On the Coupling Constants, Geometric Probability and Complex Domains*

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By recurring to Geometric Probability methods it is shown that the coupling constants, $\alpha_{EM}, \alpha_W, \alpha_C$, associated with the electromagnetic, weak and strong (color) force are given by the *ratios* of measures of the sphere S^2 and the Shilov boundaries $Q_3 = S^2 \times RP^1$, *squashed* S^5, respectively, with respect to the Wyler measure $\Omega_{Wyler}[Q_4]$ of the Shilov boundary $Q_4 = S^3 \times RP^1$ of the poly-disc D_4 (8 real dimensions). The latter measure $\Omega_{Wyler}[Q_4]$ is linked to the geometric coupling strength α_G associated to the gravitational force. In the conclusion we discuss briefly other approaches to the determination of the physical constants, in particular, a program based on the Mersenne primes p-adic hierarchy. The most important conclusion of this work is the role played by higher dimensions in the determination of the coupling constants from pure geometry and topology alone and which does *not* require to invoke the anthropic principle.

1 Geometric probability

Geometric Probability [1] is the study of the probabilities involved in geometric problems — the distributions of length, area, volume, etc. for geometric objects under stated conditions. One of the most famous problem is the Buffon's Needle Problem of finding the probability that a needle of length l will land on a line, given a floor with equally spaced parallel lines a distance d apart. The problem was posed by the French naturalist Buffon in 1733. For $l < d$ the probability is

$$P = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \frac{l\,|\cos\theta|}{d} = \frac{4l}{2\pi d}\int_0^{\frac{\pi}{2}} \cos\theta = \frac{2l}{\pi d} = \frac{2ld}{\pi d^2}. \quad (1.1)$$

Hence, the Geometric Probability is essentially the *ratio* of the areas of a rectangle of length $2d$, and width l and the area of a circle of radius d. For $l > d$, the solution is slightly more complicated [1]. The Buffon needle problem provides with a numerical experiment that determines the value of π empirically. Geometric Probability is a vast field with profound connections to Stochastic Geometry.

Feynman long ago speculated that the fine structure constant may be related to π. This is the case as Wyler found long ago [2]. We will take the fine structure constant based on Feynman's physical interpretation of the electron's charge as the probability amplitude that an electron emits/absorbs a photon. The clue to evaluate this probability within the context of Geometric Probability theory is provided by the electron self-energy diagram. Using Feynman's rules, the self-energy $\Sigma(p)$ as a function of the electron's incoming/outgoing energy-momentum p_μ is given by the integral involving the photon and electron propagator along the internal lines

$$-i\,\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu \frac{i}{\gamma^\rho (p_\rho - k_\rho) - m}\, \frac{-ig_{\mu\nu}}{k^2}\, \gamma^\nu. \quad (1.2)$$

The integral is taken with respect to the values of the photon's energy-momentum k^μ. By inspection one can see that the electron self-energy is proportional to the fine structure constant $\alpha_{EM} \sim e^2$, the square of the probability amplitude (in natural units of $\hbar = c = 1$) and physically represents the electron's emission of a virtual photon (off-shell, $k^2 \neq 0$) of energy-momentum k_ρ at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the Geometric Probability that an electron emits a photon at $t = -\infty$ (infinite past) and re-absorbs it at a much later time $t = +\infty$ (infinite future). The off-shell (virtual) photon associated with the electron self-energy diagram *asymptotically* behaves on-shell at the very moment of emission ($t = -\infty$) and absorption ($t = +\infty$). However, the photon can remain off-shell in the intermediate region between the moments of emission and absorption by the electron. The fact that Geometric Probability is a classical theory does not mean that one cannot derive the fine structure constant (which involves the Planck constant) because the electron self-energy diagram is itself a quantum (one-loop) Feynman process; i. e. one can recur to Geometric Probability to assign proper geometrical measures to Feynman diagrams, not unlike the Twistor-diagrammatic version of the Feynman rules of QFT.

The topology of the boundaries (at conformal infinity) of the past and future light-cones are spheres S^2 (the celestial sphere). This explains why the (Shilov) boundaries are essential mathematical features to understand the geometric derivation of all the coupling constants. In order to describe the physics at infinity we will recur to Penrose's ideas [12] of conformal compactifications of Minkowski spacetime by attaching the light-cones at conformal infinity. Not unlike the one-point compactification of the complex plane by adding the points at infinity leading to the Gauss-Riemann sphere.

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The conformal group leaves the light-cone fixed and it does not alter the causal properties of spacetime despite the rescalings of the metric. The topology of the conformal compactification of real Minkowski spacetime $\bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ is precisely the same as the topology of the Shilov boundary Q_4 of the 4 complex-dimensional poly-disc D_4. The action of the discrete group Z_2 amounts to an antipodal identification of the future null infinity $\mathcal{I}^+$ with the past null infinity $\mathcal{I}^-$; and the antipodal identification of the past timelike infinity i^- with the future timelike infinity, i^+, where the electron emits, and absorbs the photon, respectively.

Shilov boundaries of homogeneous (symmetric spaces) complex domains, G/K [9]–[11] are not the same as the ordinary topological boundaries (except in some special cases). The reason being that the action of the isotropy group K of the origin is not necesarily *transitive* on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries which implement the Maldacena-'T Hooft-Susskind holographic principle [15] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the poly-disc D_4 of 4 complex dimensions is an 8 real-dim Hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic *curved* phase space associated with the electron (a particle) moving in a $4D$ Anti de Sitter space AdS_4. The poly-disc is a Hermitian symmetric homogeneous coset space associated with the $4D$ conformal group $SO(4, 2)$ since $D_4 = SO(4, 2) / SO(4) \times SO(2)$. Its Shilov boundary *Shilov* $(D_4) = Q_4$ has precisely the *same* topology as the $4D$ conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$. For more details about Shilov boundaries, the conformal group, future tubes and holography we refer to the article by Gibbons [14] and [9], [18].

