Strings and Membranes from Einstein Gravity, Matrix Models and W_∞ Gauge Theories as paths to Quantum Gravity

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Abstract

It is shown how $w_\infty, w_{1+\infty}$ Gauge Field Theory actions in $2D$ emerge directly from $4D$ Gravity. Strings and Membranes actions in $2D$ and $3D$ originate as well from $4D$ Einstein Gravity after recurring to the *nonlinear* connection formalism of Lagrange-Finsler and Hamilton-Cartan spaces. Quantum Gravity in $3D$ can be described by a W_∞ Matrix Model in $D = 1$ that can be solved *exactly* via the Collective Field Theory method. We describe why a quantization of $4D$ Gravity could be attained via a $2D$ Quantum W_∞ gauge theory coupled to an infinite-component scalar-multiplet. A proof that non-critical W_∞ (super) strings are devoid of BRST anomalies in dimensions $D = 27$ ($D = 11$), respectively, follows and which coincide with the critical (super) membrane dimensions $D = 27$ ($D = 11$). We establish the correspondence between the states associated with the quasi finite highest weights irreducible representations of $W_\infty, \bar{W}_\infty$ algebras and the quantum states of the continuous Toda molecule. Schroedinger-like QM wave functional equations are derived and solutions are found in the zeroth order approximation. Since higher-conformal spin W_∞ symmetries are very relevant in the study of $2D$ W_∞ Gravity, the Quantum Hall effect, large N QCD, strings, membranes, ... it is warranted to explore further the interplay among all these theories.

Keywords: Quantum Gravity, W_∞-gravity, W_∞ Gauge Theories, Higher spins, Holography, Moyal Brackets, Collective Field Theory, Strings, Branes, Matrix Models.

1 Gravity as Gauge Theories of Diffeomorphisms and Hidden Symmetries of M theory

In this introductory section we will review the work of [2] and afterwards we will discuss the recent work related the Hidden Symmetries of M theory.

1.1 Gravity in $D = m + n$ as an m-dim Gauge Theory of diffeomorphisms of an internal n-dim space and Holography

Some time ago Park [1] showed that $4D$ Self Dual Gravity is equivalent to a WZNW model based on the group $SU(\infty)$. Namely, $4D$ Self Dual Gravity is the non-linear sigma model based in $2D$ whose target space is the "group manifold" of area-preserving diffs of another $2D$-dim manifold. Roughly speaking, this means that the effective $D = 4$ manifold, where Self Dual Gravity is defined, is "spliced" into two $2D$-submanifolds: one submanifold is the original $2D$ base manifold where the non-linear sigma model is defined. The other $2D$ submanifold is the target group manifold of area-preserving diffs of a two-dim sphere S^2.

The authors [2] went further and generalized this particular Self Dual Gravity case to the full fledged gravity in $D = 2 + 2 = 4$ dimensions, and in general, to *any* combinations of $m + n$-dimensions. Their main result is that $m + n$-dim Einstein gravity can be identified with an m-dimensional generally invariant gauge theory of $Diffs\ N$, where N is an n-dim manifold. Locally the $m + n$-dim space can be written as $\Sigma = \mathcal{M} \times \mathcal{N}$ and the metric G_{AB} decomposes as:

$$G_{AB} = \begin{pmatrix} g_{\mu\nu}(x,y) \ + \ e^2 g_{ab}(x,y)\ A^a_\mu(x,y)\ A^b_\nu(x,y) & eA^a_\mu(x,y)\ g_{ab}(x,y) \\ eA^a_\mu(x,y)\ g_{ab}(x,y) & g_{ab}(x,y) \end{pmatrix}, \tag{1.1}$$

The connection $A^a_\mu(x,y)$ is the *nonlinear* connection formalism of Lagrange-Finsler and Hamilton-Cartan spaces [6], [60], [61], [62]. The decomposition (1.1) must *not* be confused with the Kaluza-Klein reduction where one imposes an isometry restriction on the γ_{AB} that turns A^a_μ into a gauge connection associated with the gauge group G generated by isometry. Dropping the isometry restrictions allows *all* the fields to depend on *all* the coordinates x, y. Nevertheless $A^a_\mu(x,y)$ can still be identified as a connection associated with the infinite-dim gauge group of $Diffs\ N$. The gauge transformations are now given in terms of Lie-brackets and Lie derivatives:

$$\delta A^a_\mu = -\frac{1}{e} D_\mu \xi^a = -\frac{1}{e}(\partial_\mu \xi^a - e[A_\mu, \xi]^a) = -\frac{1}{e}(\partial_\mu - e\mathcal{L}_{A_\mu})\xi^a,$$

$$A_\mu \equiv A^a_\mu \partial_a,$$

$$\mathcal{L}_{A_\mu}\xi^a \equiv [A_\mu, \xi]^a,$$

$$\delta g_{ab} = -[\xi, g]_{ab} = \xi^c \partial_c g_{ab} + g_{ac}\partial_b \xi^c + g_{cb}\partial_a \xi^c,$$

$$\delta g_{\mu\nu} = -[\xi, g_{\mu\nu}]. \quad (1.2)$$

In particular, if the relevant algebra is the area-preserving diffs of S^2, given by the suitable basis dependent limit $SU(\infty)$ [49], one induces a natural Lie-Poisson structure generated by the gauge fields A_μ. The Lie derivative of f along a vector ξ is the Lie bracket $[\xi, f]$, which coincides in this case with the Poisson bracket $\{\xi, f\}$. This implies that the Lie brackets of two generators of the area-preserving diffs S^2 is given precisely by the generator associated with their respective Poisson brackets (a Lie-Poisson structure):

$$[L_f, L_g] = L_{\{f,g\}}. \quad (1.3)$$

This relation is derived by taking the vectors ξ_1^a, ξ_2^a, along which we compute the Lie derivatives, to be the symplectic gradients of two functions $f(\sigma^1, \sigma^2), g(\sigma^1, \sigma^2)$:

$$\xi_1^a \ = \ \Omega^{ab}\ \partial_b f, \ \ \xi_2^a \ = \ \Omega^{ab}\ \partial_b g, \ \ L_{\xi_1} \ = \ \Omega^{ab}\ (\partial_b f)\ \partial_a, \ \ L_{\xi_2} \ = \ \Omega^{ab}\ (\partial_b g)\ \partial_a \quad (1.4)$$

such that $[L_f, L_g] = L_{\{f,g\}}$. When nontrivial topologies are involved one must include harmonic forms ω into the definition of ξ^a [15], [16] allowing central terms for the algebras. This relation can be extended to the volume-preserving diffs of N by means of the Nambu-Poisson brackets:

$$\{A_1, A_2, A_3, ...A_n\} = Jacobian = \frac{\partial(A_1, A_2, A_3, ..., A_n)}{\partial(\sigma^1, \sigma^2, ...\sigma^n)} \Rightarrow$$

$$[L_{A_1}, L_{A_2}, ..., L_{A_n}] = L_{\{A_1, A_2, ..., A_n\}}, \quad (1.5)$$

which states that the Nambu-commutator of n-generators of the volume-preserving diffs of $\mathcal{N}$ is given by the generator associated with their corresponding Nambu-Poisson brackets.

Using eq-(1.1) the authors [2] have shown that the curvature scalar $R^{(m+n)}$ in $m+n$-dim decomposes into:

$$R^{(m+n)} \ = \ g^{\mu\nu} R^{(m)}_{\mu\nu} \ + \ \frac{e^2}{4} g_{ab} F^a_{\mu\nu} F^b_{\rho\tau} g^{\mu\rho} g^{\nu\tau} \ + \ g^{ab} R^{(n)}_{ab} \ +$$

$$\frac{1}{4} g^{\mu\nu}\ g^{ab}\ g^{cd} [\ (D_\mu g_{ac})(D_\nu g_{bd}) \ - \ (D_\mu g_{ab})(D_\nu g_{cd})\ ] +$$

$$\frac{1}{4} g^{ab} g^{\mu\nu} g^{\rho\tau} [\ \partial_a g_{\mu\rho} \partial_b g_{\nu\tau} \ - \ \partial_a g_{\mu\nu} \partial_b g_{\rho\tau}\ ] \quad (1.6)$$

plus total derivative terms given by

$$\partial_\mu (\ \sqrt{|det\ g_{\mu\nu}|}\ \sqrt{|det\ g_{ab}|}\ J^\mu\ ) \ - \partial_a (\ \sqrt{|det\ g_{\mu\nu}|}\ \sqrt{|det\ g_{ab}|}\ e A^a_\mu J^\mu) \ +$$

$$\partial_a(\sqrt{|det\ g_{\mu\nu}|}\ \sqrt{|det\ g_{ab}|}\ \ J^a\ ), \tag{1.7}$$

with the currents:

$$J^\mu = g^{\mu\nu} g^{ab} D_\nu g_{ab}, \quad J^a = g^{ab} g^{\mu\nu} \partial_b g_{\mu\nu}, \tag{1.8}$$

$$S \ = \ \frac{1}{2\kappa^2} \int \ d^m x\ d^n y\ \sqrt{|det(g_{\mu\nu})|}\ \sqrt{|det(g_{ab})|}\ R^{(m+n)}(x,y). \tag{1.9}$$

Therefore, Einstein gravity in $m+n$-dim describes an m-dim generally invariant field theory under the gauge transformations or Diffs $\mathcal{N}$. Notice how A^a_μ couples to the graviton $g_{\mu\nu}$, meaning that the graviton is charged /gauged in this theory and also to the g_{ab} fields. The "metric" g_{ab} on N can be identified as a non-linear sigma field whose self interaction potential term is given by: $g^{ab}R^{(n)}_{ab}$. The currents J^μ, J^a are functions of $g_{\mu\nu}, A_\mu, g_{ab}$. Their contribution to the action is essential when there are boundaries involved; i.e. like in the AdS/CFT correspondence.

When the internal manifold $\mathcal{N}$ is a homogeneous compact space one can perform a harmonic expansion of the fields w.r.t the internal y coordinates, and after integrating w.r.t these y coordinates, one will generate an infinite-component field theory on the m-dimensional space. A reduction of the Diffs $\mathcal{N}$, via the inner automorphims of a subgroup G of the Diffs $\mathcal{N}$, yields the usual Einstein-Yang-Mills theory interacting with a nonlinear sigma field. But in general, the theory described in (1.9) is by far *richer* than the latter theory. A crucial fact of the decomposition (2.6, 2.7) is that *each* single term in (1.6, 21.7) is by itself independently invariant under Diffs $\mathcal{N}$. The second term of (1.6), for example,

$$\frac{1}{16\pi G}\ \sqrt{|det(g_{\mu\nu})|}\ \sqrt{|det(g_{ab})|}\ \frac{e^2}{4} g_{ab} F^a_{\mu\nu} F^b_{\rho\tau} g^{\mu\rho} g^{\nu\tau}, \tag{1.10}$$

is precisely the one that is related to the large N limit of $SU(N)$ YM [29].

The decomposition of the higher-dim Einstein-Hilbert action shown in eq-(1.6, 1.7) required to use a non-holonomic basis of derivatives $\partial_\mu - eA^a_\mu\partial_a$ and ∂_a that allows a diagonal decomposition of the metric and simplifies the computation of all the geometrical quantities. In this sense, the lower m-dimensional spacetime gauged "Ricci scalar" term $g^{\mu\nu}(x,y)R^{(m)}_{\mu\nu}(x,y)$ and the internal space "Ricci scalar" term $g^{ab}(x,y)R^{(n)}_{ab}(x,y)$ are obtained. In the special case when $g_{\mu\nu}(x)$ depends solely on x and $g_{ab}(y)$ depends on y then the spacetime gauged "Ricci scalar" coincides with the ordinary Ricci scalar $g^{\mu\nu}(x)R^{(m)}_{\mu\nu}(x)$ and the internal space "Ricci scalar" becomes the true Ricci scalar of the internal space. However, the gauge field $A_\mu(x,y)$ still retains its full dependence on both variables x, y.

We have shown [13] that in this particular case the $D = m+n$ dimensional gravitational action restricted to $AdS_m \times S^n$ backgrounds admits a *holographic* reduction to a lower $d = m$-dimensional Yang-Mills-like gauge theory of diffs of

S^n, interacting with a charged/gauged nonlinear sigma model plus boundary terms, by a simple tuning of the radius of S^n and the size of the throat of the AdS_m space. Namely, in the case of $AdS_5 \times S^5$, the holographic reduction occurs if, and only if, the size of the AdS_5 throat *coincides* precisely with the radius of S^5 ensuring a *cancellation* of the scalar curvatures $g^{\mu\nu} R^{(m)}_{\mu\nu}$ and $g^{ab} R^{(n)}_{ab}$ in eq-(1.6) [13]:

$$R^{(10)} \ = \ \frac{e^2}{4} \ g_{ab}(y) \ F^a_{\mu\nu}(x,y) \ F^b_{\rho\tau}(x,y) \ g^{\mu\rho}(x) \ g^{\nu\tau}(x) \ +$$

$$\frac{1}{4} g^{\mu\nu}(x) \ g^{ab}(y) \ g^{cd}(y) \ [\ (D_\mu g_{ac}) \ (D_\nu g_{bd}) - \ (D_\mu g_{ab}) \ (D_\nu g_{cd}) \ ]. \qquad (1.11)$$

plus total derivative terms (boundary terms)

$$D_\mu \ g_{ab} \ = \ \partial_\mu g_{ab} \ + \ [A_\mu, g_{ab}].$$

where the Lie-bracket is

$$[\ A_\mu \ , \ g_{ab} \ ] \ = \ (\partial_a \ A^c_\mu(x^\mu, y^a)) \ g_{bc}(x^\mu, y^a) \ + \ (\partial_b \ A^c_\mu(x^\mu, y^a)) \ g_{ac}(x^\mu, y^a) \ +$$
$$A^c_\mu(x^\mu, y^a) \ \partial_c \ g_{ab}(x^\mu, y^a). \qquad (1.12)$$

and the Yang-Mills like field strength is

$$F^a_{\mu\nu} \ = \ \partial_\mu A^a_\nu \ - \partial_\nu A^a_\mu \ - \ [A_\mu, A_\nu]^a \ =$$
$$\partial_\mu A^a_\nu \ - \partial_\nu A^a_\mu \ - \ A^c_\mu \partial_c A^a_\nu \ + A^c_\nu \partial_c A^a_\mu. \qquad (1.13)$$

Eq-(1.11) is nothing but the *holographic* reduction of the $D = 10$-dim pure gravitational action to a 5-dim Yang-Mills-like action (of diffeomorphisms of the internal S^5 space) interacting with a charged nonlinear sigma model (involving the g_{ab} field) plus boundary terms. The previous argument can also be generalized to gravitational actions restricted to de Sitter spaces, like $dS_m \times H^n$ backgrounds as well, where H^n is an internal hyperbolic noncompact space of constant negative curvarture, and dS_m is a de Sitter space of positive constant scalar curvature. The decomposition (1.11) provided a very straightfoward explanation of why AdS spaces played a crucial importance in the Maldacena AdS/CFT duality conjecture, because the algebra of area-preserving diffs of the sphere is isomorphic to the large N (basis dependent) limit of $SU(N)$, as shown by Hoppe long ago [49]; i.e. why higher-dim gravity admits a holographic reduction to a lower-dim $SU(\infty)$ YM theory. It is unfortunate that the important work of [2], [3], [4], [5] that already contained the seeds of the holographic principle was largely ignored by the physics community.

Introducing the light-cone coordinates u, v such that

$$u \ = \ \frac{1}{\sqrt{2}} \ (x^0 + x^1), \quad v \ = \ \frac{1}{\sqrt{2}} \ (x^0 - x^1). \qquad (1.14)$$

and define

$$A^a_u \;=\; A^a_+ \;=\; \frac{1}{\sqrt{2}}\,(A^a_0 + A^a_1), \quad A^a_v \;=\; A^a_- \;=\; \frac{1}{\sqrt{2}}\,(A^a_0 - A^a_1). \qquad (1.15)$$

the Polyakov ansatz is [52]

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 2h_{++} \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} -2h_{++} & -1 \\ -1 & 0 \end{pmatrix}, \quad det\ g_{\mu\nu} = -1. \qquad (1.16)$$

$$g_{ab} \;=\; e^{\sigma}\ \rho_{ab}; \quad det\ \rho_{ab} \;=\; 1. \qquad (1.17)$$

The covariant derivative of a tensor *density* ρ_{ab} with weight 1 is

$$D_\mu\ \rho_{ab} \;=\; \partial_\mu\ \rho_{ab}\ - [\ A_\mu\ ,\ \rho\ ]_{ab}\ + (\partial_c A^c_\mu)\ \rho_{ab} \;=$$

$$\partial_\mu\ \rho_{ab}\ - A^c_\mu \partial_c\ \rho_{ab}\ - (\partial_a A^c_\mu)\ \rho_{cb}\ - (\partial_b A^c_\mu)\ \rho_{ac}\ + (\partial_c A^c_\mu)\ \rho_{ab}. \qquad (1.18)$$

the covariant derivative on the scalar density $\Omega = e^{\sigma}$ of weight -1 is

$$D_\mu \Omega \;=\; \partial_\mu \Omega\ - A^a_\mu \partial_a \Omega\ - (\partial_a A^a_\mu)\Omega\ \Rightarrow$$

$$D_\mu \sigma \;=\; \partial_\mu \sigma\ - A^a_\mu \partial_a \sigma\ - (\partial_a A^a_\mu). \qquad (1.19)$$

after factoring the e^{σ} terms. Notice the extra term $w(\rho_{ab})(\partial_c A^c_\mu)\rho_{ab}$ in the definition of the covariant derivative acting on a tensor density ρ_{ab} whose weight is $w(\rho_{ab}) = 1$. Similarly there is an extra term $-(\partial_a A^a_\mu)\Omega$ in the covariant derivative of the scalar density Ω of weight -1. The Yang-Mills like field strength is

$$F^a_{+-} \;=\; \partial_+ A^a_-\ - \partial_- A^a_+\ -\ [A_+, A_-]^a \;=$$

$$\partial_+ A^a_-\ - \partial_- A^a_+\ -\ A^c_+ \partial_c A^a_-\ + A^c_- \partial_c A^a_+. \qquad (1.20)$$

The gauged-Ricci scalar becomes [3], [4], [5]

$$\sqrt{det\ g_{ab}}\, g^{\mu\nu}\ R_{\mu\nu} \rightarrow 2h_{++}\ e^{\sigma}\ [\ D^2_- \sigma + \frac{1}{2}(D_-\sigma)^2 + \frac{1}{4}\ \rho^{ab}\ \rho^{cd}\ (D_-\rho_{ac})\ (D_-\rho_{bd})\ ]. \qquad (1.21)$$

The Polyakov ansatz (1.16) leads to

$$det\ g_{\mu\nu} = -1\ \Rightarrow\ g^{\mu\nu}\ \partial_a\ g_{\mu\nu} \;=\; 2(-det\ g_{\mu\nu})^{-1/2}\ \partial_a(-det\ g_{\mu\nu})^{1/2} \;=\; 0. \qquad (1.22)$$

and one can verify that

$$g^{ab}\ g^{\mu\nu}\ g^{\alpha\beta}\ (\partial_a g_{\mu\alpha})\ (\partial_b g_{\nu\beta}) \;=\; 0. \qquad (1.23)$$

vanishes identically.

