On Generalized Yang-Mills Theories and Extensions of the Standard Model in Clifford (Tensorial) Spaces

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Abstract

We construct the Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories and the Standard Model that allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 Tev regime. We proceed with a detailed discussion of the unique $D_4 - D_5 - E_6 - E_7 - E_8$ model of Smith based on the underlying Clifford algebraic structures in $D = 8$, and which furnishes all the properties of the Standard Model and Gravity in four-dimensions, at low energies. A generalization and extension of Smith's model to the full Clifford-space is presented when we write explictly all the terms of the extended Clifford-space Lagrangian. We conclude by explaining the relevance of multiple-foldings of $D = 8$ dimensions related to the modulo 8 periodicity of the real Cliford algebras and display the interplay among Clifford, Division, Jordan and Exceptional algebras, within the context of $D = 26, 27, 28$ dimensions, corresponding to bosonic string, M and F theory, respectively, advanced earlier by Smith. To finalize we describe explicitly how the $E_8 \times E_8$ Yang-Mills theory can be obtained from a Gauge Theory based on the Clifford (16) group.

1 Introduction : Novel Physics in Tensorial Spaces

1.1 The Extended Relativity Theory in Clifford Spaces

The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [2]. For a comprehensive review we refer to [1] . A natural generalization of the notion of a space-time interval in Minkowski space to C-space is :

$$dX^2 = d\Omega^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + ... \qquad (1-1)$$

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The Clifford valued poly-vector:

$$X = X^M E_M = \Omega \ \mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... x^{\mu_1 \mu_2 .... \mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} .... \wedge \gamma_{\mu_D}. \qquad (1-2)$$

denotes the position of a polyparticle in a manifold, called Clifford space or C-space. The series of terms in (2) terminates at a *finite* value depending on the dimension D. A Clifford algebra $Cl(r,q)$ with $r+q=D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime :

$$1/2\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}. \qquad (1-3)$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$. The multi-graded basis elements E_M of the Clifford-valued poly-vectors are

$$E_M \ \equiv \ \mathbf{1}, \quad \gamma^\mu, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \wedge ..... \wedge \gamma^{\mu_D}. \quad (1-4)$$

It is convenient to order the collective M indices as $\mu_1 < \mu_2 < \mu_3 < ...... < \mu_D$.

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x_{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x_{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetimr background.

This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, ...D-2$. The $p=0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}$.... have been *coined* in the string-brane literature [56] as the *holographic* areas, volumes, ...projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes.

The classification of Clifford algebras $Cl(r,q)$ in $D = r+q$ dimensions (modulo 8) for different values of the spacetime signature r, q is discussed, for example, in the book of Porteous [57]. All Clifford algebras can be understood in terms of $CL(8)$ and the $CL(k)$ for k less than 8 due to the modulo 8 Periodicity theorem

$$CL(n) = CL(8) \times Cl(n-8)$$

. $Cl(r,q)$ is a matrix algebra for even $n = r+q$ or the sum of two matrix algebras for odd $n = r+q$. Depending on the signature, the matrix algebras may be real, complex, or quaternionic. For furher details we refer to [57].

If we take the differential dX and compute the scalar product among two polyvectors $< dX^\dagger dX >_{scalar}$ [8] , [9] , [58] we obtain the C-space extension of the particles proper time in Minkowski space. The symbol X^+ denotes the *reversion* operation and involves *reversing* the order of all the basis γ^μ elements in the expansion of X . It is the analog of the transpose (Hermitian) conjugation : $(\gamma^\mu \wedge \gamma^\nu)^\dagger = \gamma^\nu \wedge \gamma^\mu$, etc... Therefore, the inner product can be rewritten as the scalar part of the geometric product as $< X^\dagger X >_{scalar}$. The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$< X'^\dagger X' >_{scalar} = < (R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1} >_{scalar} =$$

$$< R X^\dagger X R^{-1} >_{scalar} = < X^\dagger X >_{scalar} =$$

$$(\Omega)^2 + \Lambda^{2D-2}(x_\mu x^\mu) + \Lambda^{2D-4}(x_{\mu\nu} x^{\mu\nu}) + ... + (x_{\mu_1\mu_2.....\mu_D})(x^{\mu_1\mu_2.....\mu_D}) \qquad (1-5)$$

we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to match units. The Planck scale can be set to unity for convenience.

This condition $R^\dagger = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R in eq-(1-5) because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p})^\dagger = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} ..... \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p} \quad (1-6)$$

Hence only those terms that *change* sign (under the reversal operation) are permitted in the exponential defining $R = exp[\theta^A E_A]$. For example, in $D = 4$, in order to satisfy the condition $R^\dagger = R^{-1}$, one must have from the behavior under the reversal operation expressed in eq-(1-6) that :

$$R = exp\ [\theta^{\mu_1\mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} + \theta^{\mu_1\mu_2\mu_3} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}]. \qquad (1-7)$$

such that

$$R^\dagger = exp\ [\theta^{\mu_1\mu_2} (\gamma_{\mu_1} \wedge \gamma_{\mu_2})^\dagger + \theta^{\mu_1\mu_2\mu_3} (\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3})^\dagger] =$$
$$exp\ [-\theta^{\mu_1\mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} - \theta^{\mu_1\mu_2\mu_3} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}] = R^{-1}. \qquad (1-8)$$

These transformations are the analog of Lorentz transformations in C-spaces which transform a poly-vector X into another poly-vector X' given by $X' = R X R^{-1}$. The theta parameters $\theta^{\mu_1\mu_2}, \theta^{\mu_1\mu_2\mu_3}$ are the C-space version of the Lorentz rotations/boosts parameters. The ordinary Lorentz rotation/boosts involves only the $\theta^{\mu_1\mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2}$ terms, because the Lorentz algebra generator can be represented as $\mathcal{M}^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$. The $\theta^{\mu_1\mu_2\mu_3} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$ are the C-space corrections to the ordinary Lorentz transformations when $D = 4$.

The above transformations are *active transformations* since the transformed Clifford number X' (polyvector) is different from the "original" Clifford number X. Considering the transformations of components we have $X' = X'^M E_M = L^M{}_N X^N E_M = R X R^{-1}$, from which we can deduce that the basis poly-vectors transform as $L^M{}_N E_M = R E_N R^{-1}$ so that

$$L^M{}_N = \langle E^M\ R E_N R^{-1} \rangle_{scalar} \equiv < E^M E'_N >_{scalar} \qquad (1-9)$$

For example, in $D = 4$ an ordinary boost with parameter $\theta^t_{x^2}$ along the x^2 direction is tantamount of a "rotation" with an imaginary angle along the $x^1 - x^2$ plane where x^1 denotes the time coordinate and x^2, x^3, x^4 are the spatial coordinates. In C-space one must have as well a "rotation" along the $x^1 - x^{12}$ directions with generalized boost parameter $\theta^t_{12} = \theta^1_{12}$. Hence one has the generalized C-space transformations

$$(t)' = L^t_M(\theta^{t1};\theta^{t12})(X^M) = L^t_t\ t\ + L^t_x\ x\ + L^t_{12}\ x^{12}. \quad (1-10a)$$

$$(x)' = L^x_M(\theta^{t1};\theta^{t12})(X^M) = L^x_t\ t\ + L^x_x\ x\ + L^x_{12}\ x^{12}. \quad (1-10b)$$

$$(x^{12})' = L^{x^{12}}_M(\theta^{t1};\theta^{t12})(X^M) = L^{x^{12}}_t\ t\ + L^{x^{12}}_x\ x\ + L^{x^{12}}_{12}\ x^{12}. \quad (1-10c)$$

notice the presence of the *extra* terms containing the area coordinates x^{12} in the transformations for the t, x variables, which are not present in the standard Lorentz transformations. Also, there is an extra dependence on the boost parameter $\theta^t_{12} = \theta^1_{12}$ in the generalized Lorentz matrices L^M_N. In the more general case, when there are more non-vanishing *theta* parameters , the indices M of the X^M coordinates must be *restricted* to those directions in C-space which involve the t, x^1, x^{12}, x^{123}..... directions as required by the C-space poly-particle dynamics.

The C-space invariant proper time associated with a polyparticle motion is then :

$$< dX^\dagger dX >_{scalar} = d\Sigma^2 = (d\Omega)^2 + \Lambda^{2D-2}dx_\mu dx^\mu + \Lambda^{2D-4}dx_{\mu\nu}dx^{\mu\nu} + .. \quad (1-11)$$

Here we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops,..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkowski space interval:

$$ds^2 = c^2dt^2 - dx_i dx^i. \quad (1-12)$$

A similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [2]

Another possibility is to *complexify* the C-space polyvector valued coordinates $=$ $Z = Z^A E_A = X^A E_A + iY^A E_A$ and the boosts/rotation parameters θ allowing the unitarity condition $\bar{U}^\dagger = U^{-1}$ to hold in the generalized Clifford unitary transformations $Z' = UZU^\dagger$ associated with the complexified polyvector $Z = Z^A E_A$ such that the interval

$$<\ d\bar{Z}^\dagger\ dZ\ >_s\ =\ d\bar{\Omega}d\Omega + d\bar{z}^\mu dz_\mu + d\bar{z}^{\mu\nu}dz_{\mu\nu} + d\bar{z}^{\mu\nu\rho}dz_{\mu\nu\rho} + ... \quad (1-13)$$

remains invariant (upon setting the Planck scale $\Lambda = 1$).

The unitary condition $\bar{U}^\dagger = U^{-1}$ under the *combined* reversal and complex-conjugate operation will constrain the form of the complexified boosts/rotation parameters θ^A appearing in the rotor : $U = exp[\ \theta^A E_A\ ]$. The theta parameters θ^A are either purely real or purely imaginary depending if the reversal ${E_A}^\dagger = \pm E_A$, to ensure that an overall *change* of sign occurs in the terms $\theta^A E_A$ inside the exponential defining U so that $\bar{U}^\dagger = U^{-1}$ holds and the norm $< \bar{Z}^\dagger Z >_s$ remains invariant under the analog of unitary transformations in *complexified* C-spaces. These techniques are not very different from Penrose Twistor spaces. As far as we know a Clifford-Twistor space construction of C-spaces has not been performed so far.