The role of the conformal group in gravity in these expressions (besides the holographic bulk/boundary *AdS*/*CFT* duality correspondence [15]) stems from the MacDowell Mansouri-Chamseddine-West formulation of gravity based on the conformal group $SO(3, 2)$ which has the same number of 10 generators as the $4D$ Poincaré group. The $4D$ vielbein e^a_μ which gauges the spacetime translations is identified with the $SO(3, 2)$ generator $A^{[a5]}_\mu$, up to a crucial scale factor R, given by the size of the Anti de Sitter space (de Sitter space) throat. It is known that the Poincaré group is the Wigner-Inonu group contraction of the de Sitter Group $SO(4,1)$ after taking the throat size $R = \infty$. The spin-connection ω^{ab}_μ that gauges the Lorentz transformations is identified with the $SO(3, 2)$ generator $A^{[ab]}_\mu$. In this fashion, the e^a_μ, ω^{ab}_μ are encoded into the $A^{[mn]}_\mu$ $SO(3, 2)$ gauge fields, where m, n run over the group indices 1, 2, 3, 4, 5. A word of caution, gravity is a gauge theory of the full diffeomorphisms group which is infinite-dimensional and which includes the translations. Therefore, strictly speaking gravity is not a gauge theory of the Poincaré group. The Ogiovetsky theorem shows that the diffeomorphisms algebra in $4D$ can be generated by an infinity of *nested* commutators involving the $GL(4, R)$ and the $4D$ Conformal Group $SO(4, 2)$ generators.

In [19] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal $SO(3, 2)$ group. The AdS_4 space is a natural vacuum of the theory. The vacuum energy density was *derived* to be precisely the geometric-mean between the UV Planck scale and the IR throat size of de Sitter (Anti de Sitter) space. Setting the throat size to coincide with the future horizon scale (of an accelerated de Sitter Universe) given by the Hubble scale (today) R_H, the geometric mean relationship yields the observed value of the vacuum energy density $\rho \sim (L_P R_H)^{-2} = (L_P)^{-4} (L_P^2/R_H^2) \sim 10^{-120} M^4_{Planck}$. Nottale [24] gave a different argument to explain the small value of ρ based on Scale Relativistic arguments. It was also shown in [19] why the Euclideanized AdS_{2n} spaces are $SO(2n-1, 2)$ instantons solutions of a non-linear sigma model obeying a double self duality condition.

A typical objection to the possibility of being able to derive the values of the coupling constants, from pure thought alone, is that there are an infinite number of possible analytical expressions that accurately reproduce the values of the couplings within the experimental error bounds. However, this is not our case because once the gauge groups $U(1)$, $SU(2)$, $SU(3)$ are known there are *unique* expressions stemming from Geometric Probability which furnish the values of the couplings. Another objection is that it is a meaningless task to try to derive these couplings because these are not constants per se but vary with respect to the energy scale. The running of the coupling constants is an *artifact* of the perturbative Renormalization Group program. We will see that the values of the couplings derived from Geometric Probability are precisely those values that correspond to the natural physical scales associated with the EM, Weak and Strong forces.

Another objection is that physical measurements of irrational numbers are impossible because there are always experimental limitations which rule out the possibility of actually measuring the *infinite* number of digits of an irrational number. This experimental constraint does not exclude the possibility of deriving exact expressions based on π as we shall see. We should not worry also about obtaining numerical values within the error bars in the table of the coupling constants since these numbers are based on the values of *other* physical constants; i. e. they are based on the particular *consensus* chosen for all of the other physical constants.

In our conventions, $\alpha_{EM} = e^2/4\pi = 1/137.036\ldots$ in the natural units of $\hbar = c = 1$, and the quantities $\alpha_{weak}, \alpha_{color}$ are the Geometric Probabilities $\tilde{g}^2_w, \tilde{g}^2_c$, after *absorbing* the factors of 4π of the conventional $\alpha_W = (g^2_w/4\pi)$, $\alpha_C = (g^2_c/4\pi)$ definitions used in the Renormalization Group (RG) program.

2 The fine structure constant

In order to define the Geometric Probability associated with this process of the electron's emission of a photon at i^- ($t = -\infty$), followed by an absorption at i^+ ($t = +\infty$), we must take into account the important fact that the photon is on-shell $k^2 = 0$ *asymptotically* (at $t = \pm\infty$), but it can move off-shell $k^2 \neq 0$ in the intermediate region which is represented by the *interior* of the $4D$ conformally compactified real Minkowski spacetime which agrees with the Shilov boundary of D_4 (the four-complex-dimensional poly-disc) $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$. The Q_4 has four-real-dimensions which is half the real-dimensions of D_4 (2×4=8).

The measure associated with the celestial spheres S^2 (associated with the future/past light-cones) at timelike infinity i^+, i^-, respectively, is $V(S^2) = 4\pi r^2 = 4\pi$ $(r = 1)$. Thus, the *net* measure corresponding to the two celestial spheres S^2 at timelike infinity $i^\pm$ requires an overall factor of 2 giving $2V(S^2) = 8\pi$ $(r = 1)$. The factor of $8\pi = 2 \times 4\pi$ can also be interpreted in terms of the two-helicity degrees of freedom, corresponding to a spin 1 massless photon, assigned to the area of the celestial sphere. The Geometric Probability is defined by the ratio of the (dimensionless volumes) measures associated with the celestial spheres S^2 at i^+, i^- timelike infinity, where the photon moves on-shell, relative to the Wyler measure $\Omega_{Wyler}[Q_4]$ associated with the full *interior* region of the conformally compactified $4D$ Minkwoski space $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$, where the massive electron is confined to move, as it propagates from i^- to i^+, (and *off-shell* photons can also live in):

$$\alpha_{EM} = \frac{2V(S^2)}{\Omega_{Wyler}[Q_4]} = \frac{8\pi}{\Omega_{Wyler}[Q_4]} = \frac{1}{137.03608\ldots} \tag{2.1a}$$

after inserting the Wyler measure

$$\Omega_{Wyler}[Q_4] = \frac{V(S^4)V(Q_5)}{[V(D_5)]^{\frac{1}{4}}} = \frac{8\pi^2}{3}\,\frac{8\pi^3}{3}\left(\frac{\pi^5}{2^4 \times 5!}\right)^{-\frac{1}{4}}. \tag{2.1b}$$

The Wyler measure $\Omega_{Wyler}[Q_4]$ [2] is *not* the standard measure (dimensionless volume) $V(Q_4) = 2\pi^3$ calculated by Hua [10] but requires some elaborate procedure.