To sum up, after a laborious calculation Yoon [3], [4], [5] arrived finally at the expression for the Lagrangian density

$$\mathcal{L} \ = \ -\frac{1}{2}\, e^{2\sigma}\ \rho_{ab}\ F^a_{+-}\ F^b_{+-} \ + \ e^{\sigma}\ \mathcal{R}_2 \ + \ e^{\sigma}\ D_+\sigma\ D_-\sigma \ -$$

$$\frac{1}{2}\, e^{\sigma}\ \rho^{ab}\ \rho^{cd}\ (D_+\rho_{ab})\ (D_-\rho_{cd}) \ + \ \frac{1}{2}\, e^{\sigma}\ \rho^{ab}\ \rho^{cd}\ (D_+\rho_{ac})\ (D_-\rho_{bd}) \ +$$

$$2h_{++}\ e^{\sigma}\ [\ D^2_-\sigma \ + \frac{1}{2}(D_-\sigma)^2 \ + \frac{1}{4}\ \rho^{ab}\ \rho^{cd}\ (D_-\rho_{ac})\ (D_-\rho_{bd})\ ] \qquad (1.24)$$

plus surface terms. At each point x^μ of the $2D$ base space $\mathcal{M}$, the quantity $\mathcal{R}_2 = g^{ab}R_{ab}$ can be interpreted as the "scalar curvature" of the internal space or fiber $\mathcal{N}_2$ at x^μ. Since $g_{ab}(x^\mu, y^a)$ depends on both the base space and internal space coordinates, the integral $\int d^2y\ e^{\sigma}\mathcal{R}_2$ is no longer given in terms of the Euler class topological invariant associated with the 2-dim surface $\mathcal{N}_2$. The scalar curvature $g^{ab}R_{ab}$ is interpreted now as the potential $V(g_{ab})$ for the self-interacting non-linear sigma field g_{ab}.

The gauged-Ricci scalar $g^{\mu\nu}R_{\mu\nu}$ of the $2D$ base spacetime $\mathcal{M}$ leads to the those terms multiplying the scalar h_{++} in (1.24) such that h_{++} acts as a Lagrange multiplier enforcing the constraint

$$D^2_-\sigma \ + \frac{1}{2}(D_-\sigma)^2 \ + \frac{1}{4}\ \rho^{ab}\ \rho^{cd}\ (D_-\rho_{ac})\ (D_-\rho_{bd}) \ = \ 0. \qquad (1.25)$$

and, in this fashion, the contribution of the base space time gauged-Ricci scalar $g^{\mu\nu}R_{\mu\nu}$ *decouples* (on-shell) from the final expression. In the quantum theory one must implement the constraint (1.25) as an operator which annihilates the quantum states.

The area-preserving diffs algebra is generated by vector fields ξ^a tangent to the surface $\mathcal{N}_2$ and which are divergence-free $\partial_a\xi^a = 0$. The condition $\partial_a A^a_\pm = 0$ is invariant under area-preserving diffs, thus by imposing the divergence free condition $\partial_a A^a_\pm = 0$ one will have invariance under area-preserving diffs and such that the covariant derivatives acting on the tensor density ρ_{ab} and scalar σ in eqs-(1.18, 1.19) are now given by

$$D_\pm\sigma \ = \ \partial_\pm\sigma \ - A^a_\pm\partial_a\sigma. \qquad (1.26)$$

$$D_\pm\rho_{ab} \ = \ \partial_\pm\rho_{ab} \ - \ [\ A_\pm,\ \rho\ ]_{ab}. \qquad (1.27)$$

Under infinitesimal variations, the fields transform

$$\delta\sigma \ = \ -[\xi\ ,\sigma] \ = \ -\xi^a\partial_a\sigma, \quad \partial_a\xi^a = 0. \qquad (1.28)$$

$$\delta\rho_{ab} \ = \ -[\xi\ ,\ \rho]_{ab} \ = \ -\xi^c\partial_c\rho_{ab} \ - (\partial_a\xi^c)\rho_{cb} \ - (\partial_b\xi^c)\rho_{ac}. \qquad (1.29)$$

$$\delta A^a_+ \ = \ -D_+\xi^a \ = \ -\partial_+\xi^a \ + \ [A_+\ ,\ \xi]^a. \qquad (1.30)$$

$$\delta A^a_- \ = \ -\partial_-\xi^a\ . \qquad (1.31)$$

since δA^a_- is given by a total derivative one can choose the light-cone gauge $A^a_- = 0$ leaving $A^a_+ \neq 0$.

In the next sections we will choose the internal space metric to be conformally flat $\phi_{ab} = e^{\sigma}\delta_{ab}$. The covariant derivative of δ_{ab} is

$$D_\mu \; \delta_{ab} \;\; = \;\; - (\partial_a A_{\mu b}) \;\; - (\partial_b A_{\mu a}). \tag{1.32}$$

such that the terms

$$g^{\mu\nu} \; \delta^{ab} \; \delta^{cd} \; (D_\mu \; \delta_{ab}) \; (D_\nu \; \delta_{cd}) = \; 4 \; g^{\mu\nu} \; (\partial_a A^a_\mu) \; (\partial_c A^c_\nu) \; = \; 0. \tag{1.33}$$

as a result of the area-preserving invariant condition $\partial_a A^a_\mu = 0$. Therefore, the relevant Lagrangian density to study will be

$$\mathcal{L} \;\; = \;\; -\frac{1}{2} \; e^{2\sigma} \; F^a_{+-} \; F^a_{+-} \;\; + \;\; e^{\sigma} \; D_+\sigma \; D_-\sigma \;\; + \;\; e^{\sigma} \; V(\sigma). \tag{1.34}$$

with $\mathcal{R}_2 = V(\sigma)$.

1.2 The Canonical Decomposition of Gravity and Hidden Symmetries of M theory

Performing a $1+10$ decomposition of $11D$ Gravity from the most general expression (1.6) simplifies considerably and yields [2]

$$R^{(1+10)} \;\; = \; g^{ab} R^{(10)}_{ab} \; + \frac{1}{4} g^{tt} \; g^{ab} \; g^{cd} [\; (D_t g_{ac})(D_t g_{bd}) - (D_t g_{ab})(D_t g_{cd}) \;]. \tag{1.35}$$

plus total derivative terms (boundary terms). The internal indices range is $a, b = 1, 2, 3, ...10$. The last expression can be written in terms of the extrinsic curvature K_{ab} associated with the embedding of the 10-dim internal space $\mathcal{N}_{10}$ in $11D$.

$$R^{(1+10)} \;\; = \; g^{ab} R^{(10)}_{ab} \;\; + \;\; K_{ab} K^{ab} - (K^a_a)^2, \quad K^a_a = g^{ab} K_{ab}. \tag{1.36}$$

where the extrinsic curvature is given by the Lie derivative along the normal $\vec{n}$ to the $\mathcal{N}_{10}$ surface [2]

$$K_{ab} \;\; = \;\; \frac{1}{2} \mathcal{L}_n g_{ab} \;\; = \;\; \frac{1}{\sqrt{|g_{tt}|}} \; (\partial_t g_{ab} \;\; - \;\; \mathcal{L}_\tau g_{ab}). \tag{1.37}$$

after performing the decomposition of ∂_t into normal $n^i \partial_{n^i}$ and tangential derivatives $\tau^a \partial_a$ to the internal 10-dim surface $\mathcal{N}_{10}$, respectively, given by

$$\partial_t \;\; = \;\; \sqrt{|g_{tt}|} \; n^i \partial_{n^i} \;\; + \;\; \tau^a \partial_a. \tag{1.38}$$

Therefore, the canonical decomposition $1+10$ of $11D$ Gravity is automatically recovered from the most general expression (1.6) after the decomposition of the metric (1.1) based on the *nonlinear* connection $A^a(x,y)$ formalism. Nonlinear connections are essential ingredients of Lagrange-Finsler and Hamilton-Cartan spaces [6], [60], [61], [62]

Therefore, the decomposition $1+10$ of $11D$ Gravity displayed in eqs-(1.35, 1.36) can be interpreted as a *one*-dim *gauged nonlinear sigma model* valued in the diffs of the internal 10-dim space $\mathcal{N}_{10}$ and under the influence of the scalar potential $V = g^{ab} R^{(10)}_{ab}$. The scalar fields are the components of g_{ab} and the base space is one-dim and represented by the temporal coordinate t.

Having found that $11D$ gravity can be recast as a one-dim gauged nonlinear sigma model subjected to a potential, the next step is to construct the supersymmetric extension and view $11D$ Supergravity (SUGRA) as a supersymmetric gauged one-dim nonlinear sigma model. Such description of $11D$ SUGRA in terms of a one-dim nonlinear sigma model has been studied in recent years within the framework of infinite-dim hyperbolic Kac-Moody algebras corresponding to the extended root lattices of the Exceptional Lie algebra E_8, known as E_9, E_{10}, E_{11} [7], Hyperbolic2.

A connection between the Belinsky-Khalatnikov-Lifshitz-like "chaotic" structure of generic cosmological singularities in $11D$ supergravity and the hyperbolic Kac-Moody algebra E_{10} was found in [7], [8] . This intriguing connection suggests the existence of a hidden "correspondence" between supergravity (or even M-theory) and null geodesic motion on the infinite-dimensional coset space $\frac{E_{10}}{K(E_{10})}$. This gravity/coset correspondence would offer a new view of the (quantum) fate of space (and matter) at cosmological singularities. The hidden symmetries of the fermionic sector of $11D$ supergravity, and the role of the compact subalgebra $K(E_{10})$ was also studied and a consistent model of a massless spinning particle on an $\frac{E_{10}}{K(E_{10})}$ coset manifold was found whose dynamics can be mapped onto the fermionic and bosonic dynamics of $11D$ supergravity in the near space-like singularity limit. This E_{10} -invariant super-particle dynamics might provide the basis of a new definition of M-theory, and describe the "de-emergence" of space-time near a cosmological singularity. The role of E_{11} in Higher Spin Theories and symmetries of M theories were studied by [10], [11]. A nice review of Hyperbolic Kac-Moody algebras, Cosmological Billiards and $11D$ SUGRA can be found in [8].

The relationship between coset models based on the infinite-dim $Diffs\ (\mathcal{N}_{10})$ algebra and the Hyperbolic Kac-Moody E_{10} algebras, to our knowledge, has not been investigated. The commutators of the general linear and conformal algebra $[GL(D), SO(D,2)]$ do *not* close and the infinite nested family of multiple commutators generates the infinite-dim $Diffs(R^D)$ algebra in D-dimensions [12]. The commutators $[A_t, g_{ab}]$ present in the derivatives $D_t g_{ab}$ of eq-(1.36) are expressed as the Lie derivative of g_{ab} along the vector field A^a_t, the generators of diffs of the internal 10-sim space $\mathcal{N}_{10}$. Such Lie derivative can be expanded into an infinite number of terms (or levels) given by the modes of $A^a_t(t; y^a) = \sum A^a_{t,\{m\}}(t)\ y^{m_1} y^{m_2} ...... y^{m_n}$ with $m = m_1 + m_2 + ... + m_n$. Each level m denoted by m is related to a series of *nested* commutators of $[GL(D), SO(D,2)]$. It is in this fashion how one can make contact with the coset models based on $\frac{E_{10}}{K(E_{10})}$ that admit a similar level by level structure related to a gradient expansion of the fields.

A decomposition of $R^{(11)}$ in terms of $R^{(0+11)}$ is reminiscent of a dimensional reduction to a *point* where the infinite-dim diffs algebra is that of the internal 11-dim space $\mathcal{N}_{11}$. Such algebra $Diffs(\mathcal{N}_{11})$ might bear some relationship to the E_{11} algebra.

2 A $w_\infty, w_{1+\infty}$ Gauge Field Theory in $2D$ from $4D$ Gravity and Moyal star products deformations

It is well known that the Yang-Mills theory action in lower dimensions can be obtained from the Einstein-Hilbert Gravitational action in higher dimensions. In this section we will show how the $w_\infty, w_{1+\infty}$ Gauge Field Theory actions in $2D$ emerge directly from $4D$ Gravity via eq-(1.34). Moyal star products deformations of these actions follow. It is important to remark that the main findings of this section are new.

Zamolodchikov [14] was the first to pioneer the theory of higher conformal spin algebras W_N, $N = 2, 3, 4, ..., N$ in $2D$ that are the higher conformal spin extensions of the Virasoro algebra that arise in various physical systems as $2D$ quantum gravity, the quantum Hall effect, the membrane, the large N QCD, gravitational instantons, topological QFT, etc.... see [17] for an extensive review and references. The $w_{1+\infty}$ algebra is isomorphic to the area-preserving diffs algebra of the cylinder $S^1 \times R^1$:

$$[v^i_m \ , \ v^j_n] \ = \ [(j+1)m \ - \ (i+1)n] \ v^{i+j}_{m+n}. \tag{2.1}$$

where the index $i, j = -1, 0, 1, 2, ...$ is related to the $su(1,1)$ conformal spin $s = 1, 2, 3, ...$ and m, n label their respective Fourier modes. The spin $s = 1$ correspond to an extra spin 1 current. The w_∞ algebra is the area-preserving diffs algebra of the two-dim plane and is comprised of higher spin generators whose conformal spin range is $s = 2, 3, 4, ...$ and it is a subalgebra of $w_{1+\infty}$; whereas $su(\infty)$ is the area-preserving diffs algebra of a sphere S^2. A realization of the higher conformal spin generators of $w_{1+\infty}$ is

$$v^l_m \ = \ -i \ e^{im\theta} \ y^l \ [-im \ y \ \partial_y \ + (l+1) \ \partial_\theta]. \tag{2.2}$$

A complete set of functions (not orthogonal) on the cylinder $S^1 \times R^1$ is

$$u^l_m \ = \ -ie^{im\theta} \ y^{l+1}; \quad -\infty \ \leq \ m \ \leq \ \infty, \quad l \ \geq \ -1. \tag{2.3}$$

where the conformal $su(1,1)$-spin s in $D = 2$ is given by $s = l + 2 \geq 1$.

The $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian density was constructed by [35]

$$\mathcal{L} \;=\; \sum_{\vec{i},\vec{j}} \; (\Phi^6(x))^{-\vec{i}-\vec{j}} \; \mathcal{F}^{\vec{i}}_{+-}(x) \; \mathcal{F}^{\vec{j}}_{+-}(x) \; +$$

$$\sum_{\vec{k}} \; (\mathcal{D}_+ \Phi^{-\vec{k}}(x)) \; (\mathcal{D}_- \Phi^{\vec{k}}(x)) \; + \; V(\Phi^{\vec{k}}(x)). \qquad (2.6)$$

The gauge field $A^{\vec{k}}_\mu$ is Hemitian (w.r.t a well defined scalar product) $(A^{\vec{k}}_\mu)^* = A^{-\vec{k}}_\mu = A_{\mu,\vec{k}}$ and belongs to the adjoint representation $V_{\alpha,\beta}$ constructed by Feigin-Fuks-Kaplansky (FFK) [36], [37] with $\alpha = 1, \beta = 0$. $\Phi^{\vec{k}}$ is an infinite-component complex scalar multiplet belonging to the infinite-dim vector representation $V_{\alpha,\beta}$ with $\alpha = -1/2, \beta = 0$. In order to write invariant actions based on a scalar product the weights must obey $\alpha^* + \alpha + 1 = 0$ and $\beta^* - \beta = 0$ where α^*, β^* are the weights of the *dual* representation $V^*_{\alpha,\beta} = V_{-1-\alpha,-\beta}$. For further details we refer to [35]. The gauge invariant Lagrangian based on the Virasoro w_2 algebra involving only the conformal spin 2 current (stress energy tensor) was constructed by [34] and can be obtained from the w_∞ Lagrangian by a simple truncation.