Another alternative is to define the polyrotations by $R = exp\ (\Theta^{AB}[E_A, E_B])$ where the commutator $[E_A, E_B] = F_{ABC}E_C$ is the C-space analog of the $i[\gamma_\mu, \gamma_\nu]$ commutator which is the generator of the Lorentz algebra, and the theta parameters Θ^{AB} are the C-space analogs of the rotation/boots parameters $\theta^{\mu\nu}$. The diverse parameters Θ^{AB} are purely real or purely imaginary depending whether the reversal $[E_A, E_B]^\dagger = \pm[E_A, E_B]$ to ensure that $R^\dagger = R^{-1}$ so that the scalar part $< X^\dagger X >_s$ remains invariant under the transformations $X' = RXR^{-1}$. This last alternative seems to be more physical because a poly-rotation should map the E_A direction into the E_B direction in C-spaces, hence the meaning of the generator $[E_A, E_B]$ which extends the notion of the $[\gamma_\mu, \gamma_\nu]$ Lorentz generator. The introduction of gravity in curved C-spaces involves area, volume, hypervolume "metrics" and leads to Higher Derivative Gravity with Torsion. We refer to the review [1] and [2], [4], [7] for further details about the Extended Relativity Theory in Clifford spaces.

1.2 Higher Rank Tensor Gauge Symmetries and Higher Spin Theories

Having discussed briefly the Extended Relativity principle in Clifford spaces (antisymmetric tensorial backgrounds) where the analog of "photons" corresponds to *tensionless* p-branes [2] , [1] , [10] we may proceed with the study of higher-rank tensor gauge symmetries and higher spin theories within a completely different context and perspective.

An extension of the algebra of Abelian gauge transformations found in the study of *tensionles* strings to the Nonabelian case was advanced recently by Savvidy [11]. Yang-Mills theory becomes a member of a larger family comprised of tensor-gauge bosons of arbitrary large number of integer spins $s = 1, 2, 3, ...\infty$. It leads to a natural inclusion of the Standard Model into a larger theory in which the vector gauge bosons and fermions are part of a low-spin subgroup of an infinite enlarged family of tensor-gauge bosons and spinor-tensorial particles with arbitrary higher (half) integer spins. Although there is no experimental evidence of the existence of such tensor-gauge bosons and spinor-tensorial particles at the energy of Gev, string theory predicts the existence of fundamental particles of arbitrary large spins and masses of the order of the Planck mass where the multiplicity of the particles grows exponentially. On the other hand, the number of particles in the tensionless strings with a perimeter action has linear growth.

Formulations of conformal Higher Spin theories [12] based on twistor-particle dynamics in *tensorial* spaces initiated by Fronsdal [13] , have captured a lot of interest recently. Fronsdal conjectured that four-dim conformal higher spin field theory can be realized as an ordinary field theory on a ten-dim tensorial manifold parametrized by the coordinates $x^{\alpha\beta} = \frac{1}{2}x^\mu\gamma_\mu^{\alpha\beta} + \frac{1}{4}y^{\mu\nu}\gamma_{[\mu\nu]}^{\alpha\beta}$, where x^μ are associated with the four coordinates of conventional $4D$ spacetime and $y^{\mu\nu} = -y^{\nu\mu}$ describe six spinning degrees of freedom. An infinite tower of fields of increasing spin is obtained rather than an infinite tower of massive states as in the conventional Kaluza-Klein mechanism. In $D = 3, 4, 6, 10$ dimensions the conformal higher spin fields constitute the quantum spectrum of a twistor-like particle propagating in tensorial spaces of corresponding dimensions [12]. One can notice that a string propagating in the latter dimensions, has for transverse degrees of freedom $D-2 = 1, 2, 4, 8$ which precisely match the degrees of freedom of the real, complex, quaternion and octonion normed-division algebras .

The Higher spin theories literature is very vast , see [64], [65], [66] , [14], [64] and references therein. Consistent interactions of massless higher spin theories with gravity are possible in Anti de Sitter backgrounds provided the value of the spin is arbitrary large $s = 2, 3, 4, ...\infty$. Interactions of massive bosonic higher spins in D-dim have recently been reviewed by [14] . Roughly speaking, higher-spin theories bear many similiarties with string-field theory and W_∞ strings. Clifford Spaces are more $fundamental$ than these tensorial spaces (have a richer structure) because they require polyvector coordinates (antisymmetric tensors) of variable rank (greater than two) until the rank saturates the value of the spacetime dimension.

Higher spin symmetries of the curved target spacetime backgrounds where W_∞ strings propagate, in contrast to the higher conformal-spin symmetry of the two-dim world sheet of strings, have been investigated thoroughly by [30]. For example, higher spin algebras based on noncommutative star products in Anti de Sitter space haven been instrumental to construct higher spin massless gauge theories in AdS backgrounds . Vasiliev's construction of higher spin gauge theories and their couplings to higher spin matter currents on AdS spaces can be attained by introducing a suitable noncommutative but associative Vasiliev star product on an auxiliary (commuting) Grassmaniann even " fermionic phase space " whose deformation parameter is the inverse length scale characterizing the size of AdS_4's throat $\lambda = r^{-1}$. The Vasiliev star product encoding the nonlinear and nonlocal higher spin fields dynamics is defined taking advantage of the local isomorphism between $so(3, 2) \sim sp(4, R)$. It has the same form as the Baker integral representation of the star product. We should emphasize that one must *not* confuse Vasiliev's defomation of the $SO(3, 2)$ algebra using the AdS throat-size as deformation parameter, with the Moyal star products in phase spaces whose the deformation parameter is the Planck constant $\hbar$. Calixto has recently studied higher-dim extensions of W_∞ symmetries based on higher-spin $U(2, 2)$ fields in AdS spaces that are very relevant to radiation phenomena [34].

This in conjunction with the fact that Anti de Sitter spaces are required in Vasiliev's construction, may be very relevant in understanding more features about the AdS/CFT duality conjecture. W_∞ algebras were essential to identify the missing states in the AdS/CFT correspondence [19]. Higher Derivative Gravity is also very relevant in the AdS/CFT correspondence [20]. These higher spin algebras have been instrumental lately in [21] to construct $N = 8$ Higher-Spin Supergravity in AdS_4 which has been conjectured to be the true field theory limit of M theory on $AdS_4 \times S^7$. W_∞ symmetries that are higher conformal spin extensions of the Zamolodchikov W_3 algebra [31] have been studied extensively by numerous authors , for example, by [28], [29], [35], [36], [32], [37], [38], [22], [23], [24], [39], [40], [42], [43]. In particular, W_∞ symmetries appear in the physics of membranes as well because *noncritical* W_∞ strings [41] (with ghosts and Liouville sectors) behave like membranes ($3D$ theories) in their critical $D = 27$ ($D = 11$) dimensions [25] , [27].

Using a BRST analysis, it was shown [27] that a nilpotent BRST charge operator associated with the noncritical W_∞ superstring can be constructed by adjoining a $q = N + 1$ unitary superconformal model of the super W_N algebra to a critical W_∞ superstring spectrum in the $N = \infty$ limit. Therefore, we have an anomaly-free *noncritical* W_∞ superstring in $D = 11$. Similar BRST analysis followed for the bosonic *noncritical* W_∞

string and we found that $D = 27$ was the required critical dimension of the target spacetime. $D = 27$ is the dimension of the alledgedly anomaly-free bosonic membrane as was shown by Marquard, Scholl and Kaiser.

Hence, the massless spectrum sector of membranes living in the three-dim boundary of AdS_4 bears a relationship to the massless spectrum of (non-critical) W_∞ strings. The latter are effectively $3D$ theories living in the three-dim conformal boundary of AdS_4. Consequently, noncritical W_∞ strings in $AdS_4 \times S^7$ backgrounds are the sought-after higher conformal-spin gauge theories associated with the three-dimensional conformal group $SO(3,2)$ of the boundary of AdS_4 and which have a one-to-one correspondence to Vasiliev's higher-spin massless gauge field theories in AdS_4 spaces.

4-dim Gauge theories based on gauging the Virasoro algebra (diffs of a circle) have been constructed by [16] and 4-dim gauge theories based on the w_∞ (area-preserving diffs of the plane) algebras have been constructed by [17] using the Feigin-Fuks-Kaplansky representation of w_∞ algebras. Higgs matter fields in the adjoint representation were introduced also with the typical quartic potential terms which generated an infinite tower of massive spin 2 fields (massive higher spin fields in the case of w_∞ gauge theories) after an sponteaneous symmetry breaking. The action of Zhao [17] based on the w_∞ symmetry constitutes a first step to build higher spins extensions of the Standard Model . It has been speculated by many authors that the tower of massive higher spin particles, after a symmetry breaking via the Higgs mechanism of the massless states of W_∞ strings, furnish the infinite tower of massive string states of increasing spin (Regge trajectories).

Having presented this introduction on the extended Relativity in Clifford (tensorial) spaces and higher-spin theories, W_∞ algebras, W_∞ strings,....in the next sections we will describe the Clifford-space (tensorial spaces) generalizations of Yang-Mills theories and the Standard Model that will allow us to predict the existence of new particles and tensor-gauge fields of higher spin beyond the 10 Tev regime (related to the observed value of the vacuum energy density, cosmological constant). We conclude with a detailed outline of the $D_4 - D_5 - E_6 - E_7 - E_8$ model of Smith [50] based on the underlying Clifford algebraic structure in $D = 8$ and which furnishes all the properties of the Standard Model and Gravity in four-dim at low energies. We finalize by presenting the interplay among Clifford, Division, Jordan and Exceptional algebras within the context of $D = 26, 27, 28$ dimensions corresponding to bosonic string, M and F theory, respectively and explain how the $E_8 \times E_8$ Yang-Mills theory can be embedded into a $Cl(16)$ Gauge Theory.

