MN} \ \partial_{X^M} \ \partial_{Y^N} \ \right) \ A_1(X) \ A_2(Y)|_{X=Y=Z} \ =$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!} \ \Omega^{M_1 N_1} \ \Omega^{M_2 N_2} \ .... \ \Omega^{M_n N_n} \ (\partial^n_{M_1 M_2 ......M_n} \ A_1) \ (\partial^n_{N_1 N_2 ......N_n} \ A_2) + ... \tag{29}$$

where the ellipsis in (29) are the terms involving derivatives acting on Ω^{MN} and

$$\partial^n_{M_1 M_2 ......M_n} \ A_1(Z) \ \equiv \ \partial_{M_1} \ \partial_{M_2} \ ...... \ \partial_{M_n} \ A_1(Z). \tag{30a}$$

$$\partial^n_{N_1 N_2 ......N_n} \ A_2(Z) \ \equiv \ \partial_{N_1} \ \partial_{N_2} \ ...... \ \partial_{N_n} \ A_2(Z). \tag{30b}$$

Derivatives on Ω^{mn} appear in the ordinary Moyal star product when Ω^{mn} depends on the phase space coordinates. For instance, the Moyal star product when the symplectic structure $\Omega^{mn}(\vec{q}, \vec{p})$ is *not* constant is given by

$$A \ * \ B = \ A \ exp \left(\ \frac{i\hbar}{2} \ \Omega^{mn} \ \overleftarrow{\partial}_m \ \overrightarrow{\partial}_n \ \right) \ B \ =$$

$$A \ B + i\hbar \ \Omega^{mn} \ (\partial_m A \ \partial_n B) + \ \frac{(i\hbar)^2}{2} \ \Omega^{m_1 n_1} \ \Omega^{m_2 n_2} \ (\partial^2_{m_1 m_2} \ A) \ (\partial^2_{n_1 n_2} \ B) \ +$$

$$\frac{(i\hbar)^2}{3} [\ \Omega^{m_1 n_1} \ (\partial_{n_1} \ \Omega^{m_2 n_2}) \ (\partial_{m_1} \partial_{m_2} A \ \partial_{n_2} B \ - \ \partial_{m_2} A \ \partial_{m_1} \partial_{n_2} B \ ) \ ] + O(\hbar^3). \tag{31}$$

Due to the derivative terms $\partial_{n_1} \Omega^{m_2 n_2}$ the star product is associative up to second order only [6] $(f * g) * h \ = \ f * (g * h) + O(\hbar^3)$. Hence, due to the derivatives terms acting on $\Omega^{MN}(X)$ in (29), the star product will *no* longer be associative beyond second order.

The correct noncommutative and *associative* star product [12] associated with a Lie-algebraic structure for the noncommutative (C-space) coordinates requires the use of the Baker-Campbell-Hausdorff formula

$$exp(A) \ exp(B) \ = \ exp \left(\ A + B + \frac{1}{2} [A, B] + \frac{1}{12} (\ [A, [A, B]] - [B, [A, B]] \ ) \ + ... \right). \tag{32a}$$

and is given by

$$(\ A_1 \ * \ A_2 \ )(X) \ = \ exp \left(\ \frac{1}{2} \ X^M \Lambda_M [\ i \ \partial_Y ; \ i \ \partial_Z \ ] \ \right) \ A_1(Y) \ A_2(Z)|_{X=Y=Z}. \tag{32b}$$

where the expression for the bilinear differential polynomial $\Lambda_M[i\partial_Y; i\partial_Z]$ in eq-(32b) can be *read* from the Baker-Campbell-Hausdorff formula

$$e^{K_M \hat{X}^M} \; e^{P_N \hat{X}^N} \;=\; e^{\hat{X}^M (\; K_M \;+\; P_M \;+\; \Lambda_M [K,P] \;)}. \qquad (32c)$$

and is given in terms of the structure constants $[X^N, X^Q] = f_M^{NQ} X^M$, after setting $K_N = i\ \partial_{Y^N},\ P_Q = i\ \partial_{Z^Q}$, by the following expression

