On Timelike Naked Singularities associated with Noncompact Matter Sources

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Abstract

We show the existence of *timelike* naked singularities which are *not* hidden by a horizon and which are associated to spherically symmetric (noncompact) matter sources extending from $r = 0$ to $r = \infty$. Our *asymptotically flat* solutions do represent observable *timelike* naked singularities where the scalar curvature $\mathcal{R}$ and volume mass density $\rho(r)$ are both *singular* at $r = 0$. To finalize we explain the Finsler geometric origins behind the matter field configuration obeying the weak energy conditions and that leads to a timelike naked singularity.

Long ago Penrose [1] proposed the Cosmic Censorship Conjecture (CCC) stating that singularities which form in a gravitational collapse and consistent with Einstein's [2] field equations should never be visible to an outside observer or they should be hidden inside a horizon. Recently, Deshingkar [3] studied singularities which can form in a spherically symmetric gravitational collapse of a general matter field obeying the weak energy condition. He showed that no energy can reach an outside observer from a null naked singularity. That means they will not be a serious threat to the Cosmic Censorship Conjecture (CCC). For timelike naked singularities, where only the central shell gets singular, the redshift is always finite and they can in principle, carry energy to a faraway observer. Hence for proving or disproving CCC the study of timelike naked singularities is more important. The results of [3] were very general and independent of the initial data and the form of the matter.

The purpose of this letter is to show the existence of *timelike* naked singularities which are *not* hidden by a horizon and which are associated to spherically symmetric *noncompact* matter sources extending for $r = 0$ to $r = \infty$. Our *asymptotically flat* solutions do represent observable *timelike* naked singularities where the scalar curvature $\mathcal{R}$ and volume mass density $\rho(r)$ are both *singular* at $r = 0$ while the metric remains finite at $r = 0$. The Einstein field equations [2] associated with the signature $(+,-,-,-)$ in natural units $\hbar = c = 1$ are

$$G_{00} = \mathcal{R}_{00} \ - \frac{1}{2} \ g_{00} \ \mathcal{R} \ = \ 8\pi G \ T_{00}; \quad \mathcal{R}_{ij} \ - \frac{1}{2} \ g_{ij} \ \mathcal{R} \ = \ 8\pi G \ T_{ij} \tag{1}$$

T_{ij} are the stress energy tensor elements comprised of a radial and tangential pressures similar to a self-gravitating *anisotropic* fluid studied by [8]

$$\rho(r) \ = \ - \ p_r(r), \quad p_{tangential} \ = \ p_\theta \ = \ p_\phi \ . \tag{2}$$

The components of the mixed stress energy tensor [1] are $T^\mu_\mu = (\rho(r), -p_r, -p_\theta, -p_\phi)$. The radial pressure $p_r = -\rho$ is negative, pointing radially inwards towards the center $r = 0$ consistent with the self-gravitating picture of the droplet. The continuity equation when $p_r(r) = -\rho(r)$ yields

$$\nabla_r \ T^r_r \ = \ 0 \ \Rightarrow \ p_\theta \ = \ p_\phi \ = \ - \ \rho(r) - \frac{r}{2} \ \frac{d\rho}{dr}. \tag{3}$$

which is the relationship between the density and pressure as shown in particular by [8]. Instead of choosing a delta function point mass source [13] or a smeared delta function given by the Gaussian [8]

$$\rho(r) \ = \ M_o \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ \Rightarrow \ lim_{\sigma\to 0} \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ \to \frac{\delta(r)}{4\pi r^2}. \tag{4}$$

we will choose instead the following volume mass density density distribution associated with a *noncompact* source extending from $r = 0$ to $r = \infty$

$$\rho(r,\sigma) \ = \ \frac{M_o}{4\pi r^2} \ \frac{3 \ \sigma^3}{2} \ \frac{1}{r^4 \ [\ 1 + (\sigma/r)^3 \ ]^{3/2}} \ ; \quad \sigma > 0. \tag{5}$$

which belongs to the most general two-parameter family of volume mass density distributions

$$\rho(r,\sigma,\kappa) \ = \ \frac{M_o}{4\pi r^2} \ \frac{\kappa \ \sigma^\kappa}{2} \ \frac{1}{r^{\kappa+1} \ [\ 1 + (\sigma/r)^\kappa \ ]^{3/2}} \ ; \quad \kappa > 2. \tag{6}$$

parametrized by the length scale σ and the exponent $\kappa > 2$. At the end of this work we shall explain the Finsler geometric origins behind the matter configuration of eq-(5) that leads to timelike naked singularities.