The field strength in the adjoint representation of FFK is

$$\mathcal{F}^{\vec{k}}_{+-} \;=\; \partial_+ \mathcal{A}^{\vec{k}}_- \;-\; \partial_- \mathcal{A}^{\vec{k}}_+ \;-ie\;[\;\mathcal{A}_+,\;\mathcal{A}_-\;]^{\vec{k}}. \qquad (2.7)$$

The commutator of the gauge fields in the adjoint representation is [35]

$$[\;\mathcal{A}_+,\;\mathcal{A}_-\;]^{\vec{k}} \;=\; [\;m_1\;(k_2+2)\;-\;(m_2+1)\;k_1\;]\;\mathcal{A}^{\vec{m}}_+\;\mathcal{A}^{\vec{k}-\vec{m}}_-. \qquad (2.8)$$

where $\vec{k}$ denotes a two-dim lattice index

$$\vec{k} \;=\; (k_1,k_2),\;\; \vec{m} \;=\; (m_1,m_2), \qquad (2.9a)$$

and their values are constrained by

$$k_2 \geq -1;\;\; m_2 \geq -1;\;\; -\infty \;\leq\; k_1 \;\leq\; \infty;\; -\infty \;\leq\; m_1 \;\leq\; \infty. \qquad (2.9b)$$

since the conformal $su(1,1)$-spin s associated with the $2D$ higher conformal spin generators $v^{k_2}_{k_1}$ of the $w_{1+\infty}$ algebra is given by $s = k_2 + 2 \geq 1$ such that $s = 1,2,3,......$ Whereas, the index k_1 labels the infinite Fourier modes associated with each one of the conformal spin-s generators. The covariant derivative is

$$(\mathcal{D}_\pm \Phi^{\vec{k}}) \;=\; \partial_\pm \Phi^{\vec{k}} + ie\,[\,(m_2+1)(\frac{m_1}{2}-k_1) - (\frac{m_2}{2}-k_2)m_1\,]\,\mathcal{A}^{\vec{m}}_\pm\;\Phi^{\vec{k}-\vec{m}}. \quad (2.10)$$

The integration of the Yang-Mills-like terms of eq-(1.34) w.r.t the *internal* coordinates of the two-dim surface $\mathcal{N}_2$ furnishes the correspondence with the terms of the $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian [35] associated with the two-dim base spacetime $\mathcal{M}_2$. Integrating over a cylinder $S^1 \times R^1$ whose base S^1 has unit radius yields

$$\int dy\, d\theta\, e^{2\sigma(x;y,\theta)} \,[\, F^y_{+-}\, F^y_{+-} + F^\theta_{+-}\, F^\theta_{+-}\,] \leftrightarrow \sum_{\vec{i},\vec{j}} (\Phi^6(x))^{-\vec{i}-\vec{j}}\, \mathcal{F}^{\vec{i}}_{+-}(x)\, \mathcal{F}^{\vec{j}}_{+-}(x). \tag{2.11}$$

where one has set $\rho_{ab} = \delta_{ab}$. The scalar kinetic terms correspondence based on eq-(1.34) is

$$\int dy\ d\theta\ \ e^{\sigma(x;y,\theta)}\ D_+\sigma\ D_-\sigma\ \leftrightarrow\ \sum_{\vec{k}} (\mathcal{D}_+\Phi^{-\vec{k}}(x))\ (\mathcal{D}_-\Phi^{\vec{k}}(x)). \tag{2.12}$$

And

$$\int dy\ d\theta\ e^\sigma\ g^{ab}\ R_{ab}\ \leftrightarrow\ V(\Phi^{\vec{k}}). \tag{2.13}$$

yields the correspondence with the self-interacting scalar potential $V(\Phi^{\vec{k}})$.

The resulting integrals in eqs-(2.11, 2.12, 2.13) yield the functional relations among the infinite component fields $\sigma_{lm}(x^\mu),\ A^a_{\pm,lm}(x^\mu)$ in the decomposition

$$\sigma\ =\ \sigma(x^\mu, y, \theta)\ =\ \sum_{lm} \sigma_{lm}(x^\mu)\ e^{im\theta}\ y^{l+1}. \tag{2.14a}$$

$$A^a_\pm\ =\ A^a_\pm(x^\mu, y, \theta)\ =\ \sum_{lm} A^a_{\pm,lm}(x^\mu)\ e^{im\theta}\ y^{l+1}. \tag{2.14b}$$

with $s = l + 2 \geq 1,\quad l = -1, 0, 1, 2, 3, ..., -\infty \leq m \leq \infty$, and the fields

$$\Phi^{\vec{k}}(x^+, x^-),\ \mathcal{A}^{\vec{k}}_\pm(x^+, x^-),\quad \vec{k} = (k_1, k_2). \tag{2.15}$$

associated with the FFK representations of the $w_{1+\infty}$ algebra and which appear in the $2D$ Lagrangian density of the $w_{1+\infty}$-gauge field theory [35]. Therefore, the 1+1-dim Lagrangian density of the $w_{1+\infty}$ gauge theory is inherently present in the $1+1$-dim description of the algebraically special class of space-times in 4-dim that contain a twist-free null vector field [3].

More precisely, one may define the Lagrangian density of a Yang-Mills-like $w_\infty, w_{1+\infty}$ gauge field coupled to a scalar field $\mathbf{\Phi}$ valued in the *adjoint* representation of $w_\infty, w_{1+\infty}$ and subject to a self-interacting scalar potential $V(\mathbf{\Phi})$ given by

$$\mathcal{L}\ =\ Trace\ [\ -\frac{1}{2}\ \mathbf{F}_{+-}\ \mathbf{F}_{+-}\ +\ D_+\mathbf{\Phi}\ D_-\mathbf{\Phi}\ +\ V(\mathbf{\Phi})\ ]. \tag{2.16}$$

The trace operation given by an infinite sum over all the generators of $w_\infty, w_{1+\infty}$ can be replaced by an integration over the internal y^1, y^2 coordinates of the internal two-dim surface $\mathcal{N}$ leading to an expression of the form

$$\mathcal{L}\ =\ \int d^2y\ e^\sigma\ [\ -\frac{1}{2}\ e^\sigma\ F^a_{+-}\ F^a_{+-}\ +\ (D_+\ \sigma)\ (D_-\ \sigma)\ +\ V(\sigma)\ ]. \tag{2.17}$$

with $\mathcal{R}_2 = V(\sigma)$ after performing a similar correspondence as given by eqs-(2.11, 2.12, 2.13).

We finalize this section with an analysis of the Moyal Star Product Deformations of $su(\infty), w_{1+\infty}, w_\infty$ Gauge Theories that will allow to build a Moyal star product deformation of the Lagrangians displayed in eqs-(2.16, 2.17). The authors [25] have shown that upon quantization of field theories exhibiting symmetries provided by classical algebras $w_{1+\infty},\ w_\infty$ these symmetries get *deformed* to the quantum algebras $W_{1+\infty},\ W_\infty$ whose commutation relations are

$$[V^i_m\ ,\ V^j_n]\ =\ \sum_l\ g^{ij}_{2l}(m,n)\ V^{I+j-2l}_{m+n}\ +\ c_i(m)\ \delta^{ij}\ \delta_{m+n,0}. \tag{2.18}$$

where the structure constants $g^{ij}_{2l}(m,n)$ are complicated expressions given in terms of the generalized Saalschutzian hyper-geometric functions, binomial coefficients, ... and the c_i are the central charges associated with all of the higher spin sectors [22], [23], [24], [25], [26], [53]. The deformation of the classical algebras $w_{1+\infty},\ w_\infty$ can be obtained from a Moyal-Fedosov-Kontesevich star product deformation program as shown by [18], [19], [20], [21] and in this fashion one may generate the structure constants $g^{ij}_{2l}(m,n)$ that were originally obtained by a tour de force method [22], [23], [24], [25]. In addition, there are also *modifications* in the central charges where the central charge term present only in the Virasoro sector [18] is extended to *all* of the higher conformal spin sectors of the quantum $W_\infty, W_{1+\infty}$ algebras. The origin of the modifications of the central charge terms is due to universal gauge anomalies of the algebras [53].

The ordinary Moyal star-product of two functions in phase space $f(x,p), g(x,p)$ is :

$$(f * g)(x,p) = \sum_s \frac{\hbar^s}{s!} \sum_{t=0}^{s} (-1)^t C(s,t)\ (\partial_x^{s-t}\partial_p^t f(x,p))\ (\partial_x^t \partial_p^{s-t} g(x,p)) \tag{2.19}$$

where $C(s,t)$ is the binomial coefficient $s!/t!(s-t)!$. In the $\hbar \rightarrow 0$ limit the star product $f * g$ reduces to the ordinary pointwise product fg of functions. The Moyal product of two functions of the $2n$-dim phase space coordinates (q_i, p_i) with $i = 1, 2...n$ is:

$$(f * g)(x,p) = \sum_i^n \sum_s \frac{\hbar^s}{s!} \sum_{t=0}^{s} (-1)^t C(s,t)\ (\partial_{x_i}^{s-t}\partial_{p_i}^t f(x,p))\ (\partial_{x_i}^t \partial_{p_i}^{s-t} g(x,p)) \tag{2.20}$$

The noncommutative, associative Moyal bracket is defined:

$$\{f, g\}_{MB} = \frac{1}{i\hbar}(f * g - g * f). \tag{2.21}$$

In the $\hbar \to 0$ limit the star product $f * g$ reduces to the ordinary pointwise product fg of functions and the Moyal bracket reduces to the Poisson one. Thus, the Moyal deformations of the Yang-Mills-like terms are

$$\int dy\ d\theta\ \ e_*^{2\sigma}\ *\ [\ F^y_{+-}\ *\ F^y_{+-}\ +\ F^\theta_{+-}\ *\ F^\theta_{+-}\ ]. \tag{2.22}$$

$$\mathcal{F}_{+-}\ =\ \partial_+\mathcal{A}_-\ -\ \partial_-\mathcal{A}_+\ +\ \{\ \mathcal{A}_+,\ \mathcal{A}_-\ \}_{MB}. \tag{2.23}$$

$$D_\pm\sigma\ =\ \partial_\pm\sigma\ +\ \{\ A_\pm\ ,\ \sigma\ \}_{MB}. \tag{2.24}$$

$$D_\pm\ e_*^\sigma\ =\ \partial_\pm\ e_*^\sigma\ +\ \{\ A_\pm\ ,\ e_*^\sigma\ \}_{MB}. \tag{2.25}$$

Due to the fact that for higher derivatives

$$\partial^n_{y^a}\ e^{\sigma(x^\mu,y^a)}\ \neq\ e^{\sigma(x^\mu,y^a)}\ \partial^n_{y^a}\ \sigma(x^\mu,y^a)\ \Rightarrow$$

$$\{\ A_\pm\ ,\ e^\sigma\ \}_{MB}\ \neq\ e^\sigma\ \{\ A_\pm\ ,\ \sigma\ \}_{MB}. \tag{2.26}$$

and

$$\{\ A_\pm\ ,\ e_*^\sigma\ \}_{MB}\ \neq\ e_*^\sigma\ \{\ A_\pm\ ,\ \sigma\ \}_{MB}. \tag{2.27}$$

the correct Moyal deformations of the scalar kinetic terms are

$$\int dy\ d\theta\ \ e_*^\sigma\ *\ [\ (e_*^{-\sigma}\ *\ D_+\ e_*^\sigma)\ *\ (e_*^{-\sigma}\ *\ D_-\ e_*^\sigma)\ ]. \tag{2.28}$$

where the star-deformed exponential function is defined by

$$e_*^\sigma\ =\ 1\ +\ \sigma\ +\ \frac{1}{2!}\sigma*\sigma\ +\frac{1}{3!}\sigma*\sigma*\sigma\ +\ .... \tag{2.29}$$

There are ordering ambiguities in the definition of eq-(2.28) that for the moment shall not concern us. The star-deformed potential $V_*(\sigma)$ is defined by star-deformed Taylor expansion of the original potential $V(\sigma)$

$$V_*(\sigma(x,y)) \equiv \sum_n\ g_n\ (\sigma)_*^n\ =\ \sum_n\ g_n\ \sigma*\sigma*\sigma\ *.....*\sigma. \tag{2.30}$$

where the couplings g_n are obtained by taking the n-th derivatives of $V(\sigma)$ w.r.t σ and evaluated at $\sigma = 0$

$$g_n \equiv \frac{1}{n!}\frac{\partial^n V(\sigma)}{\partial\sigma^n}(\sigma = 0). \tag{2.31}$$

The Moyal deformed-action S_* is highly nontrivial. The leading terms $\hbar^0$ coincide with the undeformed action based on the Poisson bracket algebra of area-preserving diffs of the two-dim internal $\mathcal{N}_2$ surface. In the case that the internal two-dim space has the topology of a sphere, this Poisson bracket algebra is isomorphic to the basis-dependent limit of the $N \to \infty$ limit of $SU(N)$

[49]. For arguments refuting the ismorphism behind the large N limits of $su(N)$ algebras and the area-preserving diffs of a sphere S^2 see [50], [51] . The Moyal-deformations of the area-preserving-diffs S^2 symmetry transformations that leave invariant the Moyal-deformed gravitationally induced action-density $\mathcal{L}_*(x)$ are given by

$$\delta \mathcal{A}_\mu(x,y) \;=\; -[\;\partial_\mu \xi(x,y) \;-\{\mathcal{A}_\mu(x,y), \xi(x,y)\}_{MB}]. \qquad (2.32a)$$

where we have set $e = 1$ for convenience.

$$\delta \mathcal{F}_{\mu\nu}(x,y) = -\{\xi(x,y), \mathcal{F}_{\mu\nu}(x,y)\}_{MB}. \qquad (2.32b)$$

$$\delta \sigma(x,y) = -\{\xi(x,y), \sigma(x,y)\}_{MB}. \qquad (2.32c)$$

$$\delta D_\mu \sigma = -\{\xi(x,y), D_\mu \sigma\}_{MB}. \qquad (2.32d)$$

$$\delta V_*(\sigma) = -\{\xi, V_*(\sigma)\}_{MB}. \qquad (2.32e)$$

and the variation of $\mathcal{L}_*(x)$ is given by a sum of *total derivatives* that vanishes after integration by parts since the internal sphere has no boundaries

$$\delta L_*(x,y) = -\{\xi, L_*(x,y)\}_{MB} \;\Rightarrow\; \delta \mathcal{L}_*(x) = \int\; d^2y\; \delta L_*(x,y) \;=$$

$$-\int\; d^2y\; \{\xi, L_*(x,y)\}_{MB} \;=\; \int\; (\textit{sum of total derivatives}) \;=\; 0. \qquad (2.33)$$

To show this requires the use of the Liebnitz property of the Moyal Brackets

$$\{\xi\;,\; F_{\mu\nu}\; F^{\mu\nu}\}_{MB} \;=\; \{\xi\;,\; F_{\mu\nu}\}_{MB}\; F^{\mu\nu} \;+\; F^{\mu\nu}\; \{\xi\;,\; F_{\mu\nu}\}_{MB}. \qquad (2.34)$$

and

$$\int\; d^2y\, F_{\mu\nu} * F^{\mu\nu} \;=\; \int\; d^2y\, (F_{\mu\nu}\, F^{\mu\nu} + \textit{total derivatives}) \;=\; \int\; d^2y\, F_{\mu\nu}\, F^{\mu\nu} \;\Rightarrow$$

$$\delta \int\; d^2y\; F_{\mu\nu} \;*\; F^{\mu\nu} \;=\; \delta \int\; d^2y\; F_{\mu\nu}\; F^{\mu\nu} \;=$$

$$\int\; d^2y\; \{\xi\;,\; F_{\mu\nu}\}_{MB}\; F^{\mu\nu} \;+\; F_{\mu\nu}\; \{\xi\;,\; F^{\mu\nu}\}_{MB} \;=$$

$$\int\; d^2y\; \{\xi\;,\; F_{\mu\nu}\; F^{\mu\nu}\}_{MB} \;=\; \int\; \textit{sum of total derivatives} \;=\; 0. \qquad (2.35)$$

if there are no boundaries or if the fields vanish fast enough at infinity. Similar results follow for the kinetic terms. In general, the generators of $w_\infty, w_{1+\infty}$ admit a parametrization in terms of an infinity family of functions f as

$$L_f \ = \ \omega^{ab} \ \partial_b f \ \partial_a, \quad \omega^{ab} = symplectic \ structure. \tag{2.36}$$

where the Lie-Poisson structure is deformed into a Lie-Moyal one upon quantization

$$[L_f \ , \ L_g] \ = \ L_{\{f,g\}} \ \rightarrow \ [L_f \ , \ L_g]_* \ = \ L_{\{f,g\}_*} \ . \tag{2.37}$$

For instance, When the topology of the internal two-dim surface is that of a cylinder $S^1 \times R^1$ one may expand the function f and generators L_f as

$$f(y,\theta) \ = \ f_{lm} \, e^{im\theta} \, y^{l+1}; \ \ L_f \ = \ f_{lm} \, v^{lm} \ = \ f_{lm} \ e^{im\theta} \, [m \, y^{l+1} \, \partial_y + i(l{+}1) \, y^l \, \partial_\theta]. \tag{2.38}$$

from which one may read the commutation relations of the (deformed)currents v^l_m, V^l_m from the Lie-Poisson and Lie-Moyal algebraic structures upon deformation quantization. Similar results follow for the sphere and the two-dim plane by choosing the appropriate basis of functions. The algebras admit central charges or not depending on the genus of the two-dim surfaces [15],[16], [17].

3 $4D$ Quantum Gravity via $2D$ Quantum W_∞ Gauge Theories, Collective Fields and Matrix Models

In this section we shall show how Quantum Gravity in $D = 3$ can be described by a W_∞ Matrix Model in $D = 1$ that can be solved *exactly*. $4D$ Quantum Gravity is more complicated, nevertheless its quantization program can be attained from the perspective of a $2D$ Quantum W_∞ gauge theory coupled to an infinite-component scalar-multiplet whose action is described by eqs-(2.16, 2.17); i.e. Quantization of Einstein Gravity in $4D$ admits a reformulation in terms of a $2D$ Quantum W_∞ gauge theory coupled to an infinite family of scalars.

It has been known for some time [45], [46], that the bosonization program of non-relativistic fermions in one space dimension can be used to describe the low energy excitations of a Fermi gas in terms of a Fermi fluid of various shapes with the *same area* as the ground state configuration if one insists in fermion number conservation. The Fermi fluid exists in the 2-dim phase space of the single fermion and changes in the state of the Fermi theory correspond to $area - preserving$ shape changes of the Fermi fluid.