$$\Lambda_M[K,P] \;=\; K_N \; P_Q \; f_M^{NQ} \;+ \frac{1}{6} \; K_{N_1} \; P_{Q_1} \; (P_{N_2} - K_{N_2}) \; f_S^{N_1 Q_1} \; f_M^{S N_2} \;+$$

$$\frac{1}{24} \; (P_{N_2} \; K_{Q_2} \;+\; K_{N_2} \; P_{Q_2}) \; K_{N_1} \; P_{Q_1} \; f_{S_1}^{N_1 Q_1} \; f_{S_2}^{S_1 N_2} \; f_M^{S_2 Q_2} \;+\; .... \qquad (32d)$$

When the star product is *associative* and noncommutative, with the fields and their derivatives vanishing fast enough at infinity, one has

$$\int \; A \, * B \;=\; \int \; A \; B \;+\; \textit{total derivative} \;=\; \int \; A \; B. \qquad (33a)$$

$$\int \; A \, * \; B \, * \; C \;=\; \int \; A \, (B \, * \; C) \;+\; \textit{total derivative} \;=\; \int \; A \, (B \, * \; C) \;=$$

$$\int \; (B \, * \; C) \; A \;=\; \int \; (B \, * \; C) \, * \; A + \textit{total derivative} \;=\; \int \; B \, *C \, * \; A \quad (33b)$$

therefore, when the star product is associative and the fields and their derivatives vanishing fast enough at infinity (or there are no boundaries) one has

$$\int \; A \; * \; B \; *C \;=\; \int \; B \; *C \; * \; A \;=\; \int \; C \; *A \; * \; B \; . \qquad (33c)$$

The relations (33) are essential in order to construct invariant actions under star gauge transformations.

The C-space differential form associated with the polyvector-valued Clifford gauge field is

$$\mathbf{A} \;=\; \mathcal{A}_M \; dX^M \;=\; \Phi \; d\sigma \;+\; \mathcal{A}_\mu \; dx^\mu \;+\; \mathcal{A}_{\mu\nu} \; dx^{\mu\nu} \;+\; .... \;+$$

$$\mathcal{A}_{\mu_1 \mu_2 .....\mu_d} \; dx^{\mu_1 \mu_2 ....\mu_d}.. \qquad (34a)$$

where $\Phi = \Phi^A \ \Gamma_A,\ \mathcal{A}_\mu = \mathcal{A}_\mu^A \ \Gamma_A,\ \mathcal{A}_{\mu\nu} = \mathcal{A}_{\mu\nu}^A \ \Gamma_A, ...$ The C-space differential form associated with the polyvector-valued field-strength is

$$\mathbf{F} \;=\; F_{MN} \; dX^M \;\wedge\; dX^N \;=\; F_{0 \;\mu} \; d\sigma \;\wedge\; dx^\mu \;+\; F_{0 \;\mu_1 \mu_2} \; d\sigma \;\wedge\; dx^{\mu_1 \mu_2} \;+\; ....$$

$$F_{0 \;\nu_1 \nu_2 .....\nu_d} \; d\sigma \wedge dx^{\nu_1 \nu_2 ......\nu_d} + F_{\mu\nu} \; dx^\mu \wedge dx^\nu + F_{\mu_1 \mu_2 \;\nu_1 \nu_2} \; dx^{\mu_1 \mu_2} \wedge dx^{\nu_1 \nu_2} + ...$$

$$+ \; F_{\mu_1 \mu_2 .....\mu_{d-1} \;\nu_1 \nu_2 .....\nu_{d-1}} \; dx^{\mu_1 \mu_2 ......\mu_{d-1}} \;\wedge\; dx^{\nu_1 \nu_2 ....\nu_{d-1}}. \qquad (34b)$$

The field strength is antisymmetric under the exchange of poly-vector indices $F_{MN} = -F_{NM}$. For this reason one has $F_{00} = 0$ and $F_{12....d \;12.....d} = 0$. Finally, given the noncommutative conditions on the poly-vector coordinates