2 Clifford-Space Generalized Yang-Mills Theories

2.1 Clifford-space Extension of Maxwell Electrodynamics

Finally, in this section we will review and complement the proposal of ref.[3], [1] to generalize Maxwell Electrodynamics to C-spaces, namely, construct the Clifford algebra-valued extension of the Abelian field strength $F = dA$ associated with ordinary vectors A_μ. Using Clifford algebraic methods we shall describe how to generalize Maxwell's theory of Electrodynamics asociated with ordinary point-charges to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank, not unlike the couplings of p-branes to antisymmetric tensor fields.

Based on the standard definition of the Abelian field strength $F = dA$ we shall use the same definition in terms of polyvector-valued quantities and differential operators in C-space

$$A = A_N E^N = \phi \underline{1} + A_\mu \gamma^\mu + A_{\mu\nu} \gamma^\mu \wedge \gamma^\nu + ... \qquad (2-1)$$

Once again, in order to match units in the expansion (2-1), it requires the introduction of suitable powers of a length scale parameter L that maybe equal or *not* to the Planck scale and which is conveniently set to unity. We shall study in the next sections the very natural possibility of introducing an energy scale parameter $E = 1/L$ (in units of $\hbar = c = 1$) of the order of 10 Tev and explain the origins of such energy scale .

In $D = 4$, for example, the first component in the expansion ϕ is a scalar field that has the same units as the A_μ field which is the standard Maxwell field. The units of ϕ, A_μ are $(length)^{-1} = mass$ since the connection A_μ always couples to a dx^μ infinitesimal line element in the definition of a Wilson loop . When $D \neq 4$ the units of ϕ are no longer equal to those of A_μ. For example, in $D = 2$ the scalar field ϕ is dimensionless. Therefore, in this respect $D = 4$ is very special because both ϕ and A_μ have the same units and there is no need to multiply ϕ by a length scale parameter L in order to match the units of A_μ in the definition (2-1) of the polyvector.

The third component $A_{\mu\nu}$ is a rank two antisymmetric tensor field of units $(length)^{-2}$ ....and the last component of the expansion $A_{\mu\nu\rho\tau}$ is a pseudo-scalar (dual to a scalar) with units $(length)^{-4}$. The fact that a scalar and pseudo-scalar field appear very naturally in the expansion of the C-space polyvector valued field A_N suggests that one could attempt to identify the latter fields with a dilaton-like and axion-like field, respectively.

The differential operator is the generalized Dirac operator

$$d = E^M \partial_M = \underline{1} \partial_\sigma + \gamma^\mu \partial_{x_\mu} + \gamma^\mu \wedge \gamma^\nu \partial_{x_{\mu\nu}} + ... \qquad (2-2)$$

the polyvector-valued indices $M, N....$ range from $1, 2.....2^D$ since a Clifford algebra in D-dim has 2^D basis elements. The generalized Maxwell field strength in C-space is

$$F = dA = E^M \partial_M (E^N A_N) = E^M E^N \partial_M A_N = \frac{1}{2} \{E^M, E^N\} \partial_M A_N +$$

$$\frac{1}{2} [E^M, E^N] \partial_M A_N = \frac{1}{2} F_{(MN)} \{E^M, E^N\} + \frac{1}{2} F_{[MN]} [E^M, E^N]. \qquad (2-3)$$

where one has *decomposed* the Field strength components into a symmetric plus antisymmetric piece by simply writing the Clifford geometric product of two polyvectors $E^M E^N$ as the sum of an anticommutator plus a commutator piece respectively,

$$F_{(MN)} = \frac{1}{2} (\partial_M A_N + \partial_N A_M). \qquad (2-4)$$

$$F_{[MN]} = \frac{1}{2} (\partial_M A_N - \partial_N A_M). \qquad (2-5)$$

Let the C-space Maxwell action (up to a numerical factor) be given in terms of the antisymmetric part of the field strength:

$$I[A] = \int [\mathrm{D}X] F_{[MN]} F^{[MN]}. \qquad (2-6)$$

where $[\mathrm{D}X]$ is a C-space measure comprised of all the (holographic) coordinates degrees of freedom

$$[\mathrm{D}X] \equiv (d\sigma)(dx^0 dx^1 ...)(dx^{01} dx^{02} ...)....(dx^{012...D}). \qquad (2-7)$$

The action (2-6) is invariant under the gauge transformations

$$A'_M = A_M + \partial_M \Lambda \qquad (2-8)$$

The matter-field minimal coupling (interaction term) is:

$$\int A_M dX^M = \int [\mathrm{D}X] J_M A^M, \qquad (2-9)$$

where one has reabsorbed the coupling constant, the C-space analog of the electric charge, within the expression for the A field itself. Notice that this term (2-9) has the same form as the coupling of p-branes (whose world volume is $p+1$-dimensional) to antisymmetric tensor fields of rank $p+1$.

The open line integral in C-space of the matter-field interaction term in the action is taken from the polyparticle's proper time interval S ranging from $-\infty$ to $+\infty$ and can be recast via the Stokes law solely in terms of the antisymmetric part of the field strength. This requires closing off the integration countour by a semi-circle that starts at $S = +\infty$, goes all the way to C-space infinity, and comes back to the point $S = -\infty$. The field strength vanishes along the points of the semi-circle at infinity, and for this reason the net contribution to the contour integral is given by the open-line integral. Therefore, by rewriting the $\int A_M dX^M$ via the Stokes law relation, it yields

$$\int A_M dX^M = \int F_{[MN]} dS^{[MN]} = \int F_{[MN]} X^M dX^N =$$
$$\int dS F_{[MN]} X^M (dX^N/dS). \qquad (2-10)$$

where in order to go from the second term to the third term in the above equation we have integrated by parts and then used the Bianchi identity for the antisymmetric component $F_{[MN]}$.

The integration by parts permits us to go from a C-space domain integral, represented by the Clifford-value hypersurface S^{MN}, to a C-space boundary-line integral

$$\int dS^{MN} = \frac{1}{2} \int (X^M dX^N - X^N dX^M). \qquad (2-11)$$

The pure matter terms in the action are given by the analog of the proper time integral spanned by the motion of a particle in spacetime:

$$\kappa \int dS = \kappa \int dS \sqrt{\frac{dX^M}{dS} \frac{dX_M}{dS}}. \qquad (2-12)$$

where κ is a parameter whose dimensions are $(mass)^{p+1}$ and S is the polyparticle proper time in C-space.

The Lorentz force relation in C-space is directly obtained from a variation of

$$\int dS F_{[MN]} X^M (dX^N/dS). \qquad (2-13)$$

and

$$\kappa \int dS = \kappa \int \sqrt{dX^M dX_M}. \qquad (2-14)$$

with respect tothe X^M variables:

$$\kappa \frac{d^2 X_M}{dS^2} = e F_{[MN]} \frac{dX^N}{dS}. \qquad (2-15)$$

where we have re-introduced the C-space charge e back into the Lorentz force equation in C-space. A variation of the terms in the action w.r.t the A_M field furnishes the following equation of motion for the A field:

$$\partial_M F^{[MN]} = J^N. \qquad (2-16)$$

By taking derivatives on both sides of the last equation with respect to the X^N coordinate, one obtains due to the symmmetry condition of $\partial_M \partial_N$ versus the antisymmetry of $F^{[MN]}$ that

$$\partial_N \partial_M F^{[MN]} = 0 = \partial_N J^N = 0. \qquad (2-17)$$

which is precisely the continuity equation for the current.

The continuity equation is essential to ensure that the matter-field coupling term of the action $\int A_M dX^M = \int [\mathrm{D}X] J^M A_M$ is also gauge invariant, which can be readily verified after an integration by parts and setting the boundary terms to zero:

$$\delta \int [\mathrm{D}X] J^M A_M = \int [\mathrm{D}X] J^M \partial_M \Lambda = - \int [\mathrm{D}X] (\partial_M J^M) \Lambda = 0. \qquad (2-18)$$

Gauge invariance also ensures the conservation of the energy-momentum (via Noether's theorem) defined in tems of the Lagrangian density variation. We refer to [3] for further details.

The gauge invariant C-space Maxwell action as given in eq. (2-6) is in fact only a part of a more general action given by the expression

$$I[A] = \int [\mathrm{D}X]\ F^\dagger * F = \int [\mathcal{D}X]\ \ < F^\dagger F >_{scalar}\ . \qquad (2-19)$$

This action can also be written in terms of components, up to dimension-dependent *numerical* coefficients, as [3] :

$$I[A] = \int [\mathcal{D}X]\ (F_{(MN)} F^{(MN)} + F_{[MN]} F^{[MN]}) \qquad (2-20)$$

For rigor, one should introduce the numerical coefficients in front of the F terms, noticing that the symmetric combination should have a different dimension-dependent coefficient than the anti-symmetric combination since the former involves contractions of $\{E^M, E^N\}^* \{E_M, E_N\}$ and the latter contractions of $[E^M, E^N]^* [E_M, E_N]$.

The latter action is strictly speaking not gauge invariant, since it contains not only the antisymmetric but also the symmetric part of F. It is invariant under a *restricted* gauge symmetry transformations. It is invariant (up to total derivatives) under $infinitesimal$ gauge transformations provided the symmetric part of F is divergence-free $\partial_M F^{(MN)} = 0$ [3] . This divergence-free condition has the same effects as if one were $fixing$ a gauge leaving a $residual$ symmetry of *restricted* gauge transformations such that the gauge symmetry parameter obeys the Laplace-like equation $\partial_M \partial^M \Lambda = 0$ (in order to preserve the gauge condition) Such residual (restricted) symmetries are precisely those that leave invariant the divergence-free condition on the symmetric part of F. Residual, restricted symmetries occur, for example, in the light-cone gauge of p-brane actions leaving a residual symmetry of *volume*-preserving diffs. They also occur in string theory when the conformal gauge is chosen leaving a residual symmetry under conformal reparametrizations; i.e. the so-called Virasoro algebras whose symmetry transformations are given by holomorphic and anti-holomorphic reparametrizations of the string world-sheet.