It was realized by Smith [5] that the presence of the Wyler measure in the expression for α_{EM} given by eq.-(2.1) was consistent with Wheeler ideas that the observed values of the coupling constants of the Electromagnetic, Weak and Strong Force can be obtained if the geometric force strengths (measures related to volumes of complex homogenous domains associated with the $U(1)$, $SU(2)$, $SU(3)$ groups, respectively) are all *divided* by the geometric force strength of gravity α_G (related to the $SO(3,2)$ MMCW Gauge Theory of Gravity) and which is not the same as the $4D$ Newton's gravitational constant $G_N \sim m^{-2}_{Planck}$. Hence, upon dividing these geometric force strengths by the geometric force strength of gravity α_G one is dividing by the Wyler measure factor because (see below) $\alpha_G \equiv \Omega_{Wyler}[Q_4]$.

Furthermore, the expression for $\Omega_{Wyler}[Q_4]$ is also consistent with the Kaluza-Klein compactification procedure of obtaining Maxwell's EM in $4D$ from *pure* gravity in $5D$ since Wyler's expression involves a $5D$ domain D_5 from the very start; i. e. in order to evaluate the Wyler measure Ω_{Wyler} $[Q_4]$ one requires to embed D_4 into D_5 because the Shilov boundary space $Q_4 = S^3 \times RP^1$ is *not* adequate enough to implement the action of the $SO(5)$ group, the compact version of the Anti de Sitter Group $SO(3,2)$ that is required in the MacDowell-Mansouri-Chamseddine-West (MMCW) $SO(3,2)$ gauge formulation of gravity. However, the Shilov boundary of D_5 given by $Q_5 = S^4 \times RP^1$ is adequate enough to implement the action of $SO(5)$ via isometries (rotations) on the internal symmetry space $S^4 = SO(5)/SO(4)$. This justifies the embedding procedure of $D_4 \to D_5$

The 5 complex-dimensional poly-disc $D_5 = SO(5,2)/SO(5) \times SO(2)$ is the 10 real-dim Hyperboloid $\mathcal{H}^{10}$ corresponding to the relativistic curved phase space of a particle moving in $5D$ Anti de Sitter Space AdS_5. The Shilov boundary Q_5 of D_5 has 5 real dimensions (half of the 10-real-dim of D_5). One cannot fail to notice that the hyperboloid $\mathcal{H}^{10}$ can be embedded in the 11-dim pseudo-Euclidean $R^{9,2}$ space, with two-time like directions. This is where 11-dim lurks into our construction.

Having displayed Wyler's expression of the fine structure constant α_{EM} in terms of the ratio of dimensionless measures, we shall present a Fiber Bundle (a sphere bundle fibration over a complex homogeneous domain) derivation of the Wyler expression based on the bundle $S^4 \to E \to D_5$, and explain below why the propagation (via the determinant of the Feynman propagator) of the electron through the *interior* of the domain D_5 is what accounts for the "obscure" factor $V(D_5)^{1/4}$ in Wyler's formula for α_{EM}.

We begin by explaining why Wyler's measure $\Omega_{Wyler}[Q_4]$ in eq.-(2.1) corresponds to the measure of a S^4 bundle fibered over the base curved-space $D_5 = SO(5,2)/SO(5) \times SO(2)$ and *weighted* by a factor of $V(D_5)^{-1/4}$. This $S^4 \to E \to D_5$ bundle is linked to the MMCW $SO(3,2)$ Gauge Theory formulation of gravity and explains the essential role of the gravitational interaction of the electron in Wyler's formula [5] corroborating Wheeler's ideas that one must normalize the geometric force strengths with respect to gravity in order to obtain the coupling constants. The subgroup $H = SO(5)$ of the isotropy group (at the origin) $K = SO(5) \times SO(2)$ acts naturally on the Fibers $F = S^4 = SO(5)/SO(4)$, the internal symmetric space, via isometries (rotations). Locally, and only locally, the Fiber bundle E is the product $D_5 \times S^4$.

The restriction of the Fiber bundle E to the Shilov boundary Q_5 is written as $E|_{Q_5}$ and *locally* is the product of $Q_5 \times S^4$, but this is *not* true globally unless the fiber bundle admits a global section (the bundle is trivial). So, the volume $V(E|_{Q_5})$ does not necessary always factorize as $V(Q_5) \times V(S^4)$. Setting aside this subtlety, we shall pursue a more physical route, suggested by Wyler in unpublished work [3]*, to explain the origin of the "obscure normalization" factor $V(D_5)^{1/4}$ in Wyler's measure $\Omega_{Wyler}[Q_4] = (V(S^4) \times V(Q_5)/V(D_5)^{1/4})$, which suggests that the volumes may not factorize.