(27), the components of the Clifford-algebra valued field strength $F^C_{MN}\Gamma_C$ in *Noncommutative* C-spaces are

$$F_{[MN]} \ = \ F^C_{[MN]}\ \Gamma_C \ = \ (\ \partial_M\ \mathcal{A}^C_N \ - \ \partial_N\ \mathcal{A}^C_M\ )\ \Gamma_C \ +$$

$$\frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N - \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ \{\ \Gamma_A,\ \Gamma_B\ \} + \frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N + \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ [\ \Gamma_A,\ \Gamma_B\ ]. \tag{35}$$

The commutators $[\ \Gamma_A,\ \Gamma_B\ ]$ and anti-commutators $\{\ \Gamma_A,\ \Gamma_B\ \}$ in eqs-(35), where A, B are polyvector-valued indices, can be read from the relations in eqs-(7-10) . Notice that both the standard commutators *and* anticommutators of the gammas appear in the terms containing the star deformed products of (35) and which define the Clifford-algebra valued field strength in noncommutative C-spaces; i.e. if the products of fields were to commute one would have had only the Lie algebra commutator $\mathcal{A}^A_M \mathcal{A}^J_B\ [\Gamma_A, \Gamma_B]$ pieces without the anti-commutator $\{\Gamma_A, \Gamma_B\}$ contributions in the r.h.s of eq-(35).

We should remark that one is *not* deforming the Clifford algebra involving $[\ \Gamma_A,\ \Gamma_B\ ]$ and $\{\ \Gamma_A,\ \Gamma_B\ \}$ in eq-(35) but it is the "point" product algebra $\mathcal{A}^A_M * \mathcal{A}^B_N$ of the fields which is being deformed. (Quantum) q-Clifford algebras have been studied by [9]. The symmetrized star product is

$$\mathcal{A}^A_M \ *_s\ \mathcal{A}^B_N \ \equiv\ \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N + \mathcal{A}^B_N * \mathcal{A}^A_M\right) \ = \ \mathcal{A}^A_M\ \mathcal{A}^B_N \ +$$

$$X^{\mu\nu}\ X^{\kappa\lambda}\ (\partial_\mu\ \partial_\kappa\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \mathcal{A}^B_N)\ +\ ...... \tag{36a}$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}^A_M *_a \mathcal{A}^B_N \ \equiv\ \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N - \mathcal{A}^B_N * \mathcal{A}^A_M\right) \ = \ X^{\mu\nu}\ (\partial_\mu\ \mathcal{A}^A_M)\ (\partial_\nu\ \mathcal{A}^B_N)\ +$$

$$X^{\mu\nu}\ X^{\kappa\lambda}\ X^{\alpha\beta}\ (\partial_\mu\ \partial_\kappa\ \partial_\alpha\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \partial_\beta\ \mathcal{A}^B_N)\ +\ .... \tag{36b}$$

It is important to emphasize, as it is customary in Moyal star products, that the poly-vector coordinates appearing in the r.h.s of eqs-(35-36) are treated as c-numbers (as if they were commuting) while it is the product of functions appearing in the l.h.s of (35-36) which are *noncommutative*.

Star products in noncommutative C-space lead to *many more terms* in eqs-(35-36) than in ordinary noncommutative spaces. For example, there are derivatives terms involving polyvectors which do *not* appear in ordinary noncommutative spaces, like

$$-4\ g^{\mu_1\nu_1}\ X^{\mu_2\nu_2}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu_1\nu_2}}\ \pm\ .... \tag{37a}$$

$$2\ (\ g^{\mu_2\nu}\ X^{\mu_1}\ -\ g^{\mu_1\nu}\ X^{\mu_2}\ )\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu}}. \tag{37b}$$

$$X^{\mu_1\mu_2\mu_3\nu}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2\mu_3}}\ \frac{\partial \mathcal{A}^J_B}{\partial X^{\nu}}. \tag{37c}$$

$$96\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2\mu_3\mu_4}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu_1\nu_2\nu_3\nu_4}}, \quad etc\ ...... \qquad (37d)$$