The Das-Jevicki-Sakita [43], [44] collective field theory approximation studies the fluctuations of the phase-space density and in the semi classical limit describes the low energy excitations of the Fermi fluid near the Fermi surface when one restricts the shapes of the Fermi fluid to have a quadratic profile for the Fermi energy $\mu_F = \frac{1}{2}(p^2 - q^2)$ related to an *inverted* one-dim harmonic oscillator potential. A direct proof of bosonization of non-interacting non-relativistic fermions in one space dimension was derived by Wadia et al [45], [46] by using W_∞ coherent states in the fermion path-integral. The bosonized action was derived earlier by the method of coadjoint orbits associated with the W_∞ algebra. The classical limit of the bosonized theory and the precise nature of the truncation of the full theory that leads to the Das-Jevicki-Sakita collective field theory was also described by [45], [46].

The use of W_∞ coherent states in the fermionic path-integral was made possible by the observation [41], [42] that the bosonized problem is analogous to that of a spin in a magnetic field. This system has a W_∞ spectrum generating algebra that follows from the existence of the w_∞ symmetry of the harmonic oscillator. It is natural to rewrite the collective field theory in $0+1$ dimensions as a $1+1$ relativistic field theory so the collective field theory is a theory of a massless boson that reproduces the fluctuations in the density.

The quantum algebra W_∞ may be realized [47], [48] as the algebra of modes of the Fermion bilinears : $\partial^k \Psi(z) \partial^l \Psi(z)$:. A bosonization relates the fermion-bilinears to the bosonic currents $\frac{1}{s}$: $e^{-\phi(z)} \partial^s e^{\phi(z)}$: and similarly to the left movers by replacing $z \to \bar{z}$. The key point was that although the collective field theory is *not* a free theory it has a spectrum generating algebra given by charges

$$Q_{lm} \;=\; \int dx \int_{p_-}^{p^+} dp \; (p+x)^{l+m+1} \; (p-x)^{l-m+1}. \tag{3.1}$$

that satisfy a w_∞ algebra isomorphic to the Poisson-bracket algebra of the charges $\{ Q_{lm} \;,\; Q_{l'm'} \}_{PB}$.

After this historical this preamble one may notice that the action (1.24) obtained from the decomposition of Einstein gravity in $D = 1+2$ (instead of $D = 2+2$) is much *simpler* since there are *no* Yang-Mills-like and gauged-Ricci scalar curvature terms in a one-dimensional base space $\mathcal{M}_1$, so when $\rho_{ab} = \delta_{ab}$ the Einstein-Hilbert action in $D = 1+2$ action reduces to

$$S \;=\; \int dt \; \mathcal{L} \;=\; \int dt \int dy \; d\theta \;\; e^\sigma \; [\; D_+\sigma \; D_-\sigma \;+\; \mathcal{R}_2 \;] \;=$$

$$\int dt \int dy \; d\theta \;\; e^\sigma \; [\; D_+\sigma \; D_-\sigma \;+\; V(\sigma) \;]. \tag{3.2}$$

where $\mathcal{R}_2 \;=\; V(\sigma)$. The Moyal star product deformation is

$$\int dt \, \mathcal{L}_* \;=\; \int dt \int dy \, d\theta \;\; e_*^\sigma * [\, e_*^{-\sigma} * e_*^{-\sigma} * D_+ \, e_*^\sigma * D_- \, e_*^\sigma + V_*(\sigma) \,]. \tag{3.3}$$

and has the same functional form (up to scaling factors in the integration measure) as the W_∞ and w_∞ Matrix Model Lagrangians in $D = 1$ studied by [41], [42].

$$\mathcal{L} \;=\; trace \; [\; \frac{1}{2} \; (\; \partial_t \; \mathbf{M}(t) \;+\; \{ \; \mathbf{A}_t \; (t) \;,\; \mathbf{M}(t) \; \} \;)^2 \; - V(\mathbf{M}) \;] \;=$$

$$\int d^2z \; \frac{1}{2} \; [\; \partial_t \; M(t,z,\bar{z}) \; + \; \{ \; A_t \; (t,z,\bar{z}) \; , \; M(t,z,\bar{z}) \; \}_{MB} \;]^2_* \; - V_*(M). \quad (3.4a)$$

where the infinite-dimensional trace operation is replaced by an integration $trace \rightarrow \int d^2z$ and

$$(\; \partial_t \; M \; + \; \{ \; A_t, \; , \; M \; \}_{MB})^2_* \; =$$

$$(\; \partial_t \; M \; + \; \{ \; A_t, \; , \; M \; \}_{MB}) \; * \; (\; \partial_t \; M \; + \; \{ \; A_t \; , \; M \; \}_{MB})$$

the one-dim w_∞ Matrix Model is based on the Lagrangian

$$\mathcal{L} \; = \; \int d^2y \; \frac{1}{2} \; [\; \partial_t \; M(t,y^1,y^2) + \{ \; A_t \; (t,y^1,y^2) \; , \; M(t,y^1,y^2) \; \}_{PB} \;]^2 \; - V(M). \quad (3.4b)$$

the internal coordinates y^1, y^2 of the two-dim surface $\mathcal{N}_2$ are represented by the complex coordinates $z = \frac{1}{\sqrt{2}l}(y^1 + iy^2), \bar{z} = \frac{1}{\sqrt{2}l}(y^1 - iy^2)$ associated with the coherent-states representation and l is length scale parameter . The Moyal brackets of two functions $\xi_1(z,\bar{z}), \xi_2(z,\bar{z})$ in units of $\hbar = c = 1$ is

$$\{ \; \xi_1(z,\bar{z}) \; , \; \xi_2(z,\bar{z}) \; \}_{MB} \; =$$

$$i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \; [\; \partial^n_z(\xi_1(z,\bar{z})) \; \partial^n_{\bar{z}}(\xi_2(z,\bar{z})) \; - \; \partial^n_{\bar{z}}(\xi_1(z,\bar{z})) \; \partial^n_z(\xi_2(z,\bar{z})) \;]. \quad (3.5)$$

The canonical quantization leads to the Hamiltonians expressed in terms of momentum variables

$$H \; = \; \int d^2z \; \frac{1}{2} \; (P(z,\bar{z})) \; * \; (P(z,\bar{z})) \; + \; V_*(M) \; =$$

$$\int d^2z \; \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \; [\; \partial^n_z(P(z,\bar{z})) \; \partial^n_{\bar{z}}(P(z,\bar{z})) \; + V_*(M). \quad (3.6a)$$

$$H \; = \; \int d^2y \; \frac{1}{2} \; (P(y^1,y^2)) \; (P(y^1,y^2)) \; + \; V(M). \quad (3.6b)$$

The W_∞, w_∞ gauge invariance of the actions leads to the following constraints on the state vector $|\Psi>$

$$\int d^2z \; \{ \; \xi \; , \; M \; \}_{MB} \; P(z,\bar{z}) \; |\Psi> \; = \; 0. \quad (3.7a)$$

$$\int d^2y \; \{ \; \xi \; , \; M \; \}_{PB} \; P(y^1,y^2) \; |\Psi> \; = \; 0. \quad (3.7b)$$

Kavalov and Sakita solved the problem by using the techniques based on the collective field method [43], [44] that requires a change of variables from $P(z,\bar{z}), M(z,\bar{z})$ to $\pi(x), \phi(x)$. The procedure is quite elaborate. The end result yields the following Hamiltonians for the collective field associated with the W_∞ algebra

$$H \ = \ \int \ dx \ [\ \frac{1}{2} (\partial_x \pi(x))^2 \phi(x) + \frac{\pi}{6} \ \phi(x)^3 \ + V(x) \ \phi(x) \ ] \ - \lambda (\int dx \ \phi(x) \ - \ N). \tag{3.8a}$$

and

$$H \ = \ \int \ dx \ [\ \frac{1}{2} (\partial_x \pi(x))^2 \phi(x) + \frac{\kappa}{8} \ \frac{(\partial_x \phi(x))^2}{\phi(x)} \ + V(x) \ \phi(x) \ ] - \lambda (\int dx \ \phi(x) - L^2). \tag{3.8b}$$

associated with the w_∞ algebra. N is the number of fermions, L^2 is the area of the fluid, κ is a numerical parameter and λ a Lagrange multiplier enforcing the constraints. After a suitable scaling transformations, in the $N \to \infty$ limit the excitation spectrum found by Kavalov and Sakita [41], [42] turned out to be

$$H \ = \ \frac{1}{2} \sum_{n=0}^{\infty} \ (p_n^2 \ + \ \omega_n^2 \ q_n^2), \quad [q_n, p_n] = i \delta_{mn}, \ \hbar = c = 1. \tag{3.9}$$

for the W_∞ one-dim Matrix model case the frequencies are

$$\omega_n \ = \ \frac{n\pi}{T}, \quad T \ = \ \int_{x_1}^{x_2} \ \frac{dx}{\sqrt{2(E_o - U(x))}}, \quad \frac{1}{\pi} \ \int_{x_1}^{x_2} \ dx \ \sqrt{2(E_o - U(x))} \ = \ 1. \tag{3.10a}$$

and for the w_∞ one-dim Matrix model the frequencies are obtained from the energy levels of the solutions of the Schroedinger equation

$$\omega_n \ = \ E_n - E_o, \quad [-\frac{1}{2} \partial_x^2 \ + \ V(x) \ ] \ \psi_n(x) \ = \ E_n \ \psi_n(x). \tag{3.10b}$$

where

$$U(x) \ = \ \sum_n \ N^{\frac{n}{2} - 1} g_n x^n \ = \ \sum_n \ a_n \ x^n. \tag{3.11a}$$

and

$$V(x) \ = \ \sum_n \kappa^{n-2} \tilde{g}_n x^n \ = \ \sum_n (\kappa l)^{n-2} g_n x^n \ = \ \sum_n \ b_n \ x^n. \tag{3.11b}$$

respectively. As mentioned above, the key point was that although the collective field theory is *not* a free theory it has a w_∞ spectrum generating algebra associated with the harmonic oscillator

As stated earlier, quantization of Einstein Gravity in $4D$ admits a reformulation in terms of a $2D$ Quantum W_∞ gauge theory coupled to an infinite family of scalars. The starting point is the classical $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian density constructed by [35]

$$\mathcal{L} \ = \ \sum_{\vec{i}, \vec{j}} \ (\Phi^6(x))^{-\vec{i} - \vec{j}} \ \mathcal{F}^{\vec{i}}_{+-}(x) \ \mathcal{F}^{\vec{j}}_{+-}(x) \ +$$

$$\sum_{\vec{k}} \ (\mathcal{D}_+ \Phi^{-\vec{k}}(x)) \ (\mathcal{D}_- \Phi^{\vec{k}}(x)) \ + \ V(\Phi^{\vec{k}}(x)). \tag{3.12}$$

Or one may define the Lagrangian density of a Yang-Mills-like $w_\infty, w_{1+\infty}$ gauge field coupled to a scalar field $\mathbf{\Phi}$ valued in the *adjoint* representation of $w_\infty, w_{1+\infty}$ and subject to a self-interacting scalar potential $V(\mathbf{\Phi})$ given by eqs-(2.16, 2.17)

$$\mathcal{L} \ = \ Trace \ [\ - \frac{1}{2} \ \mathbf{F}_{+-} \ \mathbf{F}_{+-} \ + \ D_+ \mathbf{\Phi} \ D_- \mathbf{\Phi} \ + \ V(\mathbf{\Phi}) \ ] \ \leftrightarrow$$

$$\mathcal{L} \ = \ \int \ d^2 y \ e^\sigma \ [\ - \frac{1}{2} \ e^\sigma \ F^a_{+-} \ F^a_{+-} \ + \ (D_+ \ \sigma) \ (D_- \ \sigma) \ + \ V(\sigma) \ ]. \tag{3.13}$$

A quantization of eqs-(3.12, 3.13) will $deform$ the classical $w_\infty, w_{1+\infty}$ symmetry algebras of the classical Lagrangian to the quantum $W_\infty, W_{1+\infty}$ symmetry of the quantum theory (a BRST quantization procedure) and such that the latter quantum algebras will be the spectrum generating algebras. Since there are an infinite number of higher conformal spin generators the highest weight representations will generate an infinite number of states at each level. Kac and Radul [58] solved this problem by constructing quasi-finite highest weight representations that were used by [59] to develop the full fledged representation theory of the quantum $W_{1+\infty}$ algebra. Free field realizations, (Super) Matrix generalizations, the structure of subalgebras such as the W_∞ algebra, determinant formulae and character formulae can be found in [59].

Quantum Effective actions for W_∞ Gravity have been known for a long time [80], [81]. They have the form of the Wess-Zumino quantum effective action of chiral W_∞ -symmetric system of matter fields coupled to a general chiral W_∞-gravity background. It is expressed as a geometric action on a *coadjoint orbit* of the deformed group of area-preserving diffeomorphisms whose underlying Lie algebra is the centrally-extended algebra of symbols of differential operators on the circle. This action was also found based on the functional integral quantizatiom method of the equation which describes the development of cross sections in a Twistor space corresponding to $4D$ Self Dual Gravity [82]. The result is an infinite sum of $2D$ anomalous effective actions associated with the central charges that correspond to the higher-conformal spin generators of W_∞. The infinite sum is basically a sum over the generalized higher order Schwarzians. The anomalous effective action corresponding to W_2 is just the $2D$ Polyakov induced gravity action in the light-cone gauge.

The knowledge of the Quantum Effective actions for W_∞ Gravity might be a guidance principle when one wishes to evaluate the quantum effective action in terms of the higher spin gauge fields associated with the actions represented in eq-(3.12, 3.13), respectively

$$e^{i\Gamma[A^{\vec{k}}_\mu]} \ = \ \int \ [D\Phi^{\vec{k}}] \ e^{iS[A^{\vec{k}}_\mu, \Phi^{\vec{k}}]}. \tag{3.14a}$$

$$e^{i\Gamma[A_\mu]} \ = \ \int \ [D\sigma] \ e^{iS[A_\mu, \sigma]}. \tag{3.14b}$$

The evaluation of the Quantum Effective actions $\Gamma[A_\mu^{\vec{k}}]$, $\Gamma[A_\mu]$ in eqs-(3.14a, 3.14b) is a daunting task.

The Bars-Witten stringy black hole in $D = 2$ has a nonlinear $\hat{W}_\infty(k = \frac{9}{4})$ for hidden symmetry [20] that can be used as its spectrum generating algebra; a W_∞ symmetry of the Nambu-Goto string in $4D$ was also found in [21] based on a $SU(2)/U(1)$ coset model. Closely related to black-holes in $3D$, Witten has shown [63] that the energy spectrum of three-dimensional gravity with negative cosmological constant associated with the BTZ black-hole can be determined exactly. Witten has argued that the dual Conformal Field Theory (CFT) is very likely to be the Monster theory of Frenkel, Lepowsky, and Meurman. The partition function was found to be a polynomial in the modular invariant Klein function $j(q)$. Manschot has shown more recently that the partition function can be obtained as a modular sum over geometries [64].

Not so long ago, the authors [65] inspired by a formal resemblance of certain q-expansions of modular forms and the master field formalism of matrix models in terms of Cuntz operators, constructed a Hermitian one-matrix model that was coined the "Modular Matrix Model" which naturally encode the Klein elliptic $j(q)$-invariant and the irreducible representations of the Fischer-Griess Monster group resulting from the Moonshine conjecture. These results relating Modular Matrix Models, quantum gravity and the Monster, in particular the role of W_∞ algebras, warrant a furher investigation. For an extensive review of $2D$ Gravity, Matrix Models and String theory see [47], [48].

Isomonodromic quantization of dimensionally reduced Gravity can be found in [57]. The relationship between W_∞ gravity (geometry) and the Fedosov deformation quantization of the $4D$ Self-Dual Gravity [38] associated with the complexified co-tangent space of a two-dim Riemann surface was studied by [40],[39]. String and p-branes actions can be obtained by a Moyal deformation quantization of (Generalized) Yang-Mills as shown in [29], [30], [31], [32], [33]. A natural Fedosov type quantization of generalized Lagrange models and gravity theories with metrics lifted on tangent bundle, or extended to higher dimensions, has been attained by Vacaru [60], [61], [62]. The constructions are possible due to a synthesis of the nonlinear connection formalism developed in Finsler and Lagrange geometries and deformation quantization methods. The connection $A_\mu(x.y)$ in the decomposition of eq-(1.1) *is* precisely the *nonlinear* connection associated with Lagrange-Finsler spaces.

Higher spin field theories in $D > 2$ have been extensively studied over the years by Vasiliev [28] and Calixto [54] has constructed Generalized W_∞-type Higher Spin Algebras in Higher dimensions $D > 2$ where non-linear realization methods [55], [56] could be used to build higher spin extensions of Gravity theories. In section **5** we will review the interplay among quantum membranes, the continuous Toda theories and non-critical W_∞ (super) strings [66] to show why non-critical W_∞ (super) strings in $D = 27(11)$ dimensions are devoid of BRST anomalies. Such $D = 27(11)$ dimensions coincide with the alleged critical dimensions of the quantum (super) membrane, respectively. To finalize, we must say that Noncommutative $4D$ Gravity based on deformed diffs and Poincare algebras developed by [67] deserves further investigation within the context of $2D$ W_∞ gauge theory.

4 Strings and Membranes from $4D$ Einstein Gravity

The findings of this Section **4** are new where we show how string and membrane actions emerge from Gravitational actions. We must emphasize that a very different construction of p-brane actions from graviton-dilaton effective actions was provided by [68] and which differs from the *nonlinear* connection formalism of eq-(1.1) (typical of Finsler geometries) leading to eq-(1.34) and that is the basis of our derivations below.

4.1 Strings in $2D$ and $3D$ from $4D$ Einstein Gravity

In eqs-(1.3-1.5) and (2.34-2.36) we described the relationship (isomorphism) between the area-preserving diffs algebra and the algebra of symplectic diffs associated with the internal two-dim space $\mathcal{N}_2$. In particular we have the correspondence between Lie and Poisson brackets that is compatible with the Lie-Poisson structure given by

$$[\ L_{A_\mu}, L_{A_\nu}\ ] \ = \ L_{[A_\mu, A_\nu]} \ \leftrightarrow \ \mathcal{L}_{\{\mathcal{A}_\mu, \mathcal{A}_\nu\}}. \tag{4.1}$$

the first term in the l.h.s denotes the Lie commutator of two Lie derivative operators along the vector fields A_μ, A_ν; the second term denotes the Lie derivative associated with the commutator (Lie bracket) of A_μ, A_ν. While the term in the r.h.s denotes the generator $\mathcal{L}$ of symplectic diffs associated with the Poisson bracket $\{\mathcal{A}_\mu, \mathcal{A}_\nu\}$.