The relevant physical feature of this measure factor $V(D_5)^{1/4}$ is that it encodes the *spinorial* degrees of freedom of the electron, like the factor of 8π encodes the two-helicity states of the massless photon. The Feynman propagator of a massive scalar particle (inverse of the Klein-Gordon operator) $(D_\mu D^\mu - m^2)^{-1}$ corresponds to the *kernel* in the Feynman path integral that in turn is associated with the Bergman kernel $K_n(z, z')$ of the complex homogenous domain D_n, proportional to the Bergman constant $k_n \equiv 1/V(D_n)$, i. e.

$$(D_\mu D^\mu - m^2)^{-1}(x^\mu) = \frac{1}{(2\pi\mu)^D} \int d^D p \, \frac{e^{-ip_\mu x^\mu}}{p^2 - m^2 + i\epsilon} \leftrightarrow \\ \leftrightarrow K_n(\mathbf{z}, \bar{\mathbf{z}}') = \frac{1}{V(D_n)} (1 - \mathbf{z}\bar{\mathbf{z}}')^{-2n}, \tag{2.2}$$

where we have introduced a momentum scale μ to match units in the Feynman propagator expression, and the Bergman Kernel $K_n(\mathbf{z}, \bar{\mathbf{z}}')$ of D_n whose dimensionless entries are $\mathbf{z} = (z_1, z_2, \ldots, z_n)$, $\mathbf{z}' = (z'_1, z'_2, \ldots, z'_n)$ is given as

$$K_n(\mathbf{z}, \bar{\mathbf{z}}') = \frac{1}{V(D_n)} (1 - \mathbf{z}\bar{\mathbf{z}}')^{-2n} \tag{2.3a}$$

$V(D_n)$ is the dimensionless Euclidean volume found by Hua $V(D_n) = (\pi^n / 2^{n-1} n!)$ and satisfies the reproducing and normalization properties

$$f(z) = \int_{D_n} f(\xi) \, K_n(z, \xi) d^n \xi d^n \bar{\xi} \,, \quad \int_{D_n} K_n(z, \bar{z}) d^n z d^n \bar{z} = 1. \tag{2.3b}$$

The *key* result that can be inferred from the Feynman propagator (kernel) $\leftrightarrow$ Bergman kernel K_n correspondence, when $\mu = 1$, is the $(2\pi)^{-D} \leftrightarrow (V(D_n))^{-1}$ correspondence; i. e. the fundamental hyper-cell in momentum space $(2\pi)^D$ (when μ=1) corresponds to the dimensionless volume $V(D_n)$ of the domain, where $D = 2n$ real dimensions. The regularized vacuum-to-vacuum amplitude of a free *real* scalar field is given in terms of the zeta function $\zeta(s) = \sum_i \lambda_i^{-s}$ associated with the eigenvalues of the Klein-Gordon operator by

$$Z = <0|0> = \sqrt{\det(D_\mu D^\mu - m^2)^{-1}} \sim \exp\left[\frac{1}{2}\frac{d\zeta}{ds}(s=0)\right]. \tag{2.4}$$

In case of a *complex* scalar field we have to *double* the number of degrees of freedom, the amplitude then factorizes into a product and becomes $Z = \det(D_\mu D^\mu - m^2)^{-1}$.

Since the Dirac operator $\mathcal{D} = \gamma^\mu D_\mu + m$ is the "square-root" of the Klein-Gordon operator $\mathcal{D}^\dagger \mathcal{D} = D_\mu D^\mu - m^2 + \mathcal{R}$ ($\mathcal{R}$ is the scalar curvature of spacetime that is zero in Minkowski space) we have the numerical correspondence

$$\sqrt{\det(\mathcal{D})^{-1}} = \sqrt{\det\left(D_\mu D^\mu - m^2\right)^{-1/2}} = \\ = \sqrt{\sqrt{\det\left(D_\mu D^\mu - m^2\right)^{-1}}} \leftrightarrow k_n^{1/4} = \left(\frac{1}{V(D_n)}\right)^{1/4}, \tag{2.5}$$

because $\det \mathcal{D}^\dagger = \det \mathcal{D}$, and

$$\det \mathcal{D} = e^{\operatorname{tr} \ln \mathcal{D}} = e^{\operatorname{tr} \ln (D_\mu D^\mu - m^2)^{1/2}} = \\ = e^{\frac{1}{2} \operatorname{tr} \ln (D_\mu D^\mu - m^2)} = \sqrt{\det(D_\mu D^\mu - m^2)} \,. \tag{2.6}$$

The vacuum-to-vacuum amplitude of a *complex* Dirac field Ψ (a fermion, the electron) is $Z = \det(\gamma^\mu D_\mu + m) = \det \mathcal{D} \sim \exp[-(d\zeta/ds)(s=0)]$. Notice the $\det(\mathcal{D})$ behavior of the fermion versus the $\det(D_\mu D^\mu - m^2)^{-1}$ behavior of a complex scalar field due to the Grassmanian nature of the Gaussian path integral of the fermions. The vacuum-to-vacuum amplitude of a Majorana (real) spinor (half of the number of degrees of freedom of a complex Dirac spinor) is $Z = \sqrt{\det(\gamma^\mu D_\mu + m)}$. Because the complex Dirac spinor encodes both the dynamics of the electron and its anti-particle, the positron (the negative energy solutions), the vacuum-to-vacuum amplitude corresponding to the electron (positive energy solutions, propagating forward in time) must be then $Z = \sqrt{\det(\gamma^\mu D_\mu + m)}$.