This Laplace-like condition on the gauge parameter is also the one required such that the action in [3] is invariant under $finite$ (restricted) gauge transformations since under such (restricted) finite transformations the Lagrangian changes by second-order terms of the form $(\partial_M \partial_N \Lambda)^2$, which are total derivatives if, and only if, the gauge parameter is restricted to obey the analog of Laplace equation $\partial_M \partial^M \Lambda = 0$

Therefore the action of eq- (2-20) is invariant under a *restricted* gauge transformation which bears a resemblance to *volume*-preserving diffeomorphisms of the p-branes action in the light-cone gauge. A lesson that we have from these considerations is that the C-space Maxwell action written in the form (2-20) automatically contains a gauge fixing term. Analogous result for *ordinary* Maxwell field is known from Hestenes work [58], although formulated in a slightly different way, namely by direclty considering the field equations without emplying the action.

2.2 Nonabelian Gauge Field Theories in Clifford-spaces and Multi-forms Fields

It remains to be seen if this construction of C-space generalized Maxwell Electrodynamics of p-forms can be generalized to the Nonabelian case when we replace ordinary derivatives by gauge-covariant ones:

$$F = dA \rightarrow F = DA = (dA + A \bullet A). \qquad (2-21)$$

We should emphasize that these results based on Geometric Algebras in Cliffod Spaces are very $differentl$ that those results obtained from ordinary tensor calculus by Savvidy [11] which obeyed more $symmetry\ restrictions$ on the tensor indices.

Given a Lie algebra $\mathbf{G}$ whose generators are T_a for $a = 1, 2, 3, ...dim$ $\mathbf{G}$ and $[T_a, T_b] = f_{ab}^c T_c$ where the structure constants f_{abc} are fully antisymmetric in their indices, the Lie-algebra valued Clifford gauge fields are defined by $\mathbf{A}(\mathbf{X}) = E^M A^a_M(\mathbf{X}) T_a$ and from which one can define the one-form $\mathbf{A} = (A^a_M(\mathbf{X}) T_a) dX^M$. The generalized Lie-algebra valued field strength is

$$\mathbf{F} = \ [F^c_{MN}(X)\ T_c]\ dX^M \wedge dX^N =$$

$$[\ \partial_{[M} A^c_{N]}(X) T_c \ + \ g\ A^a_M(X) A^b_N(X)\ f^c_{ab}\ T_c\ ]\ dX^M \wedge dX^N. \qquad (2-22)$$

In components

$$F^c{}_{[\ [\mu_1\mu_2...\mu_m]\ [\nu_1\nu_2.....\nu_n]\ ]} =$$

$$\partial_{[\mu_1\mu_2...\mu_m]}\ A^c{}_{[\nu_1\nu_2....\nu_n]} \ - \ \partial_{[\nu_1\nu_2...\nu_n]}\ A^c{}_{[\mu_1\mu_2....\mu_m]} \ + \ g\ A^a{}_{[\mu_1\mu_2...\mu_m]}\ A^b{}_{[\nu_1\nu_2....\nu_n]}\ f^c_{ab}\ . \quad (2-23a)$$

The remaining components are of the form

$$F^c_{[0N]} = F^c{}_{[\ 0\ [\nu_1\nu_2.....\nu_n]\ ]} = \partial_\sigma\ A^c{}_{[\nu_1\nu_2....\nu_n]} - \partial_{[\nu_1\nu_2...\nu_n]}\ A^c{}_0 + g\ A^a_0\ A^b{}_{[\nu_1\nu_2....\nu_n]}\ f^c_{ab}\ . \quad (2-23b)$$

where A^c_0 is the Clifford-scalar part of the Lie-algebra valued Clifford-polyvector and in general we must consider the $m = n$ and $m \neq n$ cases resulting from the mixing of different grades (ranks). The antisymmetry with respect the collective indices MN is explicit.

In order to raise, lower and contract polyvector indices in C-space it requires a generalized metric G^{MN} [1] . In flat C-space it is defined by the components :

$$G^{\mu\nu} = \eta^{\mu\nu}. \quad G^{\mu_1\mu_2\ \nu_1\nu_2} = \eta^{\mu_1\nu_1}\ \eta^{\mu_2\nu_2} \ - \ \eta^{\mu_1\nu_2}\ \eta^{\mu_2\nu_1} \quad etc.. \qquad (2-24a)$$

in addition to the scalar-scalar component $G^{\sigma\sigma} = 1$. It can be recast as :

$$G^{\mu_1\mu_2....\mu_m\ \nu_1\nu_2....\nu_m} = det\ \mathbf{G}^{\mu_I\nu_J} = \frac{1}{m!}\epsilon_{i_1i_2...i_m}\epsilon_{j_1j_2....j_m}\eta^{\mu_{i_1}\nu_{j_1}}\eta^{\mu_{i_2}\nu_{j_2}}......\eta^{\mu_{i_m}\nu_{j_m}}. \quad (2-24b)$$

where $\mathbf{G}^{\mu_I\nu_J}$ is an $m \times m$ matrix whose entries are $\eta^{\mu_i\nu_j}$ for $i, j = 1, 2, 3, ...m \leq D$ and $\mu, \nu = 1, 2, 3, ...D$.

As a result of the expression for the flat C-space metric, given by sums of antisymmetrized products of $\eta^{\mu\nu}$, the Clifford-space generalized Yang-Mills action is of the form

$$S_{YM} = -\frac{1}{2}\int\ [\mathcal{D}X]\ \sum\ trace\ [\ F^a{}_{[\ [\mu_1\mu_2...\mu_m]\ [\nu_1\nu_2.....\nu_m]\ ]}\ F^{[\ [\mu_1\mu_2...\mu_m]\ [\nu_1\nu_2......\nu_m]\ ]\ b}\ T_aT_b\ ] \ +$$

$$-\frac{1}{2}\int\ [\mathcal{D}X]\ \sum\ trace\ [\ F^a{}_{[\ 0\ [\nu_1\nu_2.....\nu_m]\ ]}\ F^{[\ [\ 0\ [\nu_1\nu_2......\nu_m]\ ]\ b}\ T_aT_b\ ] \qquad (2-25)$$

where the C-space 2^D-dim measure associated with a Clifford algebra in D-dim is once again

$$[\mathcal{D}X] = [d\sigma]\ [\mathbf{\Pi}\ dx^\mu]\ [\mathbf{\Pi}\ dx^{\mu_1\mu_2}]\ [\mathbf{\Pi}\ dx^{\mu_1\mu_2\mu_3}]....\ [dx^{\mu_1\mu_2.....\mu_d}] \qquad (2-26)$$

and the indices are ordered as $\mu_1 < \mu_2 < \mu_3....... < \mu_m$, etc...

The action (2-25) is invariant under the infinitesimal gauge transformations

$$\delta_\xi\ A^c_M = \partial_M \xi^c \ + \ g f^c_{ab}\ A^a_M \xi^b. \quad \delta_\xi\ A^c_{\mu_1\mu_2....\mu_n} = \partial_{x_{\mu_1\mu_2....\mu_n}}\ \xi^c \ + \ g f^c_{ab}\ A^a_{\mu_1\mu_2....\mu_n}\xi^b. \quad (2-27)$$

associated with a Lie-algebra valued Clifford-scalar parameter $\xi(X) = \xi^a(X) T_a$.

One of the main differences between the Clifford Geometric Algebraic approach to Generalized Yang-Mills theories and the results of [11] is that in the latter case the mixing among the different rank tensors $A^a_{\mu_1\ (\nu_1\nu_2...\nu_j)}$ (that is symmetric in the ν's indices instead of being anti-symmetric) is of the form

$$F^c_{[\mu_1\mu_2]\ (\nu_1\nu_2......\nu_n)}(x) = \partial_{\mu_1}\ A^c_{\mu_2\ (\nu_1\nu_2........\nu_n)}(x) - \partial_{\mu_2}\ A^c_{\mu_1\ (\nu_1\nu_2........\nu_n)}(x)\ +$$

$$\sum_{j=0}^{j=n}\ \sum_{Permutations}\ g f^c_{ab}\ A^a_{\mu_1\ (\nu_1\nu_2...\nu_j)}(x)\ A^b_{\mu_2\ (\nu_{j+1}\nu_{j+2}....\nu_n)}(x) \qquad (2-28)$$

where one must sum over permutations of the ν's indices such that the gauge field strength in eq-(2-28) is anti-symmetric in the first two-indices μ_1, μ_2 and symmetric with respect to the other ν's indices. Thus these (anti) symmetry properties are *very different* from those of the C-space gauge field strengths.

The infinitesimal gauge transformations in [11] are :

$$\delta\ A^c_{\mu_1\ (\nu_1\nu_2....\nu_n)}(x) =\ (\ \delta^{cb}\partial_{\mu_1} +\ g f^c_{ab}\ A^a_{\mu_1}(x))\ \xi^b_{\ (\nu_1\nu_2....\nu_n)}(x)\ +$$

$$\sum_{j=0}^{j=n}\ \sum_{Permutations}\ g f^c_{ab}\ A^a_{\mu_1\nu_1\nu_2....\nu_j}(x)\xi^b_{\nu_{j+1}\nu_{j+2}....\nu_n}(x). \qquad (2-29)$$

where the gauge parameters are tensor-valued $\xi^a,\ \xi^a_\nu,\ \xi^a_{\nu_1\nu_2}, ...$ and one must sum over all permutations of the ν's indices such that the expression (2-29) is fully symmetric with respect all the $\nu_1, \nu_2, ...\nu_n$ indices.