It is important to emphasize that one is *not* equating the nonlinear connection $A^a(x^\mu, y^a)$ with $\mathcal{A}_\mu(x^\mu, y^a)$. We are only establishing a correspondence (isomorphism) between the l.h.s and the r.h.s of eq-(4.1). One can also replace Poisson brackets for Moyal ones and obtain the isomorphism between the quantum deformed (centerless) algebras $W_\infty, W_{1+\infty}$ and Moyal deformations of $w_\infty, w_{1+\infty}$, respectively, as discussed in section **2.2**.

It is this isomorphism (correspondence) between the area-preserving diffs algebra and the algebra of symplectic diffs that permits to show why the Eguchi-Schild (ES) string action can be obtained in a straightforward manner from the expression in eqs-(1.32-1.34) by replacing Lie brackets for Poisson ones and setting $\sigma = 0$, $\rho_{ab} = \delta_{ab}$ which implies that the internal space Ricci scalar (when ϕ_{ab} solely depends on y^1, y^2) corresponding to $\phi_{ab} = e^\sigma \delta_{ab} = \delta_{ab}$ is $\mathcal{R}_2(\phi_{ab} = \delta_{ab}) = 0 \Rightarrow V(\sigma = 0) = 0$. Hence, in this particular case, eq-(1.34) gives

$$-\frac{1}{2}\int d^2y\ \mathcal{F}_{+-}\ \mathcal{F}_{+-}\ ;\quad where\ \mathcal{F}_{+-}\ =\ \partial_+\mathcal{A}_-\ -\ \partial_-\mathcal{A}_+\ +\ \{\mathcal{A}_+\ ,\ \mathcal{A}_-\}_{PB}. \tag{4.2}$$

The validity of eq-(4.2) can also be justified by our results of [29], [30], [31], [32], [33] which showed rigorously why the large N limit of quenched **QCD** (quenched to a point and a line), after performing a Moyal deformation quantization procedure of the corresponding $SU(N)$ Yang-Mills theory, furnished string and membrane actions, respectively, in the "classical" $\hbar \to 0$ limit and which was tantamount to the $N \to \infty$ limit since $\hbar \sim \frac{1}{N}$. A Moyal quantization procedure of Generalized Yang-Mills theories based on actions involving $F \wedge F \wedge F...$ yielded p-brane actions. The large N limit of Dirac-Born-Infeld models and its relationship to the Dirac-Nambu-Goto string was also provided in [33].

The large N limit $SU(N)$ in a suitable basis yields an algebra isomorphic to the area-preserving diffs of the sphere [49]. The author [50], [51] has shown that this isomorphism is not truly exact, nevertheless for our purposes this fact does not affect our findings. $w_\infty, w_{1+\infty}$ correspond to the area-preserving diffs of a plane and cylinder respectively.

When the gauge fields $\mathcal{A}_\mu(y^1, y^2)$ solely depend on the internal space $\mathcal{N}_2$ coordinates and that is also consistent with the *quenching* procedure to a *point* of the large N limit of $SU(N)$ Yang-Mills [30], the above expression becomes

$$-\frac{1}{2}\int d^2y\ \{\mathcal{A}_+\ ,\ \mathcal{A}_-\}_{PB}\ \{\mathcal{A}_+\ ,\ \mathcal{A}_-\}_{PB}. \tag{4.3}$$

which is precisely the Eguchi-Schild (ES) string action for a string moving in a flat $2D$ space-time background after establishing the gauge field/string-coordinates correspondence $\mathcal{A}_0(y^1, y^2) \leftrightarrow X_0(y^1, y^2)$, $\mathcal{A}_1(y^1, y^2) \leftrightarrow X_1(y^1, y^2)$ [29], [30]

$$S_{ES}\ =\ T\int d^2y\ \frac{1}{2}\ \{X_0\ ,\ X_1\}_{PB}\ \{X^0\ ,\ X^1\}_{PB}. \tag{4.4}$$

and upon recurring to the defining relations $X_\pm = \frac{1}{\sqrt{2}}(X_0 \pm X_1)$ and introducing the string tension $T \sim l_{Planck}^{-2}$ to render the action dimensionless. The source of the Planck scale in the string tension stems directly from the presence of the $4D$ Newton constant $G \sim l_{Planck}^2$ in the starting Einstein-Hilbert action of eq-(1.9) when $m = 2, n = 2$ given by $2\kappa^2 = 16\pi G$.

We will show how one can obtain the Eguchi-Schild string action in a flat $3D$ space-time background in the case one has small values of the internal space ($\mathcal{N}_2$) dilaton-field configurations σ such that

$$e^\sigma\ \sim\ 1+\ \sigma,\quad V(\sigma)\ \sim\ V(0)\ +\ \frac{\partial V(\sigma)}{\partial\sigma}|_{\sigma=0}\ \sigma,\quad V(\sigma=0)=0. \tag{4.5}$$

where $\sigma(y^1, y^2)$ is constrained also to depend only on the internal coordinates y^1, y^2a of $\mathcal{N}_2$. Thus, the lowest order terms in σ of the action of eq-(1.34)

$$\int d^2y\, e^\sigma\, [\, -\frac{1}{2}\, e^\sigma\, F_{+-}\, F_{+-}\ +\ (D_+\,\sigma)\,(D_-\,\sigma)\ +\ V(\sigma)\,]. \tag{4.6}$$

after performing the gauge field/string correspondence and introducing the string tension leads to the following action

$$T \int d^2y\, [\, \frac{1}{2}\, \{X_0\, ,\, X_1\}^2 + \frac{1}{2}\, \{X_0\, ,\, \sigma\}^2 - \frac{1}{2}\{X_1\, ,\, \sigma\}^2 + \frac{\partial V(\sigma)}{\partial\sigma}|_{\sigma=0}\, \sigma + ...]. \tag{4.7}$$

hence, one obtains *corrections* to the Eguchi-Schid string action in $2D$ resulting from the higher mode expansion of e^σ and the potential $V(\sigma)$. One may interpret those corrections as the terms associated with the Eguchi-Schild action for a string moving now in a $3D$ target space time background of coordinates X_0, X_1, X_2 if one identifies the extra string coordinate $X_2(y^1, y^2)$ with the scalar field $\sigma(y^1, y^2)$. It is well known in non-critical string theory that the Liouville scalar field acquires an extra space time coordinate interpretation.

Therefore, by identifying the coordinate X_2 with σ, the first 3 terms of (4.7) lead to the Eguchi-Schild string action in a flat $3D$ space time background. If the condition $\frac{\partial V(\sigma)}{\partial\sigma}(\sigma = 0) = 0$ in (4.7) holds, then one may conclude that the expansion to *lowest* order terms in the dilaton σ associated with the internal two-dim space $\mathcal{N}_2$ metric $\phi_{ab} = e^\sigma \rho_{ab} = e^\sigma \delta_{ab}$, furnish the Eguchi-Schild string action for a string moving in a flat $3D$ space-time background. To zeroth order in σ one obtains the string action in $2D$ as shown in eq-(4.4).

4.2 Membranes in $3D$ from $4D$ Einstein Gravity

To obtain membrane actions one requires a Killing symmetry reduction from $2D$ (associated to the base space time $\mathcal{M}_2$) to $1D$ by setting a dependence on x^0, y^1, y^2, with $x^0 = \frac{1}{\sqrt{2}}(x^+ + x^-)$, such as

$$\mathcal{A}_\mu\ =\ A_\mu(x^0, y^a);\quad \sigma = \sigma(x^0, y^a), \tag{4.8}$$

$$\frac{\partial A}{\partial x_+}\ =\ \frac{\partial A}{\partial x_-}\ =\ \frac{1}{\sqrt{2}}\,\frac{\partial A}{\partial x_0} \Rightarrow\ \frac{\partial A}{\partial x_1} = 0. \tag{4.9}$$

Such Killing symmetry reduction to $1D$ is essentially equivalent to the quenching procedure of the large N **QCD** to a line [31], [32]. The gauge fields/membrane coordinates correspondence is $\mathcal{A}_\mu(x^0, y^a) \leftrightarrow X_\mu(x^0, y^a)$. Because a membrane has a three-dim world volume it requires an embedding space-time dimension of $D \geq 3$. The 3 variables $X_0, X_1, X_2 = \sigma$ match the minimum number of the target space time background membrane coordinates if one identifies the coordinate x^0 with the membrane's world volume clock $\tau = x^0$. Note that one should *not* confuse the membrane space time coordinate X_0 with the clock variable $x^0 = \tau$. The former is a function of the latter $X_0 = (\tau, y^1, y^2)$

Hence, upon imposing the Killing symmetry reduction conditions $\partial_+ = \partial_- = \frac{1}{\sqrt{2}}\partial_\tau$ in eq-(1.34) gives

$$\mathcal{L}_3 = \frac{1}{2} [(\frac{\partial X_1}{\partial \tau} + \{X_0 , X_1\})^2 + (\frac{\partial \sigma}{\partial \tau} + \{X_0 , \sigma\})^2 - \{X_1 , \sigma\}^2] + ... \quad (4.10)$$

after introducing the membrane tension one can see that the first three terms in eq-(4.10) are precisely the same as those terms given by the light-cone membrane Lagrangian moving in a flat $3D$ background [49]. One is integrating now over the three variables among the four coordinates x^μ, y^a present in the $2+2$ decomposition of the initial $4D$ Einstein-Hilbert action of eq-(1.6)

$$\int dx^1 \int dx^0 \ d^2y \ \mathcal{L}_3(x^0, y^a) \ = \ L \int d\tau \ d^2y \ \mathcal{L}_3(\tau, y^a). \quad (4.11)$$

where L is the length scale along the x^1 direction. One can notice that upon setting $\frac{\partial X_1}{\partial \tau} = \frac{\partial \sigma}{\partial \tau} = 0$ in the Lagrangian (4.10) one recovers the Eguchi-Schild string action in $3D$.

To sum up, following the gauge field/membrane coordinate correspondence $\mathcal{A}_\mu(x^0, y^a) \leftrightarrow X_\mu(x^0, y^a)$, a Killing symmetry reduction condition $\partial_+ = \partial_- = \frac{1}{\sqrt{2}}\partial_\tau$ in eq-(1.34), leads to the light-cone membrane Lagrangian for a membrane moving in a flat $3D$ space-time background if one identifies the scalar field variable $\sigma(x^0 = \tau; y^a) \leftrightarrow X_2(x^0 = \tau; y^a)$ with the *extra* embedding spacetime coordinate X_2. Therefore, we have shown in this section how string and membrane actions (in lower $2D$ and $3D$ dimensions) emerge from $4D$ Gravity.

5 On $SU(\infty)$ Yang-Mills, Gravity, Membranes and W_∞ Strings

5.1 Non-critical W_∞ (super) strings and the Critical (super) Membrane Dimension

The purpose of this subsection is to review our proof [66] that non-critical W_∞ (super) strings are devoid of BRST anomalies in dimensions $D = 27, 11$, respectively, and which coincide with the the critical (super) membrane dimensions $D = 27, 11$ found by [69], [70]. We deem this review important due to the relationship between the large N limit of Self-Dual $SU(N)$ Yang-Mills and Self Dual Gravity [71]. A Killing symmetry reduction of $4D$ SD Gravity furnishes the $3D$ continuous $SL(\infty, R)$ Toda theory [1] which exhibits a w_∞ symmetry (a Killing symmetry reduction of a CP^1 loop algebra over w_∞). Conversely, the induced $2D$ quantum W_∞ gravity action in the light-cone gauge has a hidden $SL(\infty, R)$ Kac-Moody symmetry [27]. The classical geometry of $2D$ w_∞ gravity is linked to the $4D$ Self-Dual gravity associated with the $4D$ cotangent space of $2D$ Riemann surfaces [38] and admits a Fedosov deformation quantization. [40].

It was shown in [72], [73], that the effective induced action of W_N gravity in the conformal gauge takes the form of a Toda action for the scalar fields and the W_N currents take the familiar free field form. The same action can be obtained from a constrained $WZNW$ model (modulo the global aspects of the theory due to the topology). Richer structures emerge in the reduction process by a quantum Drinfeld-Sokolov reduction of the $SL(\infty, R)$ Kac-Moody algebra at the level k [74]. Each of these quantum Toda actions posseses a W_N symmetry. A Moyal quantization of the continuous Toda theory and its relation to $4D$ Self-Dual gravity and self-dual membranes was provided in [75], [76].

In what follows, by W_N string we mean the string associated to the W algebra corresponding to the A_{N-1} Lie algebra. In general, non-critical W_N strings are constructed the same way : by coupling W_N matter to W_N gravity. The matter and Liouville sector (stemming from W_N gravity) of the W_N algebra can be realized in terms of $N-1$ scalars, ϕ_k, σ_k repectively. These realizations in general have background charges which are fixed by the Miura transformations [77], [78]. The non-critical string is characterized by the central charges of the matter and Liouville sectors, c_m, c_L. To achieve a nilpotent BRST operator these central charges must satisfy :

$$c_m + c_L = -c_{ghost} = 2\sum_{s=2}^{N}(6s^2 - 6s + 1) = 2(N-1)(2N^2 + 2N + 1). \quad (5.1)$$

In the $N \to \infty$ limit a zeta function regularization yields $c_m + c_L = -2$.

The authors [78] have shown that the BRST operator can be written as a sum of nilpotent BRST operators , Q_N^n, and that a nested basis can be chosen either for the Liouville sector or the matter sector but not for both. If the nested basis is chosen for the Liouville sector then [78] found that the central charge for the Liouville sector is

$$c_L = (N-1)[1 - 2x^2 N(N+1)]. \quad (5.2)$$

were x is an *arbitrary* parameter which makes it possible to avoid the relation with the W_N minimal models if one wishes to. By choosing , if one wishes, x appropriately one can, of course, get the q^{th} unitary minimal models by fixing x^2 to be :

$$x_o^2 = -2 - \frac{1}{2q(q+1)}. \quad (5.3)$$

where q is an integer. In this case, since $c_m + c_L = -c_{gh}$, the central charge for the matter sector must be :

$$c_m = (N-1)(1 - \frac{N(N+1)}{q(q+1)}). \quad (5.4)$$

which corresponds precisely to the q^{th} minimal model of the W_N string as one intended to have by choosing the value of x_o^2. In the present case one has the freedom of selecting the minimal model since the value of q is arbitrary. If $q = N$ then $c_m = 0$ and the theory effectively reduces to that of the "critical" W_N string. Conversely, if one chooses for the nested basis that corresponding to the matter sector instead of the Liouville sector, the roles of "matter" and "Liouville" are reversed. One would then have $c_L = 0$ instead.

Noncritical strings involve two copies of the W_N algebra. One for the matter sector and other for the Liouville sector. Since W_N is nonlinear, one cannot add naively two realizations of it and obtain a third realization. Nevertheless there is a way in which this is possible [78], This was achieved by using the nested sum of nilpotent BRST operators, Q_N^n. One requires to have all the matter fields ϕ_k; plus the scalars of the Liouville sector in the nested basis , $\sigma_{n-1}, ...\sigma_{N-1}$ and the ghost and antighost fields of the spin $n, n+1, ...N$ symmetries where n ranges between 2 and N. Central charges were computed for each set of the nested set of stress energy tensors, T_N^n depending on all of the above fields which appear in the construction of the BRST charges : Q_N^n.

In order to find a space-time interpretation, the coordinates X^μ must be related to a very specific scalar field of the Liouville sector (since one decided to choose the nested basis in the Liouville sector) and that field is σ_1. It is this central charge, associated with the scalar field σ_1, that *always* appears through its energy momentum tensor in the Miura basis. Because σ_1 always appears through its energy momentum tensor, it can be replaced by an effective T_{eff} of any conformal field theory as long as it has the same value of the central charge given by $c = 1 + 12\alpha^2 \equiv 1 - 12x^2$, where α is a background charge.

$$T(\sigma_1) \ = \ -\frac{1}{2}(\frac{\partial\sigma_1}{\partial z})^2 - \alpha\frac{\partial^2\sigma_1}{\partial z^2}. \tag{5.5}$$

In particular, the stress energy tensor corresponding to D worldsheet scalars, X^μ, with a background charge vector, α_μ is

$$T_{eff} \ = \ -\frac{1}{2} \ (\partial_z X^\mu) \ (\partial_z X_\mu) \ - \alpha_\mu \ (\partial_z^2 X^\mu); \quad c_{eff} \ = \ D \ +12 \ \alpha_\mu\alpha^\mu = 1 + 12\alpha^2. \tag{5.6}$$

For example, in the critical W_∞ string case, one is bound to the unitary minimal models that are related to Toda fields [77], [78] and one must pick for central charge associated with the scalar, σ_1, the value $\alpha^2 = -x^2 = -x_o^2$ given by (5.3). Thus, in this way the explicit value of c of the critical W_∞ string is obtained

$$c_{crit} \ = \ 1 + 12 \ (\alpha_o)^2 \ = \ 1 - 12x_o^2 \ = \ 1 - 12 \ (\ -2 - \frac{1}{2q(q+1)} \ ) \ = \ 25; \quad q = N \to \infty. \tag{5.7}$$

In the case of the ordinary critical string, $W_N = W_2$, $q = N = 2$, one has

$$x_o^2 = -2 - \frac{1}{2q(q+1)} \;\Rightarrow\; -2 - \frac{1}{12} \;\Rightarrow\; c_{eff} = 1 - 12x_o^2 = 1 + 25 = 26 = c_{crit}. \tag{5.8}$$

Since the parameter x in the non-critical string case is an *arbitrary* parameter that is no longer bound to be equal to x_o, the effective central charge in the non-critical W_N string is now $1 - 12x^2$ in contradistinction to the critical W_N string case : $1 - 12x_o^2$. Therefore, if one wishes to make contact with $D = 27$ X^μ scalars instead of $D = 25$ one can choose x in such a way that it obeys $1 - 12x^2 = (1 - 12x_o^2) + c_{m_o}$ where c_{m_o} will turn out to be the central charge of the $q = N + 1$ unitary minimal model of the W_N algebra.