Therefore, to sum up, the origin of the "obscure" factor $V(D_5)^{1/4}$ in Wyler's formula is the *normalization* condition of $V(S^4) \times V(Q_5)$ by a factor of $V(D_5)^{1/4}$ stemming from the correspondence $V(D_5)^{1/4} \leftrightarrow Z = \sqrt{\det(\gamma^\mu D_\mu + m)}$ and which originates from the vacuum-to-vacuum amplitude of the fermion (electron) as it propagates forward in time in the domain D_5. These last relations emerge from the correspondence between the Feynman fermion (electron) propagator in Minkowski spacetime and the Bergman Kernel of the complex homogenous domain after performing the Wyler map between an unbounded domain (the interior of the future lightcone of spacetime) to a bounded one. In general, the Bergman Kernel gives rise to a Kahler potential $F(z, \bar{z}) = \log K(z, \bar{z})$ in terms of which the Bergman metric on D_n is given by

$$g_{i\bar{j}} = \frac{\partial^2 F}{\partial z^i \partial \bar{z}^i}. \tag{2.7}$$

We must emphasize that this Geometric probability explanation *is very different* from the interpretations provided in [2, 5, 6, 7] and properly accounts for all the numerical factors. Concluding, the Geometric Probability that an electron emits a photon at $t = -\infty$ and absorbs it at $t = +\infty$, is given by the *ratio* of the dimensionless measures (volumes):

$$\alpha_{EM} = \frac{2V(S^2)}{\Omega_{Wyler}[Q_4]} = 8\pi \frac{1}{V(S^4)} \frac{1}{V(Q_5)} [V(D_5)]^{\frac{1}{4}} = \\ = \frac{9}{8\pi^4} \left(\frac{\pi^5}{2^4 \times 5!}\right)^{1/4} = \frac{1}{137.03608\ldots} \tag{2.8}$$

*We thank Frank (Tony) Smith for this information.

in very good agreement with the experimental value. This is easily verified after one inserts the values of the Euclideanized *regularized* volumes found by Hua [10]

$$V(D_5) = \frac{\pi^5}{2^4 \times 5!}, \; V(Q_5) = \frac{8\pi^3}{3}, \; V(S^4) = \frac{8\pi^2}{3}. \quad (2.19)$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1} n!}, \qquad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad (2.10)$$

$$\begin{aligned} V(Q_n) &= V(S^{n-1} \times RP^1) = V(S^{n-1}) \times V(RP^1) = \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)}. \end{aligned} \qquad (2.11)$$

Objections were raised to Wyler's original expression by Robertson [4]. One of them was that the hyperboloids (discs) are *not* compact and whose volumes diverge because the Lobachevsky metric diverges on the boundaries of the poly-discs. Gilmore explained [4] why one requires to use the Euclideanized regularized volumes because Wyler had shown that it is possible to map an unbounded physical domain (the interior of the future light cone) onto the interior of a homogenous bounded domain without losing the causal structure and on which there exist also a complex structure. A study of Shilov boundaries, holography and the future tube can be found in [14].

Furthermore, in order to resolve the scaling problems of Wyler's expression raised by Robertson, Gilmore showed why it is essential to use *dimensionless* volumes by setting the throat sizes of the Anti de Sitter hyperboloids to $r = 1$, because this is the only choice for r where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the scaling objections against Wyler raised by Robertson were satisfactory solved by Gilmore [4]. Thus, all the volumes in this section and forth, are based on setting the scaling factor $r = 1$.

The question as to *why* the value of α_{EM} obtained in Wyler's formula is precisely the value of α_{EM} observed at the *scale* of the Bohr radius a_B, has not been solved, to our knowledge. The Bohr radius is associated with the ground (most stable) state of the Hydrogen atom [5]. The spectrum generating group of the Hydrogen atom is well known to be the conformal group $SO(4, 2)$ due to the fact that there are two conserved vectors, the angular momentum and the Runge-Lentz vector. After quantization, one has two commuting $SU(2)$ copies $SO(4) = SU(2) \times SU(2)$. Thus, it makes physical sense why the Bohr-scale should appear in this construction. Bars [16] has studied the many physical applications and relationships of many seemingly distinct models of particles, strings, branes and twistors, based on the (super) conformal groups in diverse dimensions. In particular, the relevance of two-time physics in the formulation of M, F, S theory has been advanced by Bars for some time. The Bohr radius corresponds to an energy of $137.036 \times 2 \times 13.6$ eV $\sim 3.72 \times 10^3$eV. It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of electron, and the classical electron radius are all related to each other by a successive scaling in products of α_{EM}.

To finalize this section and based on the MMCW $SO(3,2)$ Gauge Theory formulation of gravity, with a Gauss-Bonnet topological term plus a cosmological constant, the (dimensionless) Wyler measure was *defined* as the geometric coupling strength of gravity [5]:

$$\Omega_{Wyler}[Q_4] = \frac{V(S^4)V(Q_5)}{[V(D_5)]^{\frac{1}{4}}} \equiv \alpha_G \,. \qquad (2.12)$$

The relationship between α_G and the Newtonian gravitational G constant is based on the value of the coupling $(1/16\pi G)$ appearing in the Einstein-Hilbert Lagrangian $(R/16\pi G)$, and goes as follows:

$$\begin{aligned} &(16\pi G)(m^2_{Planck}) = \alpha_{EM}\alpha_G = 8\pi \Rightarrow \\ &\Rightarrow G = \frac{1}{16\pi}\frac{8\pi}{m^2_{Planck}} = \frac{1}{2m^2_{Planck}} \Rightarrow \\ &\Rightarrow G m^2_{proton} = \frac{1}{2}\left(\frac{m_{proton}}{m_{Planck}}\right)^2 \sim 5.9 \times 10^{-39} \end{aligned} \qquad (2.13)$$

and in natural units $\hbar = c = 1$ yields the physical force strength of gravity at the Planck Energy scale 1.22×10^{19} GeV. The Planck mass is obtained by equating the Schwarzschild radius $2G\,m_{Planck}$ to the Compton wavelength $1/m_{Planck}$ associated with the mass; where $m_{Planck}\sqrt{2} = 1.22 \times 10^{19}$ GeV and the proton mass is 0.938 GeV. Some authors define the Planck mass by absorbing the factor of $\sqrt{2}$ inside the definition of $m_{Planck} = 1.22 \times 10^{19}$ GeV.