From eqs-(2,3,5) one learns that the weak energy conditions $\rho \geq 0$ and $\rho + p_i \geq 0$ $(i = 1,2,3)$ are satisfied due to $d\rho/dr < 0$; whereas the strong energy conditions $\rho + \sum_i p_i \geq 0$ are only satisfied in the region $r \geq \sigma/2$, but not in the central core region $r < \sigma/2$. While on the other hand, the dominant energy conditions $\rho \geq |p_i|$ are satisfied for the values of $r \leq \sigma(5/4)^{1/3}$ but violated for the values of $r > \sigma(5/4)^{1/3}$. Thus, the matter field configuration studied here obeys only the weak energy conditions. One may be inclined to question the

[1] we have a change of sign from [8]

physical validity of having a non-compact matter source extending all the way to infinity (mass density and pressure tending to zero as $r \rightarrow \infty$) , obeying the weak energy conditions, and having a singular density and pressure at $r = 0$. By the same token, one might have been inclined to question the physical validity of the Hilbert-Schwarzschild solution in 1916 (that led to the concept of black hole) based on an infinitely compact point-mass source (of zero extension) with a singular mass density distribution at $r = 0$. For this reason, we should not discard the model studied here.

The total mass enclosed by the noncompact shell between $r = 0$ and $r = \infty$ is given by the integral of $\rho(r,\sigma)\ 4\pi r^2$:

$$M_o \ \frac{3\ \sigma^3}{2} \int_0^\infty \frac{1}{r^4\ [\ 1 + (\sigma/r)^3\ ]^{3/2}}\ dr \ = \ M_o\ [\ \frac{1}{\sqrt{1+(\sigma/r)^3}}\ ]_{r=0}^{r=\infty} \ = \ M_o \quad (7)$$

Despite that $\rho(r = 0) = \infty$, the quantity $4\pi r^2\ \rho(r) \sim r^{1/2} \rightarrow 0$ as $r \rightarrow 0$, and this is the reason why the integral (7) converges to the finite value M_o which can be seen as the net mass being delocalized all over space, rather than being concentrated at the point $r = 0$ in the delta function source case, or smeared by the Gaussian distribution as in the case studied by [8] such that ρ and the scalar curvature $\mathcal{R}$ were $finite$ at the origin $r = 0$; whereas in our case both expressions *diverge* at $r = 0$.

The metric line element which solves Einstein' s equations, due to the presence of the volume mass density $\rho(r)$ distribution and pressure given by eqs-(2,3), can be obtained by a direct application of Birkoff's theorem by evaluating the (variable) mass $M(r,\sigma)$ enclosed by a shell of radius r

$$(ds)^2 \ = \ (1 - \frac{2G\ M(r,\sigma)}{r})\ (dt)^2\ - (1 - \frac{2G\ M(r,\sigma)}{r})^{-1}\ (dr)^2\ -\ r^2\ (d\Omega)^2. \quad (8)$$

where the solid angle infinitesimal element is $(d\Omega)^2 = (d\phi)^2 + sin^2(\phi)(d\theta)^2$. The mass enclosed in a spherical shell of radius r due to the density distribution $\rho(r,\sigma)$ is

$$M(r,\sigma) \ = \ M_o\ \frac{3\sigma^3}{2} \int_0^r \frac{1}{r^4\ [\ 1 + (\sigma/r)^3\ ]^{3/2}}\ dr \ = \ M_o\ [\frac{1}{\sqrt{1+(\sigma/r)^3}}]. \quad (9)$$