If one does *not* wish to break the target space-time Lorentz invariance one *cannot* have background charges for the D X^μ coordinates. Therefore, for the case that $q = N+1 \Rightarrow c_{m_o} = \frac{2(N-1)}{N+1}$ (instead of zero) is obtained from eq-(5.4), and the effective central charge is now :

$$c_{eff} \equiv 1 - 12x^2 \;=\; (1 - 12x_o^2) + c_{m_o} \;=$$
$$[\; 26 \;-(\; 1 \;-\frac{6}{(N+1)(N+2)} \;)\;] \;+\; [\; 2\frac{N-1}{N+2} \;] \tag{5.9}$$

then one concludes from eq-(5.9) that $c_{eff} = D = 25 + 2 = 27$ is recovered in the $N \to \infty$ limit. The reason why one wrote the last term of eq-(5.8) in such a peculiar way will be clarified shortly.

In this way, by having $c_{eff} = D = 27$ in eq-(5.9) we have shown that the expected critical dimension for the bosonic membrane background, $D = 27$, has the same number of X^μ coordinates as that of a $non-critical$ W_∞ string background if one adjoins the $q = N + 1$ unitary minimal model of the W_N algebra to that of a critical W_N string spectrum in the $N \to \infty$ limit. This phenomenon is very similar to seeing the $D = 26$ critical string as a non-critical string in $D = 25$ if one *adjoins* the Liouville mode that plays the role of the extra dimension.

For these reasons, one should expect that the physical membrane spectrum in $D = 27$ should contain a sector related to a critical W_∞ string *adjoined* to a $q = N + 1$ unitary minimal model of the W_N algebra in the $N \to \infty$ limit. The critical W_∞ string [77] is a generalization of the ordinary string in the sense that instead of gauging the two-dimensional Virasoro algebra one gauges the higher conformal spin algebra generalization ; the W_∞ algebra. The spectrum can be computed exactly and is equivalent to an infinite set of spectra of Virasoro strings with unusual central charges and intercepts [77]. As stated earlier , the critical W_N string (linked to the A_{N-1} algebra) has for central charge the value ($q = N$) :

$$c \;=\; 1 - 12x_o^2 \;=\; 26 - (1 - \frac{6}{q(q+1)}) \;=\; 26 - (1 - \frac{6}{N(N+1)}) = 25 \tag{5.10}$$

Unitarity is achieved if the conformal-spin two-sector intercept is :

$$\omega_2 = 1 - \frac{k^2-1}{4N(N+1)}; \quad 1 \leq k \leq N-1. \tag{5.11}$$

An important remark is in order : we have to emphasize that one should *not* confuse c_{eff} with c_m, c_L in the same way that one must not confuse x^2 with x_o^2. The ordinary (W_2) string is a very *special* case insofar that $c_{eff} = c_m$ or c_L depending on our choice for the nested basis. The $D = 27$ X^μ spacetime interpretation of the theory is *hidden* in the stress energy tensor of the σ^1 field $T(\sigma_1) \rightarrow T(X^\mu)$ with $c_{eff} = c(D) = D = 27$. And, in addition to the 27 X^μ space-time coordinates, one still has the infinite number of scalars $\phi_1, \phi_2, ...$ and the infinite number of remaining fields , $\sigma_2, \sigma_3, ...$ of the Liouville sector. Clearly the situation is vastly more complex that the ordinary string (the W_2 string).

The connection to the unitary Virasoro minimal models was established in eq-(5.10) by setting $q = N+1$:

$$D - 2 = 25 = c_{string} - [1 - \frac{6}{q(q+1)}] = 26 - [1 - \frac{6}{(N+1)(N+2)}]. \tag{5.12}$$

This shall guide us in repeating the arguments for the supersymmetric case. Since 10 is the critical dimension of the ordinary superstring the value of the central charge when one has 10 worldsheet scalars and 10 fermions is $c_{superstring} = 10(1+\frac{1}{2}) = \frac{30}{2}$. In order to find the central charge of a critical super W_∞ string one requires to employ also the central charge of the super-Virasoro unitary minimal super-conformal models given by $c_{superconformal} = \frac{3}{2}$. Hence, the supersymmetric analog of the terms in the r.h.s of eq-(5.12) are then

$$c_{superstring} \;-\; c_{superconformal} \;=\; \frac{30}{2} \;-\; \frac{3}{2} \;= \frac{27}{2}. \tag{5.13}$$

The supersymmetric analog of the term $c_{m_o} = \frac{2(N-1)}{N+1} \rightarrow 2$ in the l.h.s of eq-(5.12) is given by $2(1+\frac{1}{2}) = 3$. One chooses the parameter x^2 such to make contact with the bosonic sector of the $q = N+1$ unitary minimal model of the super W_N algebra in the $N \rightarrow \infty$ limit. Writing down the corresponding supersymmetric analog of each single one of the terms appearing in the r.h.s of eq-(5.12), and the same for the l.h.s , one has that D X^μ and D ψ^μ (anticommuting spacetime vectors and world sheet spinors) *without* background charges yield a central charge $c_{eff} = D(1+\frac{1}{2}) = \frac{3D}{2}$. Therefore, the supersymmetric extension of each one of the corresponding terms of eq-(5.12) yields

$$c_{eff} \;-\; c_{m_o} \;=\; [\; c_{superstring} \;-\; c_{superconformal}\;] \;\Rightarrow$$

$$\frac{3D}{2} \;=\; [\; 10\;(1+\frac{1}{2}) \;-\; \frac{3}{2}\;] \;+\; 2\;(1+\frac{1}{2}) \;=\; \frac{33}{2} \;\Rightarrow\; D = 11. \tag{5.14}$$

Concluding , one obtains the expected critical dimension for the super-membrane $D = 11$ from eq-(5.14) if one *adjoins* a $q = N+1$ unitary super-conformal minimal model of the super W_N algebra to a critical super W_N string spectrum in the $N \to \infty$ limit. The bosonic case yields $D = 27$. We have shown why non-critical W_∞ (super) strings are devoid of BRST anomalies in dimensions $D = 27, 11$, respectively, and which coincide with the critical (super) membrane dimensions $D = 27, 11$ found by [69], [70]. For this reason we believe that the quantum (super) membrane should contain non-critical (super) W_∞ strings; i.e. it must be related to quantum W_∞ Gravity. Some preliminary attempts along these lines were undertaken by [79]. A W_∞ symmetry of the Nambu-Goto string in $4D$ was found by [21].

5.2 The Self Dual Membrane from $SU(\infty)$ Self Dual Yang-Mills and the Continuous Toda Molecule

It was found in [75], [76] that spherical membranes moving in flat target spacetime backgrounds admit a class of integrable solutions linked to $SU(\infty)$ SDYM equations (dimensionally reduced to one temporal dimension). After a suitable ansatz, the SDYM equations can be recast in the form of the continuous Toda molecule equations whose symmetry algebra is the dimensional reduction of the $W_\infty \oplus \bar{W}_\infty$ algebra.

Some time ago we were able to show [71] that the $D = 4$ $SU(\infty)$ (super) SDYM equations (an effective 6 dimensional theory) can be reduced to $4D$ (super) Plebanski's Self-Dual Gravitational equations with spacetime signatures $(4, 0); (2, 2)$. The symmetry algebra of $D = 4$ $SU(\infty)$ SDYM is a Kac-Moody extension of W_∞ as shown by [85]. In particular, new hidden symmetries were found which are affine extensions of the Lorentz rotations. These new symmetries form a Kac-Moody-Virasoro type of algebra. By rotational Killing-symmetry reductions one obtains the w_∞ algebra of the continuous Toda theory. For metrics with translational Killing symmetries one obtains the symmetry of the Gibbons-Hawking equations. A rotational Killing symmetry reduction of Plebanski's heavenly equations for Self-Dual Gravity in $D = 4$ yields

$$\frac{\partial^2 u(z, \bar{z}, t)}{\partial z \, \partial \bar{z}} \;=\; -\frac{\partial^2 e^u}{\partial t^2}. \tag{5.15}$$

which is the the $3D$ continuous Toda equation and a dimensional reduction of the $3D \to 2D$ continuous Toda equation is

$$\frac{\partial^2 u(\tau, t)}{\partial \tau^2} \;=\; -\frac{\partial^2 e^u}{\partial t^2}, \quad i\tau \;\equiv\; r \;=\; z + \bar{z}. \tag{5.16}$$

the $SU(\infty)$ Toda *molecule*. Eq-(5.16) is an effective $2D$ equation and in this fashion the original $3D$ membrane can be related to a $2D$ theory (where the W_∞ string lives in) after the light-cone gauge is chosen. The Lagrangian and the Plebanski second-heavenly equations for the $4D$ SD gravity can be obtained from a dimensional reduction of the $SU(\infty)$ SDYM (an effective six-dimensional one) [71],[84]

$$\mathcal{L} \;=\; \int dz \; d\tilde{z} \; dy \; d\tilde{y} \; [\; \frac{1}{2} \; (\Theta_{,y} \; \Theta_{,z} \; - \Theta_{,\tilde{y}} \; \Theta_{,\tilde{z}}) \; + \; \frac{1}{3} \; \Theta \; \{\Theta_{,y}, \; \Theta_{,\tilde{y}}\} \;]. \tag{5.17}$$

where $\Theta(z, \tilde{z}, y, \tilde{y})$ is Plebanski's second heavenly form and the Poisson brackets are taken w.r.t $y, \tilde{y}$ variables. A real slice can be taken by setting : $\tilde{z} = \bar{z}, \tilde{y} = \bar{y}$.

A rotational Killing symmetry reduction, $t \equiv y\tilde{y}$, yields the Lagrangian for the $3D$ Toda theory and a futher dimensional reduction $z + \tilde{z} = r$ gives the Toda molecule Lagrangian. One could use the original Killing symmetry reduction of Plebanski first heavenly equation due to Boyer and Finley [88] and also discussed by Park [89], that takes the original $r \equiv y\bar{y}; z, \bar{z}$ variables into the new ones $(t, w = \bar{z}, \bar{w} = z)$.; such that $r \equiv e^u(t, w, \bar{w}$ obeys the continuous Toda equation if, and only if, Ω obeys Plebanski first heavenly equation :

$$u_{,w\bar{w}} = e^u_{,tt}, \quad t \equiv r\ \Omega_{,r}, \quad (r\Omega_{,r})_{,r}\ \Omega_{,z\bar{z}} - r\ \Omega_{,rz}\ \Omega_{,r\bar{z}} = 1. \tag{5.18}$$

A solution of the Toda equation, $u = u(t, w, \bar{w})$ upon inversion yields $t = r\Omega_{,r} = f(u, w, \bar{w})$ which defines implicitly Ω in terms of u through the function (upon inversion) $f(u, w, \bar{w})$. And, finally, one makes contact with the Lagrangian of the Toda molecule [83] with unit coupling $\beta = 1$

$$\mathcal{L} = \int dt\ [\ \frac{1}{2}\ (\frac{\partial^2 x}{\partial r\ \partial t})^2 +\ e^{(\partial^2 x/\partial t^2)}\ ], \quad \rho(r,t)\ \equiv\ \frac{\partial^2 x}{\partial t^2}. \tag{5.19}$$

The general classical solution to (5.15) depending on two variables , say $r \equiv z_+ + z_-$ and t (*not* to be confused with time !) was given by Saveliev [83]. The solution is determined in terms of two arbitrary functions , $\varphi(t)$ and $d(t)$

$$exp\ [-x(r,t)]\ =\ exp\ [-x_o(r,t)]\ \{1 + \sum_{\geq 1}\ (-1)^n \sum_{\omega} \int\int ....\ exp\ [\ r \sum_{m=1}^{n}\ \varphi(t_m)]$$

$$\times \prod_{m=1}^{m=n}\ dt_m\ d(t_m)\ [\ \sum_{p=m}^{n}\ \varphi(t_p)\ ]^{-1}\ [\ \sum_{q=m}^{n}\ \varphi(t_{\omega(q)})\ ]^{-1}$$

$$\times[\ \epsilon_m(\omega)\ \delta(t - t_m) - \sum_{l=1}^{m-1}\ \delta''(t_l - t_m)\ \theta[\omega^{-1}(m) - \omega^{-1}(l)\ ]\ ]\ \}. \tag{5.20}$$

with : $\rho_o = \partial^2 x_o/\partial t^2 = r\varphi(t) + ln\ d(t)$. This defines the boundary values of the solution $x(r,t)$ in the asymptotic region $r \to \infty$. θ is the Heaviside step-function. ω is any permutation of the indices from $[2..........n] \to [j_2, ...j_n]$. $\omega(1) \equiv 1$. $\epsilon_m(\omega)$ is a numerical coefficient. See [83] for details. An expansion of (5.20) yields :

$$exp\ [-x]\ =\ exp\ [-x_o]\ \{1\ - \mu\ + \frac{1}{2}\mu^2 + ...\}. \tag{5.21}$$

where :

$$\mu\ \equiv\ \frac{d(t)\ exp\ [\ r\ \varphi(t)\ ]}{\varphi^2}. \tag{5.22}$$

In what follows we shall fix the function $d(t) = 1$.

To sum up this subsection and the relationship to the results in [75],[76] the spectrum generating symmetry algebra $W_\infty \oplus \bar{W}_\infty$ acting on the Toda molecule, stems from the bosonic sector of the self-dual $SU(\infty)$ Supersymmetric Gauge Quantum Mechanical Model associated with the light-cone gauge of the self-dual (spherical) supermembrane : a dimensionally-reduced super SDYM theory to *one* temporal dimension. The membrane's time coordinate (X^+) has a *correspondence* with the $r = z + \bar{z}$ "temporal" variable. The extra coordinate arises from the t parameter so the initial $3D$ continuous (super) Toda theory is dimensionally-reduced to a $1 + 1$ (super) Toda *molecule* : $\Theta(z, \bar{z}, t) \rightarrow \rho(r, t)$ and, in this way, an *effective* two-dimensional theory emerges [75],[76] . Hence, the intrinsic $3D$ Self Dual membrane spectrum can be obtained from the spectrum generating algebra of the effective two-dimensional (super) Toda molecule theory. The full membrane spectrum remains an open question. Perhaps (non-linear) integrable deformations beyond the selfdual theories (non-conformal field theories) might give us more clues about the full theory.

6 Quantum Self Dual Gravity and Highest Weight Representations of W_∞ via the continuous Toda Theory

Our main objective in this section is to establish the correspondence between the states associated with the quasi finite highest weights irreducible representations of $W_\infty, \bar{W}_\infty$ algebras and the quantum states of the Toda molecule. Schroedinger-like QM wave functional equations are derived and solutions are found in the zeroth order approximation. As far as we know the main results of this section have not been published before.

6.1 Highest Weight Representations

The purpose of this subsection is to obtain important information about the highest weights representations associated with the dimensional reduction of $W_\infty \oplus \bar{W}_\infty$ algebra which acts on the continuous Toda molecule as the symmetry algebra in the same way that the Virasoro algebra does for the string.

We can borrow now the results by [59],[86], [87] on the quasi-finite highest weight irreducible representations of $W_{1+\infty}$ and W_∞ algebras. The latter is a subalgebra of the former. For each highest weight state, $|\lambda >$ parametrized by a complex number λ the above authors constructed representations consisting of a finite number of states at each energy level by succesive application of ladder-like operators. A suitable differential constraint on the generating function $\Delta(x)$ for the highest weights Δ_k of the representations was necessary in order to ensure that, indeed, one has a finite number of states at each level. The highest weight states are defined :

$$W(z^n D^k)\ |\lambda> \ = \ 0,\ n\ \geq\ 1,\ k\ \geq\ 0, \quad W(D^k)\ |\lambda> \ = \ \Delta_k\ |\lambda>,\ \ k\ \geq\ 0. \tag{6.1}$$

The $W_{1+\infty}$ algebras can be defined as central extensions of the Lie algebra of differential operators on the circle. $D \equiv zd/dz$, n belongs to $\mathbf{Z}$ and k is a positive integer. The generators of the $W_{1+\infty}$ algebra are denoted by $W(z^n D^k)$; i.e. there is a one to one mapping between $z^n(z\frac{\partial}{\partial z})^k$ and the $W_{1+\infty}$ generators. The W_∞ generators are obtained from the former : $\tilde{W}(z^n D^k) = W(z^n D^{k+1})$. The commutation relations of the W_∞ are

$$[\ \tilde{W}[z^n(z\frac{\partial}{\partial z})^k],\ \tilde{W}[\ z^m(z\frac{\partial}{\partial z})^l]\ ]\ =\ \tilde{W}[z^{n+m}(z\frac{\partial}{\partial z}+m)^k(z\frac{\partial}{\partial z})^l(z\frac{\partial}{\partial z}+m)]-$$

$$\tilde{W}[z^{n+m}(z\frac{\partial}{\partial z})^k(z\frac{\partial}{\partial z}+n)^l(z\frac{\partial}{\partial z}+n)]\ +\ C\ \Psi[\ z^n(z\frac{\partial}{\partial z})^k(z\frac{\partial}{\partial z}), z^m(z\frac{\partial}{\partial z})^l(z\frac{\partial}{\partial z})\ ]. \tag{6.2}$$

The central charge term is given by the two-cocycle Ψ times the constant C. The anti-chiral $\bar{W}_\infty$ is given exactly the same by replacing everywhere $z \rightarrow \bar{z}$ and $\partial_z \rightarrow \partial_{\bar{z}}$. (There is no spin one current).