3 The weak and strong couplings

We turn now to the derivation of the other coupling constants. The Fiber Bundle picture of the previous section is essential in our construction. The Weak and the Strong geometric coupling constant strength, defined as the probability for a particle to emit and later absorb a $SU(2)$, $SU(3)$ gauge boson, can both be obtained by using the main formula derived from Geometric Probability (as ratios of dimensionless measures/volumes) after one identifies the suitable homogeneous domains and their Shilov boundaries to work with.

Since massless gauge bosons live on the lightcone, a null boundary in Minkowski spacetime, upon performing the Wyler map, the gauge bosons are confined to live on the Shilov boundary. Because the $SU(2)$ bosons $W^{\pm}$, Z^0 and the eight $SU(3)$ gluons have *internal* degrees of freedom (they carry weak and color charges) one must also include the measure associated with the their respective internal spaces; namely, the measures relevant to Geometric Probability calculations are the measures corresponding to the appropriate sphere bundles fibrations defined over the complex bounded homogenous domains $S^m \to E \to \mathcal{D}_n$.

Furthermore, the Geometric Probability interpretation for $\alpha_{weak}, \alpha_{strong}$ agrees with Wheeler's ideas [5] that one must normalize these geometric force strengths with respect to the geometric force strength of gravity $\alpha_G = \Omega_{Wyler}[Q_4]$ found in the last section. Hence, after these explanations, we will show below why the weak and strong couplings are given, respectively, by the *ratio* of the measures (dimensionless volumes):

$$\alpha_{weak} = \frac{\Omega[Q_3]}{\Omega_{Wyler}[Q_4]} = \frac{\Omega[Q_3]}{\alpha_G} = \frac{\Omega[Q_3]}{(8\pi/\alpha_{EM})}, \quad (3.1)$$

$$\alpha_{color} = \frac{\Omega[\textit{squashed}\, S^5]}{\Omega_{Wyler}[Q_4]} = \frac{\Omega[sq. S^5]}{\alpha_G} = \frac{\Omega[sq. S^5]}{(8\pi/\alpha_{EM})}. \quad (3.2)$$

As always, one must insert the values of the regularized (Euclideanized) dimensionless volumes provided by Hua [10] (set the scale $r = 1$). We must also clarify and emphasize that we define the quantities $\alpha_{weak}, \alpha_{color}$ as the probabilities $\tilde{g}_W^2$, $\tilde{g}_C^2$, by absorbing the factors of 4π in the conventional $\alpha_W = (g_W^2/4\pi)$, $\alpha_C = (g_C^2/4\pi)$ definitions (based on the Renormalization Group (RG) program) into our definitions of probability $\tilde{g}_W^2$, $\tilde{g}_C^2$.

Let us evaluate the α_{weak}. The internal symmetry space is $CP^1 = SU(2)/U(1)$ (a sphere $S^2 \sim CP^1$) where the isospin group $SU(2)$ acts via isometries on CP^1. The Shilov boundary of D_2 is $Q_2 = S^1 \times RP^1$ but is not adequate enough to accommodate the action of the isospin group $SU(2)$. One requires to have the Shilov boundary of D_3 given by $Q_3 = S^2 \times S^1/Z_2 = S^2 \times RP^1$ that can accommodate the action of the $SU(2)$ group on S^2. A Fiber Bundle over $D_3 = SO(3,2)/SO(3) \times SO(2)$ whose $H = SO(3) \sim SU(2)$ subgroup of the isotropy group (at the origin) $K = SO(3) \times SO(2)$ acts on S^2 by simple rotations. Thus, the relevant measure is related to the fiber bundle E restricted to Q_3 and is written as $V(E|_{Q_3})$.

One must notice that due to the fact that the $SU(2)$ group is a double-cover of $SO(3)$, as one goes from the $SO(3)$ action on S^2 to the $SU(2)$ action on S^2 , one must take into account an extra factor of 2 giving then

$$\begin{aligned} V(CP^1) &= V\big(SU(2)/U(1)\big) = \\ &= 2V\big(SO(3)/U(1)\big) = 2V(S^2) = 8\pi\,. \end{aligned} \quad (3.3)$$

In order to obtain the weak coupling constant due to the exchange of $W^{\pm} Z^0$ bosons in the four-point tree-level processes involving four leptons, like the electron, muon, tau, and their corresponding neutrinos (leptons are fundamental particles that are lighter than mesons and baryons) which are confined to move in the interior of the domain D_3, and can emit (absorb) $SU(2)$ gauge bosons, $W^{\pm} Z^0$, in the respective s, t, u channels, one must take into account a factor of the square root of the determinant of the fermionic propagator, $\sqrt{\det \mathcal{D}^{-1}} = \sqrt{\det\,(\gamma^\mu D_\mu + m)^{-1}}$, for each *pair* of leptons, as we did in the previous section when an electron emitted and absorbed a photon. Since there are *two* pairs of leptons in these four-point tree-level processes involving *four* leptons, one requires *two* factors of $\sqrt{\det\,(\gamma^\mu D_\mu + m)^{-1}}$, giving a net factor of $\det\,(\gamma^\mu D_\mu + m)^{-1}$ and which corresponds now to a net normalization factor of $k_n^{1/2} = (1/V(D_3))^{1/2}$, after implementing the Feynman kernel $\leftrightarrow$ Bergman kenel correspondence. Therefore, after taking into account the result of eq.-(3.3), the measure of the $S^2 \to E \to D_3$ bundle, restricted to the Shilov boundary Q_3, and weighted by the net normalization factor $(1/V(D_3))^{1/2}$, is

$$\Omega(Q^3) = 2V(S^2)\frac{V(Q_3)}{V(D_3)^{1/2}}\,. \quad (3.4)$$

Therefore, the Geometric probability expression is given by the ratio of measures (dimensionless volumes):