One should notice the differences between Clifford-valued fields $\mathbf{A}(\mathbf{X})$ and ordinary tensor fields in spacetime $A_\mu(x), A_{\mu\nu}(x), A_{\mu\nu\rho}(x), ...$ in [11]. In addition one has the different (anti) symmetry property of the indices and the different spin content of the tensors of mixed-symmetry. There are an infinite number of tensor fields in [11] versus an infinite number of *modes* associated with a finite number of 2^D components of the polyvector $\mathbf{A}(\mathbf{X})$ corresponding to the $Cl(D, R)$ algebra. A mode expansion

$$A_M(\mathbf{X}) = A_M\ (\sigma, x^\mu, x^{\mu_1\mu_2}, ..., x^{\mu_1\mu_2....\mu_d}) =$$

$$\sum A^{n_0 n_2 n_3.....n_d}_M(x^\mu)\ \mathbf{u}\ _{n_0 n_2 n_3.....n_d}\ (\sigma, x^{\mu_1\mu_2}, ..., x^{\mu_1\mu_2.....\mu_d}). \qquad (2-30)$$

yields an infinite-number of modes. To excite these modes (freezing the σ coordinate) requires energies of the order of the Planck energy 10^{19} Gev since the coordinate polyvector $\mathbf{X} = E^M X_M$ involves the Planck scale $L_P = \Lambda$ parameter in the expansion as shown explicity in eq-(1-5) . Furthermore, in C-spaces one must extend the (super) Poincare symmetry to the more general polyvector-valued (super) Poincare symmetry [53], [5]. For a discussion of these generalized (super) symmetries and their applications to M, F theory superalgebras involving tensorial charges we refer to [53], [65], [54]

On the other hand, the expansion of the Lie-algebra valued Clifford polyvector $\mathbf{A} = E^M A^a_M T_a$ into different ranks anti-symmetric tensors requires another length scale L (energy $E = 1/L$ in natural units $\hbar = c = 1$) parameter as shown in eqs-(2-1). This expansion parameter may or not be equal to the Planck (energy) scale. Below we will argue why one may set this energy expansion parameter (where the tensor corrections to the Standard Model could be relevant) to be of the order of the 10 Tev scale.

We finalize this section with a brief note about bi-forms and multi-forms. The abelian (non-abelian) Clifford-valued field strength discussed in this section is an extension (generalization) of what is called in the Mathematics literature $bi-forms, multi-forms$ [52]. A bi-form is the $GL(D, R)$ reducible tensor product space of p-forms and q-forms on the exterior algebra whose elements are

$$T = \frac{1}{p!q!} \; T_{\mu_1\mu_2....\mu_p \;\; \nu_1\nu_2...\nu_q} \; dx^{\mu_1} \wedge dx^{\mu_2} \wedge ..... \wedge dx^{\mu_p} \otimes dx^{\nu_1} \wedge dx^{\nu_2} \wedge ..... \wedge dx^{\nu_q}. \quad (2-31)$$

All the standard operations on differential forms generalize to bi-forms [52]. Tensors in representations corresponding to Young tableaux with two columns are irreducible under $GL(D, R)$. Decomposing a general bi-form into its irreducible components corresponds to the $GL(D, R)$ Young decomposition of the tensor product of a p-form with a q-form. This construction of bi-forms can be generalized also to multi-forms that are tensor products of p-forms, q-forms, r-forms...... For a detailed discussion of tensor gauge fields of mixed symmetry, massive gauge-invariant field theories and multi-form gauge theories on spaces of constant curvature, and their applications in superstrings compactifications, we refer to [52] .

The analog of multi-forms in C-spaces will be :

$$\mathbf{F} = F_{M_{n_1} M_{n_2} ....M_{n_k}} dX^{M_{n_1}} \wedge dX^{M_{n_2}} \wedge .... \wedge dX^{M_{n_k}}. \quad (2-32)$$

For example, the analog of a tri-form in C-space is

$$F_{\;[\;[\mu_1\mu_2.....\mu_p]\;[\nu_1\nu_2......\nu_q]\;[\rho_1\rho_2.....\rho_r]\;]} \;\; dx^{\mu_1\mu_2....\mu_p} \wedge dx^{\nu_1\nu_2....\nu_q} \wedge dx^{\rho_1\rho_2.....\rho_r}. \quad (2-33)$$

Therefore, one could generalize all the standard results of ordinary bi-forms and multi-forms to Clifford-spaces and extend the notion of polyvectors to $polytensors$.

3 Clifford-space Extensions of the Standard Model

3.1 Clifford-Space Generalized Actions for Bosonic Fields

The kinetic terms of a C-space scalar field action is

$$S_{scalar} = \int \; [\mathcal{D}X] \; G^{MN} \; (\partial_M \varphi) \; (\partial_N \varphi). \quad (3-1)$$

where φ is the *scalar* component of the Clifford-valued field $\mathbf{\Phi}$ that is a *section* of the Clifford-polyvector-bundle whose structure group is the generalization of the $\mathbf{GL}(dim\ \mathcal{F}, R)$ group acting on the fiber $\mathcal{F}$; namely it is the Clifford group acting on the polyvector-valued-fiber and generated by the basis elements E_A . A special case of a Clifford-polyvector-valued bundle is the Clifford-tangent-bundle when the fiber $\mathcal{F}$ has the *same* dimension as the base manifold $\mathbf{M}$. Hence, the multi-graded components of the *section* $\mathbf{\Phi}$ of the Clifford-polyvector-bundle are

$$\mathbf{\Phi}(\mathbf{X}) = \Phi^A E_A = \varphi(\mathbf{X})\ \mathbf{1} + \Phi^a(\mathbf{X})\gamma_a + \Phi^{ab}(\mathbf{X})\gamma_a \wedge \gamma_b + ... \qquad (3-2)$$

and the Clifford-gauge-covariant derivative is

$$D_M \Phi^A = \partial_M \Phi^A \ + \ \mathcal{A}^A_{BM}\ \Phi^B. \qquad (3-3)$$

where $\mathcal{A}$ is the conection associated with the Clifford-polyvector-bundle. A natural action associated with the kinetic terms of the Clifford-analog of a massless field $\mathbf{\Phi}$ is

$$S[\mathbf{\Phi}] = \int\ [\mathcal{D}X]\ G^{MN} D_M \Phi^A D_N \Phi^B \Upsilon_{AB} =$$

$$\int\ [\mathcal{D}X]\ G^{MN}\ (\partial_M \Phi^A + \mathcal{A}^A_{CM}\Phi^C)\ (\partial_N \Phi^B + \mathcal{A}^B_{DN}\Phi^D)\Upsilon_{AB}. \qquad (3-4)$$

The action above in the case that $\mathbf{\Phi}$ is a section of the Clifford-Tangent-Bundle can be rewritten as :

$$S[\mathbf{\Phi}] = \int\ [\mathcal{D}X]\ < (\mathbf{D\Phi})^\dagger (\mathbf{D\Phi}) >_0 = \int\ [\mathcal{D}X]\ <\ (E^M D_M \Phi^A E_A)^\dagger (E^N D_N \Phi^B E_B)\ >_0 . \qquad (3-5)$$

where the frame E_A of the Clifford-Tangent-Bundle is covariantly constant $D_M E_A = 0$ and

$$G^{MN} = \frac{1}{2} <\ (E^M)^\dagger E^N + E^N (E^M)^\dagger\ >_0 . \quad \Upsilon_{AB} = \frac{1}{2} <\ (E_A)^\dagger E_B + E_B (E_A)^\dagger\ >_0 . \qquad (3-6)$$

The Geometric product among the Clifford basis elements is multi-graded since it contains objects of different grade given

$$(E^M)^\dagger E^N = \{< (E^M)^\dagger E^N >_{r+s}, \quad < (E^M)^\dagger E^N >_{r+s-2}, \quad ..... < (E^M)^\dagger E^N >_{|r-s|}\ \}. \qquad (3-7)$$

when $r = s$, the scalar part coincides with

$$< (E^M)^\dagger E^N >_{|r-s|} = < E^N (E^M)^\dagger >_{|r-s|} = < (E^M)^\dagger E^N >_0 = < E^N (E^M)^\dagger >_0 . \qquad (3-8)$$

$$\partial_M \Phi^A = \{\ \frac{\partial \Phi^A}{\partial x^\mu}, \quad \frac{\partial \Phi^A}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^A}{\partial x^{\mu\nu\rho}},\ ....\ \}. \qquad (3-9)$$

where :

$$\frac{\partial \Phi^A}{\partial x^\mu} = \{\ \frac{\partial \varphi}{\partial x^\mu}, \quad \frac{\partial \Phi^a}{\partial x^\mu}, \quad \frac{\partial \Phi^{ab}}{\partial x^\mu}, \quad \frac{\partial \Phi^{abc}}{\partial x^\mu}, \quad ....\ \}. \qquad (3-10)$$

$$\frac{\partial \Phi^A}{\partial x^{\mu\nu}} = \{\ \frac{\partial \varphi}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^a}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^{ab}}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^{abc}}{\partial x^{\mu\nu}}, \quad ....\ \}. \qquad (3-11)$$

$$\frac{\partial \Phi^A}{\partial x^{\mu\nu\rho}} = \{ \frac{\partial \varphi}{\partial x^{\mu\nu\rho}}, \quad \frac{\partial \Phi^a}{\partial x^{\mu\nu\rho}}, \quad \frac{\partial \Phi^{ab}}{\partial x^{\mu\nu\rho}}, \quad \frac{\partial \Phi^{abc}}{\partial x^{\mu\nu\rho}}, \quad .... \}. \qquad (3-12)$$

etc ——————

For simplicity, one can introduce a group-multiplet of Clifford-valued non-gauge bosons $\mathbf{\Phi} = E^M \Phi^i_M$ in a flat C-space when $\mathbf{\Phi}$ carries a representation of the group G and obey the *homogeneous* transformations

$$\delta_\xi \; \Phi^i_M = \xi^a \; \Sigma^{ij}_a \; \Phi^j_M. \qquad (3-13)$$

the matrices Σ^{ij}_a correspond to the representation of the group G according to which the fields Φ^i_M transform homogeneously. The homogeneous transformations (3-13) differ from the homogeneous transformations of the bosonic fields given by [11] which require Lie-algebra-valued tensor parameters ξ^a_M, in the same manner that the inhomogeneous transformations of the antisymmetric tensor-gauge fields (2-27) differed from those in eqs-(2-29) . Hence, when Φ^i_M is now a group-multiplet of Clifford-polyvector valued fields an invariant action in flat C-space under the transformations (3-13) is

$$S(bosons) = \int \; [\mathcal{D}X] \; G^{M_1 N_1} \; G^{M_2 N_2} [\; (\delta^{ij} \partial_{M_1} + ig \mathcal{A}^a_{M_1} \; \Sigma^{ij}_a) \; \Phi^j_{M_2} \;]^\dagger [\; (\delta^{ik} \partial_{N_1} + ig \mathcal{A}^a_{N_1} \; \Sigma^{ik}_a) \; \Phi^k_{N_2} \;]. \qquad (3-14)$$

To construct Clifford-space extensions of the Standard Model requires to include the analog of the Higgs potential and to add fermions. This follows next.