There is a *subalgebra* of the noncommutative polyvector-valued coordinates algebra (27) involving only X^μ and the bivector coordinates $X^{\mu\nu}$ when the spacetime metric components $g_{\mu\nu}$ are *constant*. However, because $[X^{\mu_1\mu_2}, X^\nu] \neq 0$ one must not confuse the algebra in this case with the ordinary Θ-noncommutative algebra $[X^{\mu_1}, X^{\mu_2}] = \Theta^{\mu_1\mu_2}$ where the components of $\Theta^{\mu_1\mu_2}$ are comprised of *constants* such that $[\Theta^{\mu_1\mu_2}, X^\nu] = 0$.

The analog of a Yang-Mills action in C-spaces when the background C-space flat metric G^{MN} is X-independent is given by

$$S\ =\ \int\ [DX]\ <\ F^A_{MN}\ \Gamma_A\ *\ F^B_{PQ}\ \Gamma_B\ >\ G^{MP}\ G^{NQ}. \qquad (38)$$

where $<\ \Gamma_A\ \Gamma_B\ >$ denotes the Clifford-scalar part of the Clifford geometric product of two generators. As mentioned in the introduction suitable powers of a length scale must be included in the expansion of the terms inside the integrand in order to have consistent dimensions (the action is dimensionless). The action (38) becomes

$$\int\ [DX]\ (\ F_{MN}\ *\ F^{MN}\ +\ F^a_{MN}\ *\ F^{MN}_a\ +$$

$$F^{a_1a_2}_{MN}\ *\ F^{MN}_{a_1a_2}\ +\ ....\ +\ F^{a_1a_2.....a_d}_{MN}\ *\ F^{MN}_{a_1a_2.....a_d}\ ). \qquad (39)$$

the measure in C-space is given by

$$DX\ =\ d\sigma\ \prod dx^\mu\ \prod\ dx^{\mu_1\mu_2}\ \prod dx^{\mu_1\mu_2\mu_3}\ .....\ dx^{\mu_1\mu_2.......\mu_d}. \qquad (40a)$$

The Clifford-valued gauge field $\mathcal{A}_M$ transforms under star gauge transformations according to $\mathcal{A}'_M\ =\ U_*^{-1} * \mathcal{A}_M * U_* + U_*^{-1} * \partial_M U_*$. The field strength F transforms covariantly $F'_{MN} = U_*^{-1} * F_{MN} * U_*$ such that the action (39) is star gauge invariant. $U_* = exp_*(\xi(X)) = exp_*(\xi^A(X)\Gamma_A)$ is defined via a star power series expansion $U_* = \sum_n \frac{1}{n!}(\xi(X))^n_*$ where $(\xi(X))^n_* = \xi(X) * \xi(X) * ..... * \xi(X)$. The integral $\int F * F = \int F\ F$ + total derivatives. If the fields vanish fast enough at infinity and/or there are no boundaries, the contribution of the total derivative terms are zero.

When the star product is truly *associative* one has star gauge invariance of the action (39) under infinitesimal $\delta F = [F, \xi]_*$ transformations

$$\delta_\xi S\ =\ 2 \int\ <\ F * [F,\ \xi]_*\ >\ =\ 2 \int\ <\ F * F * \xi\ >\ -2 \int\ <\ F * \xi * F\ >. \qquad (40b)$$

If the star product is associative due to the relations in eqs-(33) one can show that eq-(40b) becomes (up to a trivial factor of 2)

$$\int F^A * F^B * \xi^C < \Gamma_A \Gamma_B \Gamma_C > - \int F^A * \xi^C * F^B < \Gamma_A \Gamma_C \Gamma_B > =$$

$$\int F^B * \xi^C * F^A < \Gamma_B \Gamma_C \Gamma_A > - \int F^A * \xi^C * F^B < \Gamma_A \Gamma_C \Gamma_B > = 0 \tag{40c}$$

so one arrives at the zero result in (40c), assuring $\delta S = 0$, after using the *cyclic* property of the scalar part of the geometric product

$$< \Gamma_A \Gamma_B \Gamma_C > = < \Gamma_B \Gamma_C \Gamma_A > = < \Gamma_C \Gamma_A \Gamma_B > \tag{40d}$$

and *relabeling* the indices $B \leftrightarrow A$ in the third term of (40c).