$$\begin{aligned} \alpha_{weak} &= \frac{\Omega[Q^3]}{\Omega_{Wyler}[Q_4]} = \frac{\Omega[Q^3]}{\alpha_G} = \frac{2V(S^2)V(Q_3)}{V(D_3)^{1/2}}\frac{\alpha_{EM}}{8\pi} = \\ &= (8\pi)(4\pi^2)\left(\frac{\pi^3}{24}\right)^{-\frac{1}{2}}\frac{\alpha_{EM}}{8\pi} = 0.2536\ldots \end{aligned} \quad (3.5)$$

that corresponds to the weak coupling constant ($g^2/4\pi$ based on the RG convention) at an energy of the order of

$$E = M = 146 \text{ GeV} \sim \sqrt{M_{W_+}^2 + M_{W_-}^2 + M_Z^2} \quad (3.6)$$

after the expressions inserted (setting the scale $r = 1$)

$$V(S^2) = 4\pi\,, \quad V(Q_3) = 4\pi^2, \quad V(D_3) = \frac{\pi^3}{24} \quad (3.7)$$

into the formula (3-5). The relationship to the Fermi coupling goes as follows (with the energy scale $E = M = 146$ GeV):

$$\begin{aligned} G_F \equiv \frac{\alpha_W}{M^2} &\Rightarrow G_F\, m_{proton}^2 = \left(\frac{\alpha_W}{M^2}\right) m_{proton}^2 = \\ &= 0.2536 \times \left(\frac{m_{proton}}{146 \text{ GeV}}\right)^2 \sim 1.04 \times 10^{-5} \end{aligned} \quad (3.8)$$

in very good agreement with experimental observations. Once more, it is unknown why the value of α_{weak} obtained from Geometric Probability corresponds to the energy scale related to the W_+, W_-, Z_0 boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of α_{color} from eq.-(3.2) after one defines what is the suitable fiber bundle. The calculation is based on the book by L. K. Hua [10, p. 40, 93]. The symmetric space with the $SU(3)$ color force as a local group is $SU(4)/SU(3) \times U(1)$ which corresponds to a bounded symmmetric domain of type $I(1,3)$ and has a Shilov boundary that Hua calls the "characteristic manifold" $CI(1,3)$. The volume $V\big(CI(m,n)\big)$ is:

$$V(CI) = \frac{(2\pi)^{mn - m(m-1)/2}}{(n-m)!\,(n-m+1)!\ldots(n-1)!} \quad (3.9)$$

so that for $m = 1$ and $n = 3$ the relevant volume is then $V(CI) = (2\pi)^3/2! = 4\pi^3$. We must remark at this point that $CI(1,3)$ is *not* the standard round S^5 but is the *squashed* five-dimensional $\tilde{S}^5$.*

The domain of which $CI(1,3)$ is the Shilov boundary is denoted by Hua as $RI(1,3)$ and whose volume is

$$V(RI) = \frac{1!\,2!\ldots(m-1)!\,1!\,2!\ldots(n-1)!\,\pi^{mn}}{1!\,2!\ldots(m+n-1)!} \quad (3.10)$$

so that for $m=1$ and $n=3$ it gives $V(RI)=1!2!\pi^3/1!2!3!= =\pi^3/6$ and it also agrees with the volume of the standard six-ball.

The internal symmetry space (fibers) is as follows $CP^2= =SU(3)/U(2)$ whose isometry group is the color $SU(3)$ group. The base space is the $6D$ domain $B_6=SU(4)/U(3)= =SU(4)/SU(3)\times U(1)$ whose subgroup $SU(3)$ of the isotropy group (at the origin) $K=SU(3)\times U(1)$ acts on the internal symmetry space CP^2 via isometries. In this special case, the Shilov and ordinary topological boundary of B_6 both coincide with the *squashed* S^5 [5].

Since Gilmore, in response to Robertson's objections to Wyler's formula [2], has shown that one must set the scale $r=1$ of the hyperboloids $\mathcal{H}^n$ (and S^n) and use *dimensionles* volumes, if we were to equate the volumes $V(CP^2)= =V(S^4, r=1)$ [5], this would be tantamount of choosing another scale [25] R (the unit of geodesic distance in CP^2) that is *different* from the unit of geodesic distance in S^4 when the radius $r=1$, as required by Gilmore. Hence, a bundle map $E\to E'$ from the bundle $CP^2\to E\to B_6$ to the bundle $S^4\to E'\to B_6$, would be required that would allow us to replace the $V(CP^2)$ for $V(S^4, r=1)$. Unless one decides to *calibrate* the unit of geodesic distance in CP^2 by choosing $V(CP^2)=V(S^4)$.

Using again the same results described after eq.-(2.2), since a quark can emit and absorb later on a $SU(3)$ gluon (in a one-loop process), and is confined to move in the interior of the domain B_6, there is *one* factor only of the square root of the determinant of the Dirac propagator $\sqrt{\det \mathcal{D}^{-1}}= =\sqrt{\sqrt{\det\,(D_\mu D^\mu - m^2)^{-1}}}$ and which is associated with a normalization factor of $k_n^{1/4}=(1/V(B_6))^{1/4}$. Therefore, the measure of the bundle $S^4\to E'\to B_6$ restricted to the *squashed* S^5 (Shilov boundary of B^6), and weighted by the normalization factor $(1/V(B_6))^{1/4}$, is then

$$\Omega[\textit{squashed}\, S^5] = \frac{V(S^4)\; V(\textit{squashed}\, S^5)}{V(B_6)^{1/4}} \quad (3.11)$$

and the ratio of measures

$$\alpha_s = \frac{\Omega[sq.S^5]}{\Omega_{Wyler}[Q_4]} = \frac{\Omega[sq.S^5]}{\alpha_G} = \frac{V(S^4)V(sq.S^5)}{V(B_6)^{1/4}}\frac{\alpha_{EM}}{8\pi} = = \left(\frac{8\pi^2}{3}\right)(4\pi^3)\left(\frac{\pi^3}{6}\right)^{-1/4}\frac{\alpha_{EM}}{8\pi} = 0.6286\ldots \quad (3.12)$$

matches, remarkably, the strong coupling value $\alpha=g^2/4\pi$ at an energy E related precisely to the pion masses [5]

$$E = 241 \text{ MeV} \sim \sqrt{m^2_{\pi^+} + m^2_{\pi^-} + m^2_{\pi^0}}. \quad (3.13)$$