3.2 Clifford-Space Generalized Actions for Fermionic Fields

We shall introduce the spinor-tensor fields $\mathbf{\Psi}^\alpha = \Psi^\alpha_M(\mathbf{X}) \; E^M$

$$\Psi^\alpha(X); \; \Psi^\alpha_\mu(X); \; \Psi^\alpha_{\mu_1\mu_2}(X); \; \Psi^\alpha_{\mu_1\mu_2\mu_3}(X); .... \; \Psi^\alpha_{\mu_1\mu_2 ... \mu_d}(X). \qquad (3-15)$$

such that the fields

$$\Psi^\alpha_M(\mathbf{X}) \;\; = \;\; \Psi^\alpha_M \; (\; \sigma, x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ..., x^{\mu_1\mu_2 .... \mu_d} \;). \qquad (3-16)$$

transform under generalized poly-rotations of the underlying C-space as

$$\delta \; \Psi^\alpha_M = L^N_M \Psi^\alpha_N. \quad L^N_M = < R E^N R^{-1} E_M >_0 . \quad R = e^{\Theta^{MN} [E_M, E_N]}. \qquad (3-17)$$

under ordinary tangent space Lorentz transformations

$$\delta \; \Psi^\alpha_M = \xi^{ab} \; [\gamma_a, \gamma_b]^\alpha_\beta \; \Psi^\beta_M. \qquad (3-18)$$

and under generalized-Lorentz tangent space transformations (poly-rotations in the Clifford-tangent space) :

$$\delta \; \Psi^\alpha_M = \xi^{AB} \; [E_A, E_B]^\alpha_\beta \; \Psi^\beta_M. \qquad (3-19)$$

One may begin with the action

$$S^{(1)}_{Fermions} = \int [\mathcal{D}X] \ \bar{\Psi}_{M_1} \ E^{M_1} \ E^{M_2} \ E^{M_3} \ (\ - i\partial_{M_2} \ + \Omega^{AB}_{M_2} \ [E_A, E_B] \ ) \ \Psi_{M_3}. \quad (3-20)$$

Ω^{AB}_M is the C-space extension of the spin-connection ω^{ab}_μ. Extended Relativity in *curved* C-spaces has been studied in [1]. In particular, we have shown why the curvature scalar in C-space admits an expansion into powers of curvature and torsion of the underlying spacetime; i,e. Gravity in C-spaces is a associated to a higher derivative gravity with torsion in the underlying spacetime. The action (3-20) is the C-space extension of the covariant massless Dirac and Rarita-Schwinger actions, given respectively (in curved spacetimes) by :

$$S_{Dirac} = \int [d^4x] \ \bar{\Psi} \ \gamma^\mu(\ - i\partial_\mu + \omega^{ab}_\mu \ [\gamma_a, \gamma_b] \ ) \ \Psi. \quad (3-21)$$

$$S_{gravitino} = \int [d^4x] \ \bar{\Psi}_\mu \ \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \ (\ i\partial_\nu + \omega^{ab}_\nu \ [\gamma_a, \gamma_b] \ ) \ \Psi_\rho. \quad (3-22)$$

with ω^{ab}_μ is the Lorentz spin connection.

Given a matrix representation Υ^{ij}_a of the group G according to which a multiplet of spinor-tensors Ψ^i_M transform allows to introduce a gauge-invariant interaction of fermions in $flat$ C-spaces (after omitting the spinor indices) :

$$S^{(2)}_{Fermions} = \int [\mathcal{D}X] \ \bar{\Psi}^i_{M_1} \ E^{M_1} \ E^{M_2} \ E^{M_3} \ (\ - i\delta^{ij}\partial_{M_2} \ + g\mathcal{A}^a_{M_2}\Upsilon^{ij}_a \ ) \ \Psi^j_{M_3}. \quad (3-23)$$

the spinor-tensor fields Ψ^j_M (omitting the spinor indices) carry now a group index i (like the quark fields which carry a $SU(3)$ colour index) and transform under gauge transformations as

$$\delta_\xi \ \Psi^i_M = -i\xi^a\Upsilon^{ij}_a \ \Psi^j_M. \quad (3-24)$$

Another possible action in $flat$ C-space is of the form :

$$\int [\mathcal{D}X] \ \bar{\Psi}^i_M \ E^N \ (\ - i\delta^{ij}\partial_N \ + gA^a_N\Upsilon^{ij}_a \ ) \ \Psi^M_j \quad (3-25)$$

but the action in (3-23) is the most general one.

3.3 Lagrangians for Clifford Space Extensions of the Standard Model

The Lagrangian of the Clifford-space extensions of the Standard Model is

$$L = L_{Bosons}[\mathbf{\Phi}] + L_{YM}[\mathbf{A}] + L_{Fermions}[\mathbf{\Psi}] - V_{Higgs}[\mathbf{\Phi}] + L_{Yukawa}[\mathbf{\Phi}, \mathbf{\Psi}]. \quad (3-26)$$

where L_{Bosons}, L_{YM} and $L_{Fermions}$ were given in eqs-(3-14, 2-25, 3-23) respectively. The Higgs potential is given by

$$V_{Higgs}(\mathbf{\Phi}) = \frac{\lambda}{2}[\ (\mathbf{\Phi})^\dagger \mathbf{\Phi} - \frac{1}{2}\eta^2\ ]^2. \qquad (3-27)$$

where we have omitted the several indices.

The Yukawa couplings of fermions to Higgs bosons that generate the fermion masses, and the subsequent breakdown of the gauge symmetry by nonzero vacuum-expectation values (vev) of the Higgs bosons, can also be generalized to Clifford-spaces. For example, in the case of the $SO(10)$ Grand Unified Models, to generate the possible mass terms for the fermions one must write down gauge-invariant Yukawa couplings of the form [59]

$$\tilde{\psi}\ BC^{-1}\gamma^\mu\ \psi\ \phi_\mu. \quad \tilde{\psi}\ BC^{-1}\gamma^\mu\gamma^\nu\gamma^\rho\ \psi\ \phi_{\mu\nu\rho}. \quad etc.... \qquad (3-28)$$

where $\tilde{\psi}$ stands for the transpose of ψ. B is the equivalent of the charge conjugation matrix for $SO(10)$, and C is the Dirac charge conjugation matrix. ϕ_μ, $\phi_{\mu\nu\rho}$ are the Higgs bosons belonging to the totally irreducible anti-symmetric 10-dim and 120-dim representations of $SO(10)$. The fermions ψ belong to the 16-dim spinor representation of $SO(10)$. The symmetry breaking patterns of the $SO(10)$ gauge symmetry down to the $SU(3)_c \times SU(2)_L \times U(1)_Y$ are very subtle. For further details we refer to [59].

To extend this construction to Clifford-spaces involves the introduction of polyvectors. In the special case that $D = 2N$, the Clifford bivectors $\Sigma^{ab} = \Gamma^a \wedge \Gamma^b$ are also the generators of the $SO(2N)$ algebra. The $SO(2N)$ generators can be rewritten also in terms of a spinorial $SU(N)$ basis. Thus, the generalized Yukawa couplings of the spinor-tensors Ψ_M to the tensorial Higgs bosons Φ_M in the Clifford spaces associated with the $CL(2N, R)$ algebra are of the form :

$$L_{Yukawa} = \ \tilde{\Psi}_{M_1}\ BC^{-1}\ \Gamma^{M_1}\ \Gamma^{M_2}\ \Gamma^{M_3}\ \Psi_{M_2}\Phi_{M_3}. \ \ \Gamma^M = \mathbf{1},\ \Gamma^\mu,\ \Gamma^{\mu_1}\wedge\Gamma^{\mu_2},\ \Gamma^{\mu_1}\wedge\Gamma^{\mu_2}\wedge\Gamma^{\mu_3}, ... \qquad (3-29)$$

We may notice that a fermionic action corresponding to the $SO(2N)$ algebra

$$I^{(1)}_{fermions} = \int_{Cliff\ \ (2N)} [\mathcal{D}X]\ \ \bar{\Psi}_{M_1}\ \Gamma^{M_1}\ \Gamma^{M_2}\ \Gamma^{M_3}\ (\ -i\partial_{M_2}\ + \mathcal{A}^{ab}_{M_2}\ [\Gamma_a, \Gamma_b]\ )\ \Psi_{M_3}. \qquad (3-30)$$

is part of the most general action based on the Clifford group $Cl(2N, R)$:

$$I^{(2)}_{Fermions} = \int_{Cliff\ \ (2N)} [\mathcal{D}X]\ \ \bar{\Psi}_{M_1}\ \Gamma^{M_1}\ \Gamma^{M_2}\ \Gamma^{M_3}\ (\ -i\partial_{M_2}\ + \mathcal{A}^{AB}_{M_2}\ [\Gamma_A, \Gamma_B]\ )\ \Psi_{M_3}. \qquad (3-31)$$

where the Γ_A, Γ_B.... are anti-symmetrized products of the ordinary $2^N \times 2^N$ gamma matrices in $D = 2N$. The C-space measure $[\mathcal{D}X]$ corresponds to the 2^{2N}-dim measure belonging to the $Cl(2N, R)$ algebra. The polyvector-valued gauge connection is $\mathcal{A}^{AB}_M$.