We can see that the metric component

$$g_{tt} \ = \ 1 - \frac{2GM_o}{r\ \sqrt{1+(\sigma/r)^3}}. \quad (10)$$

obeys the condition $g_{tt}(r = 0) = g_{tt}(r = \infty) = 1$. The metric given by eq-(8) is asymptotically flat and is *not* singular at $r = 0$, however the scalar curvature R is *singular* at $r = 0$ as we will show next. The scalar curvature can be obtained from Einstein's equations by taking the trace of $8\pi G\ (T_{\mu\nu})$ leading to

$$\mathcal{R} \;=\; -\; 8\pi G \; [\; 4\rho(r) \;+\; r \; \frac{\partial \rho}{\partial r}] \;=\; -\, 8\pi G \; T^{\mu}_{\mu} \;=\; -\, 8\pi G \; T. \qquad (11)$$

after recurring to eqs-(2,3). The expression for $\mathcal{R}$ coincides with the expression obtained directly from the metric components after a laborious but straightforward calculation

$$\mathcal{R} \;=\; -\, 2G \; [\; \frac{1}{r} \; \frac{\partial^2 M(r,\sigma)}{\partial r^2} \;+\; \frac{2}{r^2} \; \frac{\partial M(r,\sigma)}{\partial r} \;]. \qquad (12)$$

After using the relations

$$\rho(r,\sigma) \;=\; \frac{1}{4\pi r^2} \; \frac{\partial M(r,\sigma)}{\partial r}, \quad \frac{\partial^2 M(r,\sigma)}{\partial r^2} \;=\; 8\pi r \rho \;+\; 4\pi r^2 \frac{\partial \rho}{\partial r} \qquad (13)$$

one can verify that the expressions in eqs-(12,13) are the same indeed which is a corroboration of the validity of the metric solution (8) to the Einstein field equations associated to the matter distribution of eqs-(2,3). Thus, the scalar curvature associated with the mass function $M(r,\sigma)$ in eq-(10) becomes

$$\mathcal{R} \;=\; -(2GM_o) \; (\frac{3\;\sigma^3}{r^6}) \; [\; - \; \frac{1}{[\; 1 + (\sigma/r)^3 \;]^{3/2}} \;+\; \frac{9\;\sigma^3}{4\;r^3} \; \frac{1}{[\; 1 + (\sigma/r)^3 \;]^{5/2}} \;] \qquad (14)$$

and it is *singular* at $r = 0$ by inspection because $\mathcal{R}$ in eq-(15) behaves as

$$\mathcal{R} \;\sim\; -\, (2GM_o) \; (\frac{15}{4\;\sigma^{3/2}\;r^{3/2}}) \;\sim\; r^{-3/2} \rightarrow \infty \; as \; r \rightarrow 0. \qquad (15)$$

despite that the metric is *not* singular at $r = 0$, the reason being that the scalar curvature depends directly on the density which is singular $\rho(r = 0) = \infty$, but the metric depends on the mass function $M(r,\sigma)$ given by the integral (9) such that $4\pi r^2 \; \rho(r) \sim r^{1/2} \rightarrow 0$ as $r \rightarrow 0$. When $g_{tt}(r_h) = 0$ at the location of a horizon $r = r_h$ the component $g_{rr} = -(g_{tt})^{-1}$ blows up, however the singularity at $r = r_h$ is a coordinate singularity. Analytical extensions to the region $r < r_h$ where $g_{tt}(r) < 0$ is *spacelike* can be made following a similar procedure as performed by Fronsdal-Kruskal-Szekeres [6].