In ordinary commutative C-spaces one can perform the mode expansion in integer powers of the poly-vector coordinates

$$\mathcal{A}_M(X) = \mathcal{A}_M(\sigma, \mathbf{x}^\mu, x^{\mu_1\mu_2}, ..., x^{\mu_1\mu_2......\mu_d}) =$$
$$\sum_{n_I} \mathcal{A}_{M,n_I}(\mathbf{x}^\mu)\ \sigma^{n_o}\ (x^{12})^{n_{12}}\ ......\ (x^{123})^{n_{123}}\ ....\ (x^{12......d})^{n_{123......d}}. \tag{41}$$

The sum over the spacetime dependent fields $\mathcal{A}_{M,n_I}(\mathbf{x}^\mu)$ is taken over the infinite number of integer-valued modes associated with the collection set of integers

$$n_I = n_o, n_{12}, \ ......, n_{123}, \ ...., n_{1234}, \ ...., n_{12....d}. \tag{42}$$

In Noncommutative C-spaces we may replace the ordinary products of the poly-vector valued coordinates in eq-(41) for their star products.

To finalize we provide a description of QM in Noncommutative C-spaces based on Yang's Noncommutative phase space algebra [10]. There is a *subalgebra* of the C-space operator-valued coordinates which is *isomorphic* to the Noncommutative Yang's $4D$ spacetime algebra [10]. This can be seen after establishing the following correspondence between the C-space vector/bivector (area-coordinates) algebra, associated to the $6D$ angular momentum (Lorentz) algebra, and the Yang's spacetime algebra via the $SO(6)$ generators Σ^{ij} in $6D$ $(i, j = 1, 2, 3......, 6)$ as follows [11]

$$i\ \hbar\ \Sigma^{\mu\nu} \leftrightarrow i\ \frac{\hbar}{\lambda^2}\ \hat{X}^{\mu\nu}, \quad i\ \Sigma^{56} \leftrightarrow i\ \frac{R}{\lambda}\ \mathcal{N}. \tag{43a}$$

$$i\ \lambda\ \Sigma^{\mu 5} \leftrightarrow i\ \hat{X}^\mu, \quad i\ \Sigma^{\mu 6} \leftrightarrow i\ \frac{R}{\hbar}\ \hat{P}^\mu \tag{43b}$$

where the indices $\mu, \nu = 1, 2, 3, 4$. The scales λ and R are a lower and upper scale respectively, like the Planck and Hubble scale. The $SO(6)$ algebra $[\Sigma^{ij},\ \Sigma^{kl}] = -g^{ik}\Sigma^{jl} + ...$ can be recast in terms of a *noncommutative* phase space algebra as

$$[\hat{P}^\mu,\ \mathcal{N}] \ = \ -\,i\ \eta^{66}\ \frac{\hbar}{R^2}\hat{X}^\mu, \quad [\hat{X}^\mu,\ \mathcal{N}] \ = \ i\ \eta^{55}\ \frac{\lambda^2}{\hbar}\ \hat{P}^\mu. \tag{44}$$

$$[\hat{X}^\mu,\ \hat{X}^\nu] \ = \ -i\ \eta^{55}\ \hat{X}^{\mu\nu}, \quad [\hat{P}^\mu,\ \hat{P}^\nu] \ = \ -i\ \eta^{66}\ \frac{\hbar^2}{R^2\ \lambda^2}\ \hat{X}^{\mu\nu}, \quad \hat{X}^{\mu\nu} \ = \ \lambda^2\ \Sigma^{\mu\nu}. \tag{45}$$

$$[\hat{X}^\mu,\ \hat{P}^\mu] \ = \ i\ \hbar\ \eta^{\mu\nu}\ \frac{\lambda}{R}\ \Sigma^{56} \ = \ i\ \hbar\ \eta^{\mu\nu}\ \mathcal{N}, \quad [\hat{X}^{\mu\nu},\ \mathcal{N}] \ = \ 0. \tag{46}$$