The generating function $\Delta(x)$ for the weights is

$$\Delta(x)\ =\ \sum_{k=0}^{k=\infty} \Delta_k\ \frac{x^k}{k!}\ \Rightarrow\ -W(e^{xD})\ |\lambda>\ =\ \Delta(x)\ |\lambda>. \tag{6.3a}$$

The generating function $\Delta(x)$ for the weights satisfies the following differential equation resulting from the quasi-finiteness property in the number of states at each level

$$b(d/dx)\ [\ (e^x-1)\Delta(x)+C\ ]\ =\ 0. \tag{6.3b}$$

and the solution is

$$\Delta(x)\ =\ \frac{\sum_{i=1}^K\ p_i(x)\ e^{\lambda_i x}\ -\ C}{e^x-1}. \tag{6.3c}$$

where $p_i(x)$ is a polynomial of degree m_i-1. C is the central charge and $b(w)$ is the characteristic polynomial

$$b(w)\ =\ \prod\ (w-\lambda_i)^{m_i},\ \lambda_i \neq \lambda_j. \tag{6.4a}$$

Unitary representations were studied by [59],[86], [87] and the necessary and sufficient condition for a *unitary* representation is that C is a non-negative integer so the weight function that solves the differential equation (6.3b) is given by

$$\Delta(x) \;=\; \frac{\sum_{i=1}^{C}\; e^{\lambda_i x}-1}{e^x-1}, \quad \lambda_i = real \qquad (6.4b)$$

All unitary representations can be realized by tensoring C pairs of **bc** ghosts representations as displayed by eq-(6.4b). The generating function for the W_∞ case is $\tilde{\Delta}(x) = (d/dx)\Delta(x)$ and the central charge is $c = -2C$. The Verma module (a representation) is spanned by the states :

$$\mid v_\lambda > \;=\; W(z^{-n_1}D^{k_1})\; W(z^{-n_2}D^{k_2})\; ....\; W(z^{-n_m}D^{k_m})\; |\lambda > . \qquad (6.5)$$

The energy level is $\sum_{i=1}^{i=m}\; n_i$. Highest weight unitary representations for the W_∞ algebra obtained from field realizations with central charge $c = 2$ were constructed in [59],[86]. The weights associated with the highest weight state $|\lambda >$ will be obtained from the expansion in (6.2). In particular, the "energy" operator acting on $|\lambda >$ will be :

$$W(D)\; |\lambda > \;=\; \Delta_1\; |\lambda > . \qquad (6.6)$$

$L_o = -W(D)$ counts the energy level : $[L_o, W(z^nD^k)] = -n\; W(z^nD^k)$. As an example, the weight function $\Delta(x)$ corresponding to the free-field realization of W_∞ in terms of free fermions or **bc** ghosts [59] is given by

$$\Delta(x) \;=\; C\; \frac{e^{\lambda x}-1}{e^x-1} \;\Rightarrow\; \frac{\partial\Delta}{\partial\lambda} \;=\; C\; \frac{x\; e^{\lambda x}}{e^x-1}. \qquad (6.7)$$

which agrees with the expression in eq-(6.4b) when $\lambda_i = \lambda$ for all λ_i. The second term is the generating function for the Bernoulli polynomials

$$\frac{x\; e^{\lambda x}}{e^x-1} \;=\; 1 + (\lambda - \frac{1}{2})\, x + (\lambda^2 - \lambda + \frac{1}{6})\, \frac{x^2}{2!} + (\lambda^3 - \frac{3}{2}\lambda^2 + \frac{1}{2}\lambda)\, \frac{x^3}{3!} + ... \qquad (6.8)$$

Integrating (6.8) with respect to λ yields back :

$$\Delta(x) \;=\; C\; \frac{e^{\lambda x}-1}{e^x-1} \;=\; \sum_{k=0} \Delta_k\; \frac{x^k}{k!}. \qquad (6.9)$$

The first few weights (modulo a factor of C) can be read from integrating (6.8) w.r.t λ and matching the numerical values with the coefficients Δ_k in the r.h.s of (6.9)

$$\Delta_0 \;=\; \lambda,\; \Delta_1 \;=\; \frac{1}{2}(\lambda^2 - \lambda),\; \Delta_2 \;=\; \frac{1}{3}\lambda^3 \;-\frac{1}{2}\lambda^2 \;+\frac{1}{6}\lambda\; ..... \qquad (6.10)$$

The generating function for the W_∞ case is $\tilde{\Delta}(x) = \frac{d\Delta(x)}{dx} \Rightarrow \tilde{\Delta}^\lambda_k = \Delta^\lambda_{k+1}$. Now we proceed to relate the construction of quasi-finite highest weight unitary representations of W_∞ with the quantum states associated to the continuous $sl(\infty)$ Toda molecule.

6.2 Correspondence between Highest Weight representations and the quantum states of the continuous Toda molecule

We are going to construct the dimensional reduction of the $W_\infty \oplus \bar{W}_\infty$ algebra associated with the symmetries of the continuous Toda molecule. From the previous discussion we learnt that $\tilde{\Delta}_1^\lambda = \Delta_2^\lambda$ is the weight associated with the "energy" operator. In the ordinary string, W_2 algebra, the Hamiltonian is related to the Virasoro generator, $H = L_o + \bar{L}_o$ and states are built in by applying the ladder-like operators to the highest weight state, the "vacuum". In the $W_{1+\infty}, W_\infty$ case it is *not* longer true, as we shall see, that the Hamiltonian (to be given later) can be written exactly in terms of the zero modes w.r.t the $z, \bar{z}$ variables of the W_2 generator, once the realization of the W_∞ algebra is given in terms of the *dressed* continuous Toda field , $\Theta(z,\bar{z},t)$, given by Savaliev [83]. The chiral generators are

$$W_2^+ = \int^{t_o} dt_1 \int^{t_1} dt_2\, exp[-\Theta(z,\bar{z};t_1)]\frac{\partial}{\partial z} exp[\Theta(z,\bar{z};t_1) - \Theta(z,\bar{z};t_2)]\frac{\partial}{\partial z} exp[\Theta(z,\bar{z};t_2)]. \quad (6.11a)$$

$$W_n^+ = \int^{t_o} dt_1 \int^{t_1} dt_2 \,.....\int^{t_{n-1}} dt_n\, \mathcal{D}_+^{(0)}\, \mathcal{D}_+^{(1)} \,......\, \mathcal{D}_+^{(n-1)}\, exp[\Theta(z,\bar{z};t_n)]. \quad (6.11b)$$

with

$$\mathcal{D}_+^{(0)} = exp\,[-\Theta(z,\bar{z};t_1)]\frac{\partial}{\partial z};\ \mathcal{D}_+^{(m)} \equiv exp[\Theta(z,\bar{z};t_m) - \Theta(z,\bar{z};t_{m+1})\,]\,\frac{\partial}{\partial z},\quad m \geq 1. \quad (6.12)$$

The antichiral generators are obtained upon replacing $\partial/\partial z$ by $\partial/\partial \bar{z}$ in eqs-(6.11,6.12). Hence, the chiral generators have the form $W_{h,0}^+[\partial^2\rho/\partial z^2 .... \partial^h\rho/\partial z^h]$ where h are the conformal chiral weights. A similar expression for the anti-chiral generators $W_{0,\bar{h}}^-$ is obtained by replacing $\partial_z \rightarrow \partial_{\bar{z}}$. After performing a dimensional reduction from $3D \rightarrow 2D$, and by defining $r = z + \bar{z}$ one has

$$\tilde{W}_2(r,t_o) = \int^{t_o} dt_1 \int^{t_1} dt_2\, exp[-\rho(r,t_1)]\,\frac{\partial}{\partial r}\, exp\,[\rho(r,t_1) - \rho(r,t_2)]\,\frac{\partial}{\partial r}\, exp[\rho(r,t_2)]. \quad (6.13a)$$

And similar procedure applies to eqs-(6.11b, 6.12)

$$\tilde{W}_n = \int^{t_o} dt_1 \int^{t_1} dt_2 \,....\int^{t_{n-1}} dt_n\, \mathcal{D}^{(0)}\, \mathcal{D}^{(1)} \,......\, \mathcal{D}^{(n-1)}\, exp\,[\rho(r;t_n)]. \quad (6.13b)$$

with

$$\mathcal{D}^{(0)} \equiv exp\,[-\rho(r;t_1)]\frac{\partial}{\partial r}, \quad \mathcal{D}^{(m)} \equiv exp\,[\rho(r;t_m) - \rho(r;t_{m+1})]\,\frac{\partial}{\partial r}. \quad (6.13c)$$

When $\rho(r,t)$ is quantized, eqs-(6.13) involve the operator, $\hat{\rho}(r,t)$, acting on a suitable Hilbert space of states, $|\rho>$, and in order to evaluate (6.13) one needs to perform the highly complicated Operator Product Expansion among the operators $\hat{\rho}(r,t_1), \hat{\rho}(r,t_2), ...$ Since these are no longer free fields it is no longer trivial to compute the operator products of

$$\frac{\partial\rho}{\partial r}\, e^{\rho}, \quad e^{\rho(r,t_1)}\, e^{\rho(r,t_2)}\, ... \quad (6.14)$$

As we have discussed in **2** quantization deforms the classical w_∞ algebra into the W_∞ algebra. Since the w_∞ algebra has been effectively quantized the classical Poisson bracket algebra is retrieved by taking *single* contractions in the Operator Product Expansion of the operators $\hat{\rho}(r,t_1), \hat{\rho}(r,t_2), ...$ which appear in the continuous Toda field realizations of the quantum algebra $W_\infty \oplus \bar{W}_\infty$. Hence, the expectation value of the quantum $\hat{W}_2$ operator $< \rho|\, \hat{W}_2(\hat{\rho})\, |\rho >$ in the asymptotic limit $(r \to \infty)$, yields after the dimensional reduction the following results

$$\frac{\partial\rho}{\partial r} = \varphi, \quad \frac{\partial^2\rho}{\partial r^2} = \frac{\partial^2 e^{\rho}}{\partial t^2} \to 0,\ r \to \infty \quad (6.15a)$$

such that

$$lim_{r\to\infty} < \rho|\hat{W}_2|\rho > = \int_0^{2\pi} dt\ (\int_0^t dt'\ \varphi(t')\ )^2 \quad (6.15b)$$

after the normalization condition is chosen

$$< \rho'|\rho >= \delta(\rho' - \rho).\ < \rho|\rho >= 1 \quad (6.16)$$

The purpose becomes now to relate the states $|\rho >$ to the quasi-finite highest weight representations of $W_\infty \oplus \bar{W}_\infty$ generated from the highest weight states $|\lambda, \lambda^* >$. It is useful to recall the results from ordinary $2D$ conformal field theory. Given the holomorphic current generator of two-dimensional conformal transformations, $T(z) = W_2(z)$, the mode expansion is

$$W_2(z) = \sum_m W_2^m z^{-m-2} \Rightarrow W_2^m = \oint \frac{dz}{2\pi i} z^{m+2-1} W_2(z). \quad (6.17)$$

the closed integration contour encloses the origin. When the closed contour surrounds $z = \infty$, this requires performing the conformal map $z \to (1/z)$ and replacing

$$z \to (1/z),\ dz \to (-dz/z^2),\ W_2(z) \to (-1/z^2)^2 W_2(1/z) = W_2(z) + \frac{c}{12} S[z', z] \quad (6.18)$$

in the integrand. $S[z', z]$ is the Schwarzian derivative of $z' = 1/z$ w.r.t the z variable. There is also a one-to-one correspondence between local fields and states in the Hilbert space

$$|\phi > \quad \leftrightarrow \quad lim_{z,\bar{z}\rightarrow 0} \ \hat{\phi}(z,\bar{z}) \ |0,0> . \tag{6.19}$$

This is usually referred as the $| \ in >$ state. A conformal transformation $z \rightarrow 1/z; \bar{z} \rightarrow 1/\bar{z}$: defines the $< out \ |$ state at $z = \infty$

$$< out| \ = \ lim_{z,\bar{z}\rightarrow 0} \ < 0,0| \ \hat{\phi}(1/z, 1/\bar{z}) \ (-1/z^2)^h \ (-1/\bar{z}^2)^{\bar{h}}. \tag{6.20}$$

where $h, \bar{h}$ are the conformal weights of the field $\phi(z,\bar{z})$.

The analog of eqs-(6.19,6.20) is to consider the states parametrized by the functions $\varphi(t), d(t)$

$$|\rho >_{\varphi(t),d(t)} \ \equiv \ lim_{r\rightarrow\infty} \ |\rho(r,t)> \ \equiv \ |\rho(out)> .$$

$$|\rho >_{-\varphi(t),d(t)} \ \equiv \ lim_{r\rightarrow-\infty} \ |\rho(r,t)> \ \equiv \ |\rho(in)> . \tag{6.21}$$

since the continuous Toda equation is symmetric under $r \rightarrow -r$, then $\rho(-r,t)$ is also a solution and it's obtained from the general solution (5.20) by setting $\varphi \rightarrow -\varphi$ to ensure convergence at $r \rightarrow -\infty$. The state $|\rho(r,t)>$ is parametrized in terms of the functions $\varphi(t), d(t)$ resulting from the solutions (5.20) and for this reason one should always keep this in mind. The temporal evolution of the state $|\rho>$ is governed by the Hamiltonian, thus knowing the $|\rho(in)>$ state at the "time" $r = -\infty$, upon radial quantization, the Hamiltonian yields the "temporal" evolution to another value of r. What is required now is to establish the correspondence (a functor) between the representation space realized in terms of the continuous Toda field and that representation space (the Verma module) built from the highest weights $|\lambda>$

$$< \lambda|\tilde{W}(D)|\lambda> \ = \ \tilde{\Delta}_1 \equiv \Delta_2 \ \leftrightarrow \ < \rho|\hat{W}_2[\hat{\rho}(r,t)|\rho> . \tag{6.22}$$

The contour integral (6.17) means evaluating quantities for fixed "times", which in the language of the $z, \bar{z}$ coordinates, implies choosing circles of $fixed$ radius around the origin and integrating w.r.t the angular variable which is represented by t. Therefore, the conserved Noether charges (the Virsoro generators in the string case) are just the integrals of the conserved currents at fixed contour-radius (fixed "times"). The expression to evaluate in our case is the expectation value w.r.t the $| \ in>$ state, of the zero modes of the quantity

$$\hat{W}_2[f] \ = \ \int_0^{2\pi} dt' \ f(t')\hat{W}_2[\rho(r',t')]. \tag{6.24a}$$

when the real valued function, $f(t)$ can be expanded into a Fourier series as

$$f(t) = \sum_n a_n \ cos(nt). \tag{6.24b}$$

Rigorously speaking, when one writes $f(t)$ one means values at a given "time" r. Hence, by *zero* mode expectation value one means those w.r.t the angle variable t and *not* w.r.t the $z, \bar{z}$ variables associated with the $3D$ continuous Toda equation and whose dimensional reduction furnishes the $2D$ continuous Toda molecule. The zero-mode of the expectation value of eq-(6.24) w.r.t the $|in>$ state is given by

$$< \hat{W}_0^{(2)}[f] > \ = \ lim_{r \rightarrow -\infty} \ < \rho \mid [\int_0^{2\pi} dt \ a_0 \ \hat{W}_2[\hat{\rho}(r,t)] \ ] \ |\rho > . \quad (6.25a)$$

and a similar expression holds for the zero modes of the higher conformal spin generators

$$< \hat{W}_0^{(k)}[f] > \ = \ lim_{r \rightarrow -\infty} \ < \rho \mid [\int_0^{2\pi} dt \ a_0 \ \hat{W}_k[\hat{\rho}(r,t)] \ ] \ |\rho > . \quad (6.25b)$$

Due to the dimensional reduction of the algebra $W_\infty \oplus \bar{W}_\infty$, one must take a suitable real-valued linear combination of the weights of the chiral and anti-chiral algebras before we can impose relations with the expectation values of eqs-(6.25). Firstly, one imposes the conditions $(\Delta_k)^* \ = \ \bar{\Delta}_k, \ \bar{\lambda} \ = \ (\lambda)^*$ on all of the anti-chiral highest weights and afterwards one performs the linear combination of chiral and anti-chiral weights. For example, in the particular case of free Fermi fields or in the **bc** ghost system whose weights are explicitly given by eq-(6.10), one has a linear combination as follows

$$\Delta_0 \ + \ \bar{\Delta}_0 \ = \ (\lambda + \bar{\lambda}), \ \ \Delta_1 \ + \ \bar{\Delta}_1 = \ \frac{1}{2}[(\lambda^2 + \bar{\lambda}^2) - (\lambda + \bar{\lambda})]$$

$$\Delta_2 \ + \ \bar{\Delta}_2 \ = \ \frac{1}{3}(\lambda^3 + \bar{\lambda}^3) - \frac{1}{2}(\lambda^2 + \bar{\lambda}^2) + \frac{1}{6}(\lambda + \bar{\lambda}); .... \quad (6.26)$$

In general, the expression for the weights is more complicated than (6.26). One should notice that one is adding the weights as vectors in a Hilbert space and not the values of λ, λ^*. Going back to eq-(6.24), the n^{th} mode component associated with the function $f(t) = \sum a_n \ cos(nt)$ leads to

$$\hat{W}_n^{(2)}[C_r] \ = \ \int_0^{2\pi} dt \ a_n \ cos(nt) \ \hat{W}_2[\rho(r,t)]. \quad (6.27)$$

and similarly for the other generators

$$\hat{W}_n^{(s)}[C_r] \ = \ \int_0^{2\pi} dt' \ a_n \ cos(nt) \ \hat{W}_s[\rho(r,t)]. \quad (6.28)$$

where a radial quantization has been imposed such that C_r stands for a circle of fixed radius (a fixed "time") r, for all values of the "angles" t and is obtained by mapping the cylinder determined by the r, t variables, $-\infty \leq r \leq +\infty$ and $0 \leq t \leq 2\pi$ into the complex plane $Z = e^{-i(t+ir)}, \ \bar{Z} = e^{i(t-ir)}$. Notice that the variables $z, \bar{z}$ are *not* the same as $Z, \bar{Z}$ because $r = z + \bar{z}$.