We may notice that by setting $\sigma \rightarrow 0$ in eq-(14), *prior* to taking the limit $r \rightarrow 0$, yields $\mathcal{R} \rightarrow 0$ as expected since one recovers then in eq-(8) the standard Hilbert-Schwarzschild metric [4], [5] which is the static spherically symmetric solution to the *vacuum* Einstein field equations. By replacing r for $|r|$ in the Hilbert-Schwarzschild metric one recovers the required delta function terms in the scalar curvature due to a point mass delta function source as shown [13]

$$\mathcal{R} \;=\; -\, 2GM_o \; [\; \frac{\delta'(r)}{r} \;+\; 2 \; \frac{\delta(r)}{r^2} \;]. \qquad (16)$$

due to the *discontinuity* of the derivative of the modulus function $|r|$ at $r = 0$. A rigorous mathematical treatment of point-mass distributions in nonlinear theories like gravity requires the use of Colombeau's calculus [7]. We will show now that depending on the values of the ratios $\sigma/2GM$ one can have one, two and no horizons. This can be derived by solving the cubic equation

$$g_{tt}(r,\sigma) \;=\; 0 \;\Rightarrow\; r^3 \;-\; (2GM_o)^2\; r \;+\; (\sigma)^3 \;=\; 0. \tag{17}$$

The 3 roots of the cubic equation are given in terms of the quantities S, T

$$S \;=\; [\;-\frac{\sigma^3}{2} + \sqrt{\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27}}\;]^{1/3}, \quad T \;=\; [\;-\frac{\sigma^3}{2} - \sqrt{\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27}}\;]^{1/3}. \tag{18}$$

as follows

$$r_1 \;=\; S + T \;=$$

$$[\;-\frac{\sigma^3}{2} \;+\; \sqrt{\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27}}\;]^{1/3} \;+\; [\;-\frac{\sigma^3}{2} \;-\; \sqrt{\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27}}\;]^{1/3}. \tag{19a}$$

$$r_2 \;=\; -\frac{1}{2}\,(S+T) \;+\; \frac{i\sqrt{3}}{2}\,(S-T), \quad r_3 \;=\; -\frac{1}{2}\,(S+T) \;-\; \frac{i\sqrt{3}}{2}\,(S-T). \tag{19b}$$

There are 3 cases to study :

• (1) when the discriminant is equal to zero : $\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27} = 0$, then $S = T$ and all the roots are real, there is a *double* root $r_2 = r_3 > 0$ and a negative root $r_1 < 0$ that is discarded. In this critical case there is *one* horizon corresponding to the location

$$r_2 \;=\; r_3 \;=\; 2^{-1/3}\;(\frac{4}{27})^{1/6}\;(2GM_o) \;=\; \frac{2GM_o}{\sqrt{3}}. \tag{20}$$

• (2) when the discriminant $\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27} > 0$, there is one real root $r_1 < 0$ which is discarded and two *complex* conjugates roots $r_2 = (r_3)^*$. In this case there is *no* horizon since there are no positive real values of r which solve the horizon condition $g_{tt}(r) = 0$.

• (3) when the discriminant $\frac{\sigma^6}{4} \;-\; \frac{(2GM_o)^6}{27} < 0$, there are 3 real distinct roots. From the graph of the cubic polynomial in eq-(17) we may infer that there is one negative and two positive roots. The negative real root is discarded and the two positive real roots correspond to the location of *two* horizons. The 3 real roots are expressed in terms of the angle $\alpha < 0$ as [10]

$$r_1 \;=\; -\frac{2}{\sqrt{3}}\,(2GM_o)\;cos(\frac{\alpha}{3}); \quad r_2 \;=\; -\frac{2}{\sqrt{3}}\,(2GM_o)\;cos(\frac{\alpha+2\pi}{3});$$

$$r_3 \;=\; -\frac{2}{\sqrt{3}}\,(2GM_o)\;cos(\frac{\alpha+4\pi}{3}). \tag{21}$$

The sum of the roots is $r_1 + r_2 + r_3 = 0$ so the 3 roots in eq-(21) correspond to the projections onto the real axis of the 3 vertices of an equilateral triangle (the vertices are spaced 120 degrees apart). The angle $\alpha < 0$ is