The generalized bosonic action in this case is given by

$$I[\mathbf{\Phi}] = \int \ [\mathcal{D}X] \ G^{MN} D_M \Phi^A D_N \Phi^B \Upsilon_{AB} =$$

$$\int \ [\mathcal{D}X] \ G^{MN} \ (\partial_M \Phi^A + \mathcal{A}^A_{CM} \Phi^C) \ (\partial_N \Phi^B + \mathcal{A}^B_{DN} \Phi^D) \Upsilon_{AB}. \qquad (3-32)$$

The generalized Yang-Mills action is provided by eqs-(2-23, 2-25) where one replaces the structure constants f_{abc} of the Lie algebra $SO(2N)$ for those structure constants f_{ABC} associated with the full 2^{2N}-dimensional algebra corresponding to the Clifford Group $Cl(2N,R)$ and represented by the $2^N \times 2^N$ gamma matrices in $D = 2N$:

$$[\Gamma_A, \Gamma_B] = f^C_{AB} \Gamma_C. \quad \Gamma^A = \mathbf{1}, \ \Gamma^a, \ \Gamma^{a_1} \wedge \Gamma^{a_2}, \ \Gamma^{a_1} \wedge \Gamma^{a_2} \wedge \Gamma^{a_3}, \ ......\Gamma^{a_1} \wedge \Gamma^{a_2} \wedge .... \wedge \Gamma^{a_{2N}}. \qquad (3-33)$$

Therefore, the generalized Yang-Mills action related to the $Cl(2N,R)$ algebra is :

$$I_{YM} = -\frac{1}{2} \int \ [\mathcal{D}X] \ \sum \ trace \ [\ F^A_{\ [\ [\mu_1 \mu_2 ... \mu_m] \ [\nu_1 \nu_2 .....\nu_m] \ ]} \ F^{[\ [\mu_1 \mu_2 ... \mu_m] \ [\nu_1 \nu_2 ......\nu_m] \ ] \ B} \ \Gamma_A \Gamma_B \ ] \ +$$

$$-\frac{1}{2} \int \ [\mathcal{D}X] \ \sum \ trace \ [\ F^A_{\ [\ 0 \ [\nu_1 \nu_2 .....\nu_m] \ ]} \ F^{[\ [\ 0 \ [\nu_1 \nu_2 ......\nu_m] \ ] \ B} \ \Gamma_A \Gamma_B \ ] \qquad (3-34)$$

The importance of the $Cl(8,R)$ algebra in the description of the Standard Model and Gravity in $4D$, along with the Exceptional Grand Unified Theories [50] will be analyzed in full detail in section **4** . Therefore, the actions described in eqs-(3-31, 3-32, 3-34) are instrumental to the sought-after Clfford-space generalization of Smith's model [50] .

3.4 New Particles and Interactions of Tensorial Extensions of the Standard Model and the origins of the 10 Tev scale

As explained earlier, the expansion of the Lie-algebra valued Clifford polyvector $\mathbf{A} = E^M A^a_M T_a$ into different ranks anti-symmetric tensors requires another length scale L (energy $E = 1/L$ in natural units $\hbar = c = 1$) parameter as shown in eqs-(2-1). This expansion parameter may or not be equal to the Planck (energy) scale. Below we will argue why one may set this energy expansion parameter (where the tensor corrections to the Standard Model could be relevant) to be of the order of the 10 Tev scale.

In [44] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of Gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal $SO(3,2)$ group. The AdS_4 space is a natural vacuum of the theory. The vacuum energy density was *derived* (instead of postulated) to be *precisely* the geometric-mean between the UV Planck scale and the IR throat size of de Sitter (Anti de Sitter) space . Setting the throat size to coincide with the future horizon scale (of an accelerated de Sitter Universe) given by the Hubble scale (today) R_H , the geometric mean relationship yields the *observed* value of the vacuum energy density

$$\rho \sim (L_P)^{-2} (R_H)^{-2} = (L_P)^{-4} (L_P^2 / R_H^2) \sim 10^{-120} M^4_{Planck} = m^4. \qquad (3-35)$$

from which we can infer that $m = 10^{-30} M_{Planck} \sim 10^{-11}\ Gev = 10^{-2}\ ev$ which is of the order of the electron neutrino mass.

The 10 Tev energy scale can be obtained if one *postulates* the geometric-mean relationship [60]:

$$L^{-2} = m M_{Planck} = E^2 \ \Rightarrow E = \sqrt{m M_P} \sim 10\ Tev. \qquad (3-36)$$

It is desirable to derive this last geometric-mean relationship in the same way that we derived the *observed* value of the vacuum energy density as the geometric mean between the UV and IR scales [44]. Nottale [45] gave a different argument to explain the small value of ρ based on Scale Relativity theory.

• Irreducible representations of the Poincare group determine the spin content of tensor-fields of mixed symmety via the Young tableaux diagramatic techniques. The spin content of the antisymmetric tensor fields corresponding to the Clifford polyvectors is different in general from the spin content of the symmetric tensors of Savvidy [11]. Despite these differences in the spin content, we shall outline the most salient features of Savvidy's tensorial extensions of the Standard Model to get a feeling of what the first rung of the (infinite) particle ladder hierarchy looks like. Notice that at the first level the spin content $s = 3/2$ of the spinor-tensors of our model (after freezing all the modes except those stemming from x^μ) is the same as in the model of [11]. As stated earlier, in $D = 4$ we also have the scalar $A_0 = \phi$ part of the polyvector $\mathbf{A}$ with the same dimensions as that of A_μ in the expansion of eq-(2-1). It is warranted to find the physical interpretation of such scalar (a dilaton-like field).

• The first members of the new leptons of the higher-spin hierarchy $(\psi^\alpha)_\mu(x)$ are given by 6 left-handed new leptons of spin $s = 3/2$, plus 3 right-handed new leptons of spin $s = 3/2$ which are associated with the standard 6 left-handed leptons that appear in $SU(2)$ doublets $(e,\nu_e)_L$; $(\mu,\nu_\mu)_L$; $(\tau,\nu_\tau)_L$ and 3 right-handed leptons e_R, μ_R, τ_R , respectively, with their antiparticles that we shall leave for granted in our discussion here.

• There are 6 new spinor-tensors of spin $s = 3/2$ associated with the standard 6 left-handed quarks ($SU(2)$ doublets) $(u,d)_L$; $(c,s)_L$; $(t,b)_L$ and 6 right-handed quarks u_R, d_R.......

• There are 4 new tensor-gauge bosons of spin $s = 2$ associated with the vector-gauge bosons $W^\pm, Z, \gamma$ of the electroweak $SU(2)_L \times U(1)_Y$ interaction; and 8 new tensor-gauge bosons of spin $s = 2$ associated with the 8 gluons of the color $SU(3)_c$ force.

• These first order extensions of the tree level $SU(2)_L \times U(1)_Y$ Lagrangian described by Savvidy [11] furnish new interactions of the ordinary vector-gauge bosons $\gamma, Z, W^\pm$ with the new leptons $e_{3/2}$, $\nu_{3/2}$, ... (spinor-tensors of spin 3/2) . It also yields new interactions of the tensor-gauge bosons ($s = 2$) with the ordinary leptons $e_{1/2}$, $\nu_{1/2}$, ... and the new leptons $e_{3/2}, \nu_{3/2}$, ... (of spin $s = 3/2$).

• The vacuum expectation value (vev) of the scalar fields is the same as the Standard Model $< \phi >_{vev} = \eta/\sqrt{2}$. The vev of the non-gauge boson ϕ_μ is equal to zero and does not break Poincare invariance.

• The Higgs boson mass is $m_H = \lambda\eta$.

• It predicts a tree-level degeneracy of the mass spectrum of the new tensor-gauge bosons of mass $m = g\eta$.

• Decay of the standard vector-gauge bosons into new leptons through the channels :

$$\gamma,\ Z \rightarrow e^-_{3/2} + e^+_{3/2}. \quad \gamma,\ Z \rightarrow \nu_{3/2} + \bar{\nu}_{3/2}. \quad W \rightarrow \nu_{3/2} + e_{3/2}. \quad . \qquad (3-37)$$

• Decay of the tensor-gauge bosons into ordinary and new leptons through the channels :

$$Z_{\mu_1\mu_2} \ \rightarrow \ e^+_{3/2} + e^-_{1/2}. \quad Z_{\mu_1\mu_2} \ \rightarrow \ \nu_{3/2} + \bar{\nu}_{1/2}. \qquad (3-38)$$

$$W_{\mu_1\mu_2} \ \rightarrow \ e_{3/2} + \nu_{1/2}. \quad W_{\mu_1\mu_2} \ \rightarrow \ \nu_{3/2} + e_{1/2}. \quad . \qquad (3-39)$$

• The new interaction vertices yields also the new two-step process :

$$e^+_{1/2} + e^-_{1/2} \ \rightarrow \ Z \ \rightarrow \ W^+_{\mu_1\nu_1} + W^-_{\mu_1\mu_2}. \qquad (3-40)$$

followed by a final decay of the tensor-gauge bosons $W^{\pm}_{\mu_1\mu_2}$ into an ordinary and a new lepton displayed by eq- (3-39). For more details we refer to [11].

4 On Clifford-space Extensions of the Standard Model based on the Cl (8, R) Algebra

As mentioned in the last section, the importance of the $Cl(8,R)$ algebra (in $D = 8$) in obtaining the Standard Model and Gravity in four-dimensions, along with the Exceptional Grand Unified Theories [50] will be analyzed in full detail next. The actions described in eqs-(3-31, 3-32, 3-34) are instrumental in the construction of the Clfford-space generalization of Smith's model [50] described below . Since we have already displayed the actions in eqs-(3-31, 3-32, 3-34) that $define$ the Clifford space extension of Smith's model, we shall focus below on the importance of Smith's work that reproduces *all* of the features of the Standard Model (quark masses, coupling constants, Higgs mass, , Yukawa couplings, Kobayashi-Maskawa matrix parameters, ...) and Gravity in 4-dim.

Contrary to the standard lore that is not possible to obtain the $SU(3)\times SU(2)\times U(1)$ gauge field structure from a Kaluza-Klein framework in $D = 8$, Batakis [46] uncovered an extra $SU(2) \times U(1)$ gauge field structure to the $SU(3)$ gauge field stucture from a Kaluza-Klein mechanism in $M^4 \times CP^2$ provided a nontrivial torsion in the total space is incorporated. Such torsion creates a new and nontrivial possibility for the accomodation of a fully unified theory in $D = 8$ not envisioned before in the physics literature. Clifford spaces have torsion [1]. For these reasons we shall outline now the important results of Smith [50] based on Clifford algebraic structures in $D = 8$.