With these findings at hand after recurring to eqs-(6.27, 6.28) and evaluating the expectation values associated with the zero modes given by eqs-(6.25) one arrives at

$$\frac{1}{2} \ (\ \Delta_k \ + \ \bar{\Delta}_k \ ) \ = \ lim_{\ r \rightarrow -\infty} \ < \rho | \ [\ \int_0^{2\pi} \ dt \ a_0 \ \hat{W}_s [\ \hat{\rho}(r,t)] \ ] \ | \rho > . \quad (6.29)$$

with $k = 1, 2, 3, 4......\infty$ and $s = k + 1 = 2, 3, 4, ...\infty$; the coefficient a_0 is $1/2\pi$. The weight function corresponding to quasi-finite highest weight unitary representations of W_∞ was given by (6.4b)

$$\Delta(x) \ = \ \frac{\sum_{i=1}^{C} \ e^{\lambda_i x} - 1}{e^x - 1} \ = \ \sum_{k=0} \ \Delta_k[\lambda_1, \lambda_2, ..., \lambda_C] \ \frac{x^k}{k!}, \quad \lambda_i = real, \quad \lambda_i \neq \lambda_j \quad (6.30a)$$

a similar expansion follows for the anti-chiral weights, and it leads to a polynomial expression in the parameters $\lambda_i, \bar{\lambda}_i$ as follows

$$\Delta_k \ [\lambda_1, \lambda_2, ..., \lambda_C] \ + \ \bar{\Delta}_k \ [\bar{\lambda}_1, \bar{\lambda}_2, ...\bar{\lambda}_C] \ =$$

$$\sum_{n=1}^{n=k+1} \ a_n^{(k)} \ (\ \lambda_1^n \ + \ \lambda_2^n \ + \ ...... \ + \ \lambda_C^n \ ) + \bar{a}_n^{(k)} \ (\ \bar{\lambda}_1^n \ + \ \bar{\lambda}_2^n \ + \ .... \ + \ \bar{\lambda}_C^n \ ). \quad (6.30b)$$

Since the weights in the l.h.s of eq-(6.29) due to eq-(6.30b) have an explicit polynomial dependence on the $\lambda_i, \bar{\lambda}_i$ parameters, the functions $\varphi(t), d(t)$ which parametrize the space of solutions of $\rho(r,t)$ in eq-(5.20) must also encode such an explicit polynomial dependence on $\lambda_i, \bar{\lambda}_i$. One may fix the function $d(t) = 1$ leaving the $\varphi_{\lambda_i, \bar{\lambda}_i}(t)$ to be a family of functions of the form

$$\varphi_{\{\lambda_i, \bar{\lambda}_i\}}(t) \ = \ \sum_l \ \varphi_l(t) \ (\ \lambda_1^l \ + \ \lambda_2^l \ ...... \ + \ \lambda_C^l \ ) + \bar{\varphi}_l(t) \ (\ \bar{\lambda}_1^l \ + \ \bar{\lambda}_2^l .... \ + \ \bar{\lambda}_C^l \ ) =$$

$$\sum_l \sum_m \ \varphi_l^m \ cos(mt) \ (\ \lambda_1^l \ + \ \lambda_2^l ...... \ + \ \lambda_C^l \ ) + \bar{\varphi}_l^m \ cos(mt) \ (\ \bar{\lambda}_1^l \ + \ \bar{\lambda}_2^l .... \ + \ \bar{\lambda}_C^l \ ). \quad (6.31)$$

The number of coefficients $a_n^{(k)}$ corresponding to $k = 0, 1, 2, 3, ...N - 1$ is $\sum_{k=0}^{k=N-1} \ (k+1) = (N+1)(N)/2$. The number of coefficients $\varphi_l^{(m)}$ corresponding to $m = 1, 2, ...N$ and $l = 1, 2, 3, ...N$ is N^2. If one sets $\varphi_l^{(m)}$ as the components of a *symmetric* $N \times N$ matrix the number of independent components is $(N)(N+1)/2$ which matches the number of coefficients $a_n^{(k)}$ corresponding to $k = 0, 1, 2, 3, ...N - 1$. In this fashion one has a precise match among the number of coefficients $a_n^{(k)}$ and $\varphi_l^{(m)}$ for all values of N.

To sum up, the generators $\hat{W}_2, \hat{W}_3, ...$ are realized in terms of $\rho(r,t)$ by the expressions in eqs-(6.13) which depend in general on two arbitrary functions $\varphi(t), d(t)$. We have fixed the function $d(t) = 1$. Thus, an explicit realization of the $\hat{W}_2, \hat{W}_3, ...$ generators in terms $\varphi(t), d(t)$ permits the evaluation of the integrals (6.29). The eqs-(6.13, 6.30, 6.31) are required to evaluate the expectation values in eqs-(6.29) for $k = 1, 2, 3, ...$ which establish the one-to-one correspondence among the coefficients φ_l^m, $\bar{\varphi}_l^m$ associated with the modes of the family of functions $\varphi_{\{\lambda_i, \bar{\lambda}_i\}}(t)$ (6.31) and the coefficients $a_n^{(k)}$, $\bar{a}_n^{(k)}$ corresponding to the polynomial expressions of the conformal weights $\Delta_k[\{\lambda_i\}] + \bar{\Delta}_k[\{\bar{\lambda}_i\}]$ given by eq-(6.30b). Since the coefficients $a_n^{(k)}, \bar{a}_n^{(k)}$ are *not* arbitrary this implies that the modes of the family of functions $\varphi_{\{\lambda_i, \bar{\lambda}_i\}}(t)$ cannot be arbitrary either, meaning that the family of functions $\varphi_{\{\lambda_i, \bar{\lambda}_i\}}(t)$ cannot be arbitrary in the same vein that the energy levels of a harmonic oscillator or the hydrogen atom are not arbitrary either. This *is* a consequence of *quantization*.

6.3 Casimir invariant Wave Equations for the W_∞ algebra and Energy spectrum of the continuous Toda molecule

If the classical $2D$ Toda molecule is indeed an exact integrable system it must posess an infinite number of functionally independent classical integrals of motion whose Poisson Brackets are zero; i.e the conserved charges are in involution. At the Quantum level one should have an infinite number of mutually commuting operator charges obtained from the classical-quantum correspondence

$$I_n \;\rightarrow\; Q_n, \quad \{I_n,\, I_m\} \;=\; 0 \rightarrow \; \frac{1}{i\hbar}[Q_n, Q_m] = 0. \tag{6.32}$$

The quantum integrals of motion are conserved so their expectation values in the $|in>$ state given by $|\ \rho(r = -\infty, t) >$ do not depend on "time" r. The expectation values can be computed in terms of the asymptotic states; i.e. in terms of the values of the function $\varphi_{\lambda,\bar{\lambda}}(t)$ when $d(t) = 1$. The Casimirs in the classical case are [4]

$$I_n \;=\; \int_0^{2\pi} dt\ (\ \int_0^t dt'\ \varphi(t')\ )^n. \tag{6.33}$$

The explicit expression relating the infinite number of involutive conserved *quantum* charges in terms of the generators of the chiral W_∞ algebra has been given by [90]

$$\hat{Q}_2 \;=\; \oint \hat{W}_2\ dz, \quad \hat{Q}_3 \;=\; \oint \hat{W}_3\ dz, \quad \hat{Q}_4 \;=\; \oint (\ \hat{W}_4 \;-\; \hat{W}_2\ \hat{W}_2\ )\ dz.$$

$$\hat{Q}_5 \;=\; \oint (\ \hat{W}_5\ - 6\ \hat{W}_2\ \hat{W}_3\ )\ dz$$

$$\hat{Q}_6 \;=\; \oint (\ \hat{W}_6\ - 12\ \hat{W}_2\ \hat{W}_4\ - 12\ \hat{W}_3\ \hat{W}_3\ + 8\ \hat{W}_2\ \hat{W}_2\ \hat{W}_2\ )\ dz,\ ...... \tag{6.34}$$

Similar expressions hold for the antichiral algebra. In the dimensionall-reduction of the $W_\infty \oplus \bar{W}_\infty$ algebra, these expressions hold as well where now the integral expressions are those given in eqs-(6.25) by replacing the contour integrals in the complex plane $z, \bar{z}$ by integrals w.r.t. the "angular" t variable. Upon evaluation of the expectation values in the $|in>$ state $| \rho(r = -\infty, t) >$ one has expressions of the type :

$$E \ = \ I_2 = \ lim_{\ r \rightarrow -\infty} \ \ < \rho| \int \ dt \ \hat{W}_2[\rho(r,t)] \ |\rho > . \qquad (6.35a)$$

$$I_3 \ = \ lim_{\ r \rightarrow -\infty} \ \ < \rho| \int \ dt \ \hat{W}_3[\rho(r,t)] \ |\rho > . \qquad (6.35b)$$

and similar procedure with the other expectation values

$$I_4 \ = \ < \int dt \ (\hat{W}_4 - \hat{W}_2 \ \hat{W}_2) \ >, \ \ I_5 \ = \ < \int dt \ (\ \hat{W}_5 - 6 \ \hat{W}_2 \ \hat{W}_3) \ >, \ ...... \qquad (6.35c)$$

The Hamiltonian associated with the $2D$ continuous Toda field theory is [83]

$$\mathcal{H} \ = \ \int dt \ [\ -\frac{1}{2} \ \beta^2 \ (\frac{\partial^2 x}{\partial r \ \partial t})^2 \ + \ (\frac{m^2}{\beta^2}) \ exp \ (\ \beta \ \partial^2 x/\partial t^2 \ ) \ ], \quad \rho \equiv \frac{\partial^2 x}{\partial t^2} \quad (6.36)$$

The wave functional is defined by $\Psi[\rho(r,t)] \ \equiv \ < \rho(r,t) \mid \Psi >$ where the state $|\rho(r,t) >$ has an explicit dependence on the two-parameter family of functions $\varphi_{\lambda, \bar{\lambda}}(t)$ as stated earlier. From eqs-(5.19,6.36) for unit coupling $\beta = 1$ one learns that $(\partial^2 x / \partial r \partial t)$ is the momentum-conjugate variable π_ρ corresponding to the continuous Toda molecule field $\rho \ \equiv \ (\partial^2 x / \partial t^2)$; the time variable is r and the angle variable is t. Without loss of generality we will set $m = \beta = 1$ in (6.36). Therefore one has the momentum-operator correspondence given by

$$\frac{\partial^2 x}{\partial r \ \partial t} \ = \ \pi_\rho \ \Rightarrow \ \frac{\partial^2 x}{\partial r \ \partial t} \ \leftrightarrow \ - \ i\hbar \ \frac{\delta}{\delta \rho(r,t)}. \qquad (6.37)$$

there are two related functional wave equations of the form $\mathcal{H}\Psi = E\Psi$ associated with the Hamiltonian (6.36)

$$\int_0^{2\pi} \ dt \ [\ exp \ \{\rho(r,t)\} \ - \ (-i\hbar \ \frac{\delta}{\delta \rho(r,t)} \ )^2 \ - \ E \ ] \ \Psi[\ \rho(r,t) \ ] \ = \ 0. \quad (6.38a)$$

and

$$\{ \ \int_0^t \ dt' \ [\ exp \ \{\rho(r,t')\} \ - \ (-i\hbar \ \frac{\delta}{\delta \rho(r,t')} \ )^2 \ ] \ \} \ \Psi[\ \rho(r,t) \ ] \ = \ E \ \Psi[\ \rho(r,t) \ ]. \qquad (6.38b)$$

when $m = \beta = 1$. One must *not* interpret Ψ as a probability amplitude but as a field operator linked to a continuous Toda field in a given quantum state $|\rho >$

parametrized by the family of functions $\varphi_{\{\lambda_i,\bar{\lambda}_i\}}(t)$. The functional differential equation (6.38) is quite complicated. A naive zeroth-order simplification will be given shortly. This is because the Ψ can have the form $\Psi = \Psi[\rho, \rho_{t'}, \rho_{t't'}, ...]$. The equation in the momentum representation does not have that complexity but it has an exponential functional differential operator. Evenfurther, the Ψ is a non-local object. In string field theory the string field is a multilocal object that depends on all of the infinite points along the string.

One can expand $|\Psi >$ in an infinite dimensional basis spanned by the Verma module associated with the highest weight states $|\lambda >$ ($|\bar{\lambda} >$). This is very reminiscent of the string-field $\Phi[X^\mu(\sigma)] \ = \ < X^\mu| \ |\Phi >$ where the state $|\Phi >$ is comprised of an infinite array of point fields associated with the center of mass $x^\mu_o(\sigma)$ coordinates of the open-string

$$|\Phi > \ = \ \phi(x^\mu_o)|0> \ + \ A_\mu(x^\mu_o) \ a_1^{\dagger\mu}|0> \ + \ g_{\mu\nu}(x^\mu_o) \ a_1^{\dagger\mu} \ a_1^{\dagger\nu}|0> + ... \quad (6.39)$$

where the first field is the tachyon, the second is the massless Maxwell photon field, the third is the massive graviton... The oscillators play the role of ladder-like operators acting on the "vacuum"$|0>$ in the same manner that the Verma module is generated from the highest weight states $|\lambda>$ by successive applications of a series of $W(z^{-n}D^k)$ operators acting on $|\lambda >$. The state $|\rho(r,t) >$ is the relative of the string state $|X^\mu(\sigma_1,\sigma_2) >$, whereas $|\Psi >$ is the relative of the string field state $|\Phi >$. This picture is also consistent with the fact that the Liouville field in non-critical strings backgrounds can be viewed as an extra string coordinate.

Now we can write the *zeroth* order approximation of the functional wave equation (6.38) in the Schroedinger-like fashion by setting $\hbar = 1$ as follows

$$(\ \partial_y^2 \ + \ e^y \ ) \ \Psi(y) \ = \ E \ \Psi(y). \quad (6.40)$$

A change of variables $x \ = \ 2 \ e^{y/2}$ converts (6.40) into Bessel's equation

$$(\ x^2 \ \partial_x^2 \ + \ x \ \partial_x \ + \ x^2 \ - 4 \ E \ ) \ \Phi(x) = 0. \quad (6.41)$$

where $\Psi(y(x)) = \Phi(x)$ and whose solution is

$$\Phi(x) \ = \ c_1 \ \mathcal{J}_\nu(x) \ + \ c_2 \ \mathcal{J}_{-\nu}(x). \quad (6.42)$$

where $\nu \equiv 2\sqrt{E}$ and c_1, c_2 constants. The wavefunctional solution of (6.38) in the *zeroth* order approximation is then given by

$$\Psi[\ \rho(r,t) \ ] \ = \ c_1 \ \mathcal{J}_\nu(\ 2e^{\rho(r,t)/2} \ ) \ + \ c_2 \ \mathcal{J}_{-\nu}(\ 2e^{\rho(r,t)/2}). \quad (6.43)$$

This completes section **6** where the main results are displayed by eqs-(6.29,6.30,6.31) and eqs-(6.38,6.42,6.43).

7 Concluding Remarks

To recapitulate, sections **1, 3, 5** contain known material necessary to derive the results of sections **2, 4, 6** which are new. To finalize we will summarize some important points related to W_∞ strings and membranes in a wide variety of physical models .

1- The 3-dim world volume of the membrane as the boundary of a four dimensional anti-deSitter spacetime, AdS_4, [100].

2- The membrane as a coherent state of an infinite number of strings . This is reminiscent of the Sine-Gordon soliton being the fundamental fermion of the massive Thirring model, a quantum lump. The lowest fermion-antifermion bound state (soliton-antisoliton doublet) is the fundamental meson of Sine-Gordon theory. Higher level states are built from excitations of the former in the same way that infinitely many massless states can be built from just two singletons.

3- The membrane as a Matrix [104] model. Uncompactified $D = 11$ M-theory was found to have an equivalence with the $N = \infty$ limit of supersymmetric matrix quantum mechanics describing D 0 branes. Matrix models of $2D$ gravity and Toda theory have been discussed by Gerasimov et al [105] and by Kharchev et al [103].

4- W_∞ symmetry in the Edge states of Quantum Hall Fluids [95], the set of unitary highest weight irreps of $W_{1+\infty}$ have been used to algebraically characterize the low energy edge-excitations of the incompressible (area preserving) Quantum Hall Fluids.

5-Perhaps the most relevant physical applications of the membrane quantization program will be in the behaviour of black hole horizons [92]. The connection between black hole physics and non-abelian Toda theory has been studied in [93]. W gravity was formulated as chiral embeddings of a Riemann surface into CP^n. Toda theory plays a crucial role as well [94].

6- In understanding the string vacua : The ordinary bosonic string has been found to be a special vacua of the $N = 1$ superstring [96]. It appears that there is a whole hierarchy of string theories : w_2 string is a particular vacua of the w_3 string and so forth......If this is indeed correct one has then that the (super) membrane, viewed as noncritical W_∞ string theory, is, in this sense, the universal space of string theory. The fact, advocated by many, that a Higgs symmetry-breakdown-mechanism of the infinite number of massless states of the membrane generates the infinite massive string spectrum fits within this description.

7- Membranes as gauge theories of area preserving diffs [97] and as composite antisymmetric tensor field theories [98],[99]. And many more. We hope that the essential role that Self Dual $SU(\infty)$ Yang-Mills theory has played in the origins of the membrane-Toda theory, will shed more light into the origin of duality in string theory and the full membrane spectrum. In [99] the analogs of S, T duality were built in from the very start. For a review of duality in string theory [50] and the status of string solitons see [101]. As of now we must have all unitary irreps of W_∞ to construct no-ghost theorems and be able to have the OPE of the Toda exponentials to fully exploit the results of **6**.

A lot remains ahead, we hope that the contents of this work based on gauge theories of $Diffs$ (area-preserving diffs) may shed some light into the geometrical foundations of string theory. Since higher-spin W_∞ symmetries are very relevant in the study of $2D$ W_∞ Gravity, the Quantum Hall effect, large N QCD, strings, membranes, topological QFT, gravitational instantons, Noncommutative $4D$ Gravity, Modular Matrix Models and the Monster group,.... it is warranted to explore further the interplay among all these theories.

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