The last relation is the *modified* Heisenberg algebra in $4D$ since $\mathcal{N}$ does *not* commute with X^μ nor P^μ. The remaining *nonvanishing* commutation relations are

$$[\Sigma^{\mu\nu},\ \hat{X}^\rho] \ = \ -\,i\ \eta^{\mu\rho}\ \hat{X}^\nu \ +i\ \eta^{\nu\rho}\hat{X}^\mu \tag{47a}$$

$$[\Sigma^{\mu\nu},\ \hat{P}^\rho] \ = \ -\,i\ \eta^{\mu\rho}\ \hat{P}^\nu \ +i\ \eta^{\nu\rho}\hat{P}^\mu. \tag{47b}$$

$$[\Sigma^{\mu\nu},\ \Sigma^{\rho\tau}] \ = \ -\,i\ \eta^{\mu\rho}\ \Sigma^{\nu\tau} \ +i\ \eta^{\nu\rho}\ \Sigma^{\mu\tau} \ - \ .... \tag{47c}$$

the last relation is the same as that in eq-(18) after reabsorbing factors of 2 in the definition of $\Sigma^{\mu\nu}$. Eqs-(44-47) are the defining relations of the Yang's Noncommutative $4D$ spacetime algebra involving the $8D$ phase-space variables X^μ, P^μ and the angular momentum (Lorentz) generators $\Sigma^{\mu\nu}$ in $4D$. The above commutators obey the Jacobi identities. An immediate consequence of Yang's noncommutative algebra is that now one has a modified products of uncertainties

$$\Delta X^\mu\ \Delta P^\nu \ \geq \ \frac{\hbar}{2}\ \eta^{\mu\nu}\ ||<\ \Sigma^{56}\ >||; \quad \Delta X^\mu\ \Delta X^\nu \ \geq \ \frac{\lambda^2}{2}\ ||<\ \Sigma^{\mu\nu}\ >||$$

$$\Delta P^\mu\ \Delta P^\nu \ \geq \ \frac{1}{2}(\frac{\hbar}{R})^2\ ||<\ \Sigma^{\mu\nu}\ >||. \tag{48}$$

The Noncommutative phase space Yang's algebra in $4D$ can be generalized to the Noncommutative Clifford phase space algebra associated to the $4D$ spacetime after following the same prescription as in eqs-(43) by invoking higher dimensions ($12D$ in this case instead of $6D$) as follows

$$X^\mu \ \leftrightarrow \ \lambda\ \Gamma^\mu \wedge \Gamma^5, \quad P^\mu \ \leftrightarrow \ \frac{\hbar}{R}\ \Gamma^\mu \wedge \Gamma^6. \tag{49}$$

$$X^{\mu_1\mu_2} \ \leftrightarrow \ \Upsilon^{[\mu_1\mu_2]\ [57]} \ \neq \ \lambda^2\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^5 \wedge \Gamma^7$$

$$P^{\mu_1\mu_2} \ \leftrightarrow \ \Upsilon^{[\mu_1\mu_2]\ [68]} \ \neq \ (\frac{\hbar}{R})^2\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^6 \wedge \Gamma^8. \tag{50}$$

$$X^{\mu_1\mu_2\mu_3} \ \leftrightarrow \ \Upsilon^{[\mu_1\mu_2\mu_3]\ [579]} \ \neq \ \lambda^3\ \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9$$

$$P^{\mu_1\mu_2\mu_3} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3]\ [6810]} \neq (\frac{\hbar}{R})^3\ \Gamma^{\mu_1}\wedge\Gamma^{\mu_2}\wedge\Gamma^{\mu_3}\wedge\Gamma^6\wedge\Gamma^8\wedge\Gamma^{10}. \quad (51)$$

$$X^{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4]\ [57911]} \neq \lambda^4\ \Gamma^{\mu_1}\wedge\Gamma^{\mu_2}\wedge\Gamma^{\mu_3}\wedge\Gamma^{\mu_4}\wedge\Gamma^5\wedge\Gamma^7\wedge\Gamma^9\wedge\Gamma^{11}$$

$$P^{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4]\ [681012]} \neq (\frac{\hbar}{R})^4\ \Gamma^{\mu_1}\wedge\Gamma^{\mu_2}\wedge\Gamma^{\mu_3}\wedge\Gamma^{\mu_4}\wedge\Gamma^6\wedge\Gamma^8\wedge\Gamma^{10}\wedge\Gamma^{12}. \quad (52)$$