$$\alpha \;=\; cos^{-1} \;[\; - \frac{3\sqrt{3}}{2} \; (\frac{\sigma}{2GM_o})^3 \;] \;<\; 0 \;\Rightarrow\; | \; \frac{3\sqrt{3}}{2} \; (\frac{\sigma}{2GM_o})^3 \; | \;\leq\; 1. \tag{22}$$

since the range of the cosine function lies between -1 and 1. The existence of one, two and no horizons depends on the relative ratios of the length scales σ and $2GM_o$. When the net mass M_o is *smaller* (case (2)) than a certain critical value $M_{critical}$ given by the relation

$$\sigma \;=\; \frac{2^{1/3}}{\sqrt{3}} \; (2GM_{critical}) \;\Rightarrow\; \frac{\sigma}{2GM_{critical}} \;=\; \frac{2^{1/3}}{\sqrt{3}} \;\sim\; 0.727. \tag{23}$$

there is *no* horizon and in this case (2) we have an *observable naked timelike* singularity (which is not hidden behind any horizon since there is no horizon). When there is a horizon, $r = r_h$ and $g_{rr} = -(g_{tt})^{-1}$, the Hawking black hole temperature is defined by

$$T_H \;=\; \frac{1}{4\pi} \; (\frac{\partial g_{tt}(r)}{\partial r})_{r=r_h}. \tag{24}$$

In the extremal *one* horizon case we have that $g_{tt}(r = r_h) = 0$ and also $(\partial g_{tt}/\partial r) = 0$ at $r = r_h = r_2 = r_3$ such that $T_H(r = r_h) = 0$ attains the minimum zero value and the evaporation process stops, "freezes" at that stage, since one cannot go below the absolute zero temperature. One may envision a *two*-horizon black-hole scenario as discussed by [8] where the black hole, during the Hawking evaporation process, reaches a *maximal* temperature T_{max}, and from this point-on, it begins to *cool* down until it reaches the minimal *zero* temperature at the critical point and corresponding to the *one* horizon extremal solution associated to the double root $r_h = r_2 = r_3 = \frac{2GM_o}{\sqrt{3}}$ of eq-(20).

The *maximal* T_{max} occurs at the *inflection* point where $(\partial^2 g_{tt}/\partial r^2) = 0$. Such inflection points exists because $g_{tt}(r = 0) = g_{tt}(r = \infty) = 1$ and $g_{tt}(r)$ vanishes at the horizon as the graphs of [8] indicate for the Gaussian mass distribution. We have a similar behaviour in this work. However, this truncated black-hole evaporation scenario at $T_H = 0$ does *not* preclude the existence of observable timelike naked singularities (case (2)) because one may have, *ab initio*, a matter configuration $\rho(r, \sigma)$ such that M_o is already *less* than the critical mass given by eq-(23) and which does *not* admit horizons (it does *not* correspond to a black hole). Namely, we have a matter configuration in case (2) that *did not arise* as a result of a Hawking black-hole evaporation process associated to a mass distribution whose M_o was initially *greater* than the critical mass. For this reason, our solutions when $M_o < M_{critical}$ do represent observable *timelike naked singularities* since the scalar curvature and mass density are *singular* at $r = 0$ and $g_{tt}(r = 0) = 1$. Once again, we must emphasize that the fundamental difference between our solutions and those of [8], where one, two and no horizons are also found, is that we have a *singular* scalar curvature and mass density at $r = 0$ compared to a $finite$ value in [8].

To finalize, one can verify that null radial geodesics emanating from the naked singularity $r = 0$ reach an observer at $r \neq \infty$ in a finite time

$$(ds)^2 \;=\; (1 - \frac{2G\ M(r,\sigma)}{r})\ (dt)^2 \;-(1 - \frac{2G\ M(r,\sigma)}{r})^{-1}\ (dr)^2 \;= 0\ \Rightarrow$$

$$\int_0^t dt \;=\; t \;=\; \int_0^r \frac{dr}{g_{tt}(r,\sigma)} \;=\; \int_0^r \frac{dr}{1 - \frac{2GM_o}{r\ \sqrt{1+(\sigma/r)^3}}} \;=\; finite. \qquad (25)$$

since g_{tt} is *never* zero ($1/g_{tt}$ is never singular) when $M_o < M_{critical}$ (naked singularity case) , thus the integrand is well behaved everywhere so the definite integral does not diverge for a finite r and the light signal reaches the observer in a finite time.