The one-loop Renormalization Group flow of the coupling is given by [28]:

$$\alpha_s(E^2) = \alpha_s(E_0^2)\left[1 + \frac{\left(11 - \frac{2}{3}N_f(E^2)\right)}{4\pi}\alpha_s(E_0^2)\ln\left(\frac{E^2}{E_0^2}\right)\right]^{-1} \quad (3.14)$$

where $N_f(E^2)$ is the number of quark flavors whose mass $M^2 < E^2$. For the specific numerical details of the evaluation (in energy-intervals given by the diverse quark masses) of the Renormalization Group flow equation (3-14) that yields $\alpha_s(E=241\text{ MeV})\sim 0.6286$ we refer to [5]. Once more, it is unknown why the value of α_{color} obtained from Geometric Probability corresponds to the energy scale $E=241$ MeV related to the masses of the pions. The pions are the known lightest quark-antiquark pairs that feel the strong interaction.

Rigorously speaking, one should include higher-loop corrections to eq.-(3.14) as Weinberg showed [28] to determine the values of the strong coupling at energy scales $E=241$ MeV. This issue and the subtleties behind the calibration of scales (volumes) by imposing the condition $V(CP^2)=V(S^4)$ need to be investigated. For example, one could calibrate lengths in terms of the units of geodesic distance in CP^2 (based on Gilmore's choice of $r=1$) giving $V(CP^2)= =V(S^5; r=1)/V(S^1; r=1)=\pi^2/2!$ [25], and it leads now to the value of $\alpha_s=0.1178625$ which is very close to the value of α_s at the energy scale of the Z-boson mass (91.2 GeV) and given by $\alpha_s=0.118$ [28].

4 Mersenne primes p-adic hierarchy. Other approaches

To conclude, we briefly mention other approaches to the determination of the physical parameters. A hierarchy of coupling constants, including the cosmological constant, based on Seifert-spheres fibrations was undertaken by [26]. The ratios of particle masses, like the proton to electron mass ratio $m_p/m_e\sim 6\pi^5$ has also been calculated using the volumes of homogeneous bounded domains [5, 6]. A charge-mass-spin relationship was investigated in [27]. It is not known whether this procedure should work for Grand Unified Theories (GUT) based on the groups like $SU(5)$, $SO(10)$, E_6, E_7, E_8, meaning whether or not one could obtain, for example, the $SU(5)$ coupling constant consistent with the Grand Unification Models based on the $SU(5)$ group and with the Renormalization Group program at the GUT scale.

Beck [8] has obtained all of the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dim lattices (two-dim non-linear sigma-like models which resemble Feynman's chess-board lattice models) based on Stochastic Quantization methods. The results by Smith [5] (also based on Feynman's chess board models and hyper-diamond lattices) are analytical rather than being numerical [8] and it is not clear if there is any relationship between

*Frank (Tony) Smith, private communication.

these latter two approaches. Noyes has proposed an iterated numerical hierarchy based on Mersenne primes $M_p = 2^p - 1$ for *certain* values of $p = primes$ [20], and obtained a quite large number of satisfactory values for the physical parameters. An interesting coincidence is related to the iterated Mersenne prime sequence

$$\begin{aligned} &M_2 = 2^2 - 1 = 3\,, \qquad M_3 = 2^3 - 1 = 7\,, \\ &M_7 = 2^7 - 1 = 127\,, \quad 3 + 7 + 127 = 137\,, \\ &M_{127} = 2^{127} - 1 \sim 1.69{\times}10^{38} \sim \left(\frac{M_{Planck}}{m_{proton}}\right)^2. \end{aligned} \tag{4.1}$$

Pitkanen has also developed methods to calculate physical masses recurring to a p-adic hierarchy of scales based on Mersenne primes [21].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in [23]. These are numbers which can be written in terms of sums of its divisors, including unity, like $6 = 1 + 2 + 3$, and are of the form $P(p) = \frac{1}{2}\, 2^p(2^p - 1)$ if, and only if, $2^p - 1$ is a Mersenne prime. Not all values of $p = prime$ yields primes. The number $2^{11} - 1$ is not a Mersenne prime, for example. The number of generators of the anomaly free groups $SO(32)$, $E_8 \times E_8$ of the 10-dim superstring is 496 which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the $SO(2^{13}) = SO(8192)$ group related to the bosonic string compactified on the $E_8 \times SO(16)$ lattice. The number of generators of $SO(8192)$ is an even perfect number since $2^{13} - 1$ is a Mersenne prime. For an introduction to p-adic numbers in Physics and String theory see [22].

A lot more work needs to be done to be able to answer the question: is all this just a mere numerical coincidence or is it design? However, the results of the previous sections indicate that it is very *unlikely* that these results were just a mere numerical coincidence (senseless numerology) and that indeed the values of the physical constants could be actually calculated from pure thought, rather than invoking the anthropic principle; i. e. namely, based on the interplay of harmonic analysis, geometry, topology, higher dimensions and, ultimately, number theory. The fact that the coupling constants involved the ratio of measures (volumes) may cast some light on the role of the world-sheet areas of strings, and world volumes of p-branes, as they propagate in target spacetime backgrounds of diverse dimensions.

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