The indices $\mu_1, \mu_2, \mu_3, \mu_4$ range from $1,2,3,4$. The extra indices span 8 additional directions (dimensions) leaving a total dimension of $4 + 8 = 12$. The *noncommutative* Clifford phase space algebra commutators are defined in terms of the algebra

$$[\Upsilon^{MN},\ \Upsilon^{PQ}] = -i\,G^{MP}\ \Upsilon^{NQ} + i\,G^{MQ}\ \Upsilon^{NP} + i\,G^{NP}\ \Upsilon^{MQ} - i\,G^{NQ}\ \Upsilon^{MP} \quad (53)$$

The generators obey $\Upsilon^{MN} = -\Upsilon^{NM}$, and $G^{MN} = G^{NM}$ under an exchange of multi-indices $M \leftrightarrow N$.

The algebra (53) has the same structure as a *generalized spin algebra* and satisfies the Jacobi identities. We must stress that

$$[\Upsilon^{MN},\ \Upsilon^{PQ}] \neq [\ [\Gamma^M, \Gamma^N],\ [\Gamma^P, \Gamma^Q]\ ]. \quad (54)$$

except in the special case when M, N, P, Q are all *bivector* indices : hence we must *emphasize* that the generalized spin algebra (53) *is not isomorphic* to the noncommutative algebra of eqs-(15-24) ! For example, from the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[\nu_1\nu_2\nu_3]\ [6810]}] = -i\ G^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}\ \Upsilon^{[579]\ [6810]}. \quad (55a)$$

one can infer the Weyl-Heisenberg algebra commutator

$$[X^{\mu_1\mu_2\mu_3},\ P^{\nu_1\nu_2\nu_3}] = -i\ \hbar^3\ G^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}\ \Upsilon^{[579]\ [6810]}. \quad (55b)$$

From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[\nu_1\nu_2\nu_3]\ [579]}] = -i\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}. \quad (56a)$$

one can infer the commutator among the tri-vector coordinates

$$[X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}] = -i\ \lambda^6\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}. \quad (56b)$$

where $\Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}$ is a generalized angular momentum (spin) generator. From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[579]\ [6810]}] = i\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [6810]}. \quad (57a)$$

one can infer the commutator

$$[X^{\mu_1\mu_2\mu_3},\ \Upsilon^{[579]\ [6810]}] \ = \ i\ \lambda^6\ \frac{1}{\hbar^3}\ G^{[579]\ [579]}\ P^{\mu_1\mu_2\mu_3}. \qquad (57b)$$

which *exchanges* the $X^{\mu_1\mu_2\mu_3}$ for $P^{\mu_1\mu_2\mu_3}$, etc ..... Therefore, eqs-(55,56,57) are the suitable tri-vector analog of eqs-(44,45,46). Clearly, the above non-vanishing commutators $differ$ from those in eqs-(15-24) and will modify the QM wave equations when one introduces potential terms like $V(X) = g(X * X * .... * X)$. QM in ordinary (commutative) C-spaces can be found in [11].

Having provided the basic ideas and results behind polyvector gauge field theories in Noncommutative Clifford spaces, the construction of Noncommutative Clifford-space gravity as polyvector valued gauge theories of twisted diffeomorphisms in C-spaces will be the subject of future investigations. It would require quantum Hopf algebraic deformations of Clifford algebras [9]. Such theory is far richer than gravity in Noncommutative spacetimes [13].

Acknowledgments

We thank M. Bowers for her assistance.

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