There are two interesting cases for the values of the scale parameter σ that fixes the value of the critical mass, below which one has timelike naked singularities and no horizon enclosing them. If one sets σ to be of the order of the Planck scale L_P, since in four dimensions one has $G \sim L_P^2$ in natural units of $\hbar = c = 1$, one gets from eq-(23) that the critical mass is of the order of the Planck mass. If one sets σ to be the Hubble scale of the order of $10^{61} L_P$, one gets that the critical mass is of the order of the observed mass of the universe $M_{Universe} \sim 10^{61}\ m_{Planck}$. Thus, from eq-(23) one infers that the scale σ and the critical mass obey a self-similiarity behaviour (scaling invariance). As the scale σ is increased so is the value of the critical mass and vice versa. For instance, if one sets the critical mass to be of the order of the minimal mass $m \sim 1/R_{Hubble} \sim 10^{-61} m_{Planck}$ one gets that the scale σ is of the order $10^{-61} L_P$, etc....

A minimal mass in Noncommutative-geometry-inspired charged black holes in four and extra dimensions has been analyzed by [8] and also by [9]. The minimal scale and the thermodynamics of a black hole based on a generalized uncertainty relation can be found in [9], [11]. A minimal mass within the framework of the renormalization group improved Schwarzschild solution in asymptotically safe quantum gravity in four dimension can be found in [12] where one, two and no horizons are also found depending on the values of the mass.

The authors [16] have studied strong curvature singularities in gravitational collapse where pressures were allowed to be *negative* while satisfying the weak energy condition to avoid trapped surface formation (horizons). The formation of trapped surfaces in spherically symmetric (inhomogeneous dust) gravitational collapse can be viewed in terms of how much mass is there within a given area-radius of the matter cloud. In order to avoid trapped surface formation there must be a mechanism available to throw away and radiate the mass so that the total mass in a shell of comoving radius r, at an epoch t, does not exceed the size of the physical area radius $R(r,t)$ determining the size of the apparent horizon, at any given time t. It was found by [16] that the physical mechanism for the formation of a naked singularity was due to the presence of *shear* which delays the formation of the apparent horizon.

It was also found by [17] that loss of matter due to heat flux prevents the trapped surface formation and a naked singularity is formed at the end state of the gravitational collapse. The latter authors considered a scenario where the interior spacetime, described by a heat conducting fluid sphere, is matched to a Vaidya metric in higher dimensions. The non-occurrence of a horizon is due to the fact that the rate of mass loss is exactly counterbalanced by the decrease of the boundary area-radius. These results posed a counter example to the so-called cosmic censorship hypothesis. The *stability* issue of our solutions and their implications to the work of [16], [17] deserve to be analyzed further. Our results rely on the bounds of the net mass M_o (relative to the scale σ) given by eq-(23) , whereas the results of [16], [17] depend on the ratio of the mass function $M(r,t)$ to the area-radius $R(r,t)$.

To finalize, we should remark that the matter configuration of eq-(5) that leads to timelike naked singularities when $M_o < M_{critical}$ was inspired from the results of Brandt [14] based on the Finsler-geometry improved Schwarzchild solutions to a system of two coupled $fourth$ order nonlinear differential equations associated with the $modified$ Schwarzchild metric given in terms of a $length$ scale σ. It turns out that the metric (8) is not a solution of the fourth order differential equations found by Brandt [14] as I have been informed by [15]. Concluding, this is where the expression for the mass density ρ in eq-(5) leading to the timelike naked singularities originated. As far as we know there is no cosmic censorship conjecture in Finsler geometry and very little is known about gravitational collapse in Finsler geometry [2].

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