4.1 Gravity and the Standard Model from a Cl(1,7) Group Graded Structure in D = 8

We will follow very closely the main results of Smith [50] to get a representation of all the known particles and fields in Physics based on the real Clifford group $Cl(1,7)$ (one timelike and 7 spacelike directions) . The $Cl(1,7)$ is $2^8 = 256 = 16 \times 16$ dimensional and has a graded structure :

$$1 \quad 8 \quad 28 \quad 56 \quad 35+35 \quad 56 \quad 28 \quad 8 \quad 1 \qquad (4-1)$$

into a scalar, vector, bivector, 3-vector, ... pseudoscalar. The middle 70 is written as $35+35$ because it is self-dual under Hodge duality. By Hodge duality, the 1 8 28 56 35 is dual to the 35 56 28 8 1. It can be shown that the 1 8 28 56 35 correspond to physical fields in the coordinate representation while the 35 56 28 8 1 correspond to physical fields in the momentum representation and complementarity between space-time and momentum-energy is achieved by bit inversion, which interconverts between position representation and momentum representation. The model [50] does use *all* the graded parts of Cl(8), and also the spinor structure of Cl(8), but the 56 and 35 parts are not physically effective at low energies after dimensional reduction, and consequently they are not written down explicitly in the $8D$ Lagrangian below which is used to calculate force strengths, particle masses, etc in the low energy region where we do experiments today.

• The correct $4D$ spacetime signature $(1,3)$.

The model is also consistent with the quaternionic structure of conformal $Cl(2,4) = 4 \times 4$ quaternionic matrices and with the quaternionic structure of $Cl(1,3) = 2 \times 2$ quaternionic matrices, so the 4-dim physical spacetime has the *correct* signature $(1,3)$ and *not* the signature $(3,1)$ of $Cl(3,1) = 4 \times 4$ real matrices. Hence, the $(1,7)$-dimensional vector representation corresponds to an 8-dim high-energy spacetime with octonionic structure that reduces at lower energies to quaternionc structures that correspond to the $(1,3)$-dim physical spacetime and a $(0,4)$-internal symmetry space.

• Emergence of Gravity and $SU(3) \otimes SU(2) \otimes U(1)$.

There is a 28-dim bivector representation ($28 = 16 + 12$) that corresponds to the gauge symmetry Lie algebra of $Spin(1,7)$ that reduces at lower energies to **(i)** a 16-dim $U(2,2) = U(1) \otimes SU(2,2) = U(1) \otimes Spin(2,4)$ whose conformal Lie algebra structure leads to gravity (with a cosmological constant) via the MacDowell-Mansouri-Chamseddine-West mechanism, and **(ii)** a 12-dim $SU(3) \otimes SU(2) \otimes U(1)$ Standard Model symmetry group involving 12 Gauge Bosons (8 gluons, 3 weak bosons $W_{\pm}, Z_0$, and the photon) that can be represented on an internal 4-dim symmetry space by the coset structure $SU(3)/U(2) = SU(3)/SU(2) \otimes U(1)$ associated with a CP^2 projective space.

• Unified Lagrangian in $8D$.

The above structures fit together into an $8D$ ($4D$ spacetime with a $4D$ internal symmetry space) Lagrangian :

$$\int_{8D} F \ \wedge \ ^*F + + \bar{\Psi} \mathbf{d} \Psi + (d\Phi + [A, \Phi]) \ \wedge \ ^*(d\Phi + [A, \Phi]). \qquad (4-2)$$

$\bar{\Psi} = \Psi^\dagger \Gamma_0$ and $\mathbf{d} = \Gamma^M \partial_M$. that reduces to the Lagrangian of Gravity plus the Standard Model upon dimensional reduction as shown in [50]

• Hermitian Symmetric Spaces.

The geometry of these representation spaces is associated with complex homogeneous domains with Shilov boundaries. In conjunction with the combinatorial structure of the second and third generation fermions (based on paths along the internal dimensions) allows the explicit *calculation* and *derivation* of the relative force strength of all coupling constants and particle masses, [50]. Most recently the derivation of the observed coupling constants were obtained based on Geometric Probability [51]. Smith started with the structure and data of the Standard Model plus a MacDowell-Mansouri formulation of Gravity, and proceeded to construct the $D_4 - D_5 - E_6 - E_7 - E_8$ Physics Model noticing that:

• the four forces correspond to the four types of 4-dimensional quaternionic symmetric spaces;

• the relative strengths of the four forces correspond to geometric structures related to those symmetric spaces, using a generalization of the ideas of Armand Wyler;

• the Lie algebra generators of the forces correspond to the root vectors of the D_4 ($SO(8)$) Lie algebra, the adjoint representation of which can therefore be (in an unconventional way) be broken down to form all the forces of the Standard Model plus Gravity;

• the chiral-spinor representations of D_4 correspond to the first generation fermion particles and antiparticles, and to the basis $1, i, j, k, E, I, J, K$ of the octonions;

• the vector representation of D_4 corresponds to a (1,7) 8-dimensional octonionic spacetime;

• picking a particular quaternionic subspace of the octonions (freezing it out) breaks the 8-dimensional spacetime into (1,3) 4-dimensional physical spacetime plus a 4-dimensional CP^2 internal symmetry space similar to [46].

• the spacetime dimensional reduction breaks the D_4 generators into the groups of Gravity plus the Standard Model;

• the spacetime dimensional reduction produces 3 generation of fermion particles, whose relative masses can be calculated from the symmetric space geometric structures and some combinatorial relations connected with the 3-generation structure;

• the D_4 Lie algebra and its representations fit inside the $Cl(8)$ Clifford algebra; which is the fundamental building block of all Clifford algebras of arbitrary high dimension because of 8-fold periodicity, which leads to formulation of a real generalized hyperfinite Type II1 von Neumann algebra factor, a unique structure that satisfies Einstein's Criterion: "... a theorem which at present can not be based upon anything more than upon a faith in the simplicity, i.e., intelligibility, of nature: there are no arbitrary constants ... that is to say, nature is so constituted that it is possible logically to lay down such strongly determined laws that within these laws only rationally completely determined constants occur (not constants, therefore, whose numerical value could be changed without destroying the theory). ...".

To sum up : the approach of Smith [50] is fiundamentally "bottom-up" in that it begins with noticing some unusual symmetries among the known characteristics of particles and fields, but it leads to a unique real generalized hyperfinite Type II1 von Neumann algebra factor that might be called a Unique Theory of Physics at sub-Planck energy levels.

4.2 On Clifford, Division, Jordan and Exceptional Algebras

Exceptional Gauge Theories can be constructed from Clifford algebras. Models of Grand Unified Theories based on exceptional groups were initially proposed by Gursey and followed by many others. For example, early E_8 GUT models were advanced by [49], [48] with and without supersymmetry, respectively, with the prediction of additional quarks and leptons families. Physical applications of Jordan algebras to strings have been studied by many authors, in particular by [61], [62], [63]. More closely related to string, M, F theory, there is an interplay among the $Cl(8)$ algebra, Division, Jordan and Exceptional Lie algebras where $D = 26, 27, 28$ dimensions play a fundamental role.

Roughly speaking, we can interpret $D = 26, 27, 28$ dimensions as the **3**-foldings of 8-dimenions with strings, membranes and three-branes living *transversely* to these **3**-folds : $24 = 3 \times 8$; i.e. the number of *transverse* degrees for the strings, membranes and three-branes moving in $D = 26, 27, 28$ dimensions are 24 in all of these cases. The world-manifolds of a string, brane and three-brane are two, three and four-dimensional respectively. Hence, a bosonic and/or supersymmetric String, M, F theory correspond to the following dimensions :

$8k + 2$, $D = 10$ superstring for $k = 1$; and $D = 26$ Bosonic string for $k = 3$. $8k + 3$, $D = 11$ M theory for $k = 1$; and $D = 27$ Bosonic membrane for $k = 3$. $8k + 4$, $D = 12$ F theory for $k = 1$; and $D = 28$ Bosonic 3-brane for $k = 3$.

The interplay among $D = 26, 27, 28$ dimensions as explained by Smith [50] goes as follows. The 28-real-dimensional degree-4 quaternionic Jordan algebra $J_4(Q)$ of 4×4 Hermitian matrices over the Quaternions

$p \ D \ B \ A$

$D^* \ q \ E \ C$

$B^* \ E^* \ r \ F$

$A^* \ C^* \ F^* \ t$

where * denotes conjugate and p, q, r, t are real R and A, B, C, D, E, F are quaternionic-valued. The $4 \times 28 = 112$-real dimensional Quaternification of $J_4(Q)$ can be represented as the Symmetric Space $E_8/E_7 \times SU(2)$. $J_4(Q)$ contains the traceless $28 - 1 = 27$-dimensional subalgebra $J_4(Q)_{tr}$ that has the unique structure of the 27-dimensional exceptional Jordan algebra $J_3(O)$ of 3×3 Hermitian matrices over the Octonions

$p \ B \ A$

$B^* \ q \ C$

$A^* \ C* \ r$

where * denotes conjugate and p, q, r are reals R and A, B, C are Octonionic valued . The $2 \times 27 = 54$-real dimensional Complexification of $J_3(O) = J_4(Q)_{tr}$ can be represented as the Symmetric Space $E_7/E_6 \times U(1)$. $J_3(O)$ contains a traceless $27 - 1 = 26$-dimensional subalgebra $J3(O)_{tr}$ that can be represented as the Symmetric Space E_6/F_4.

In other words, the chain of dimensions $D = 26, 27, 28$ corresponding to bosonic string, M, F theory, respectively, yields Jordan algebra $J_3(O)_{tr}$, $J_3(O) = J4(Q)_{tr}, J4(Q)$ related to the exceptional Lie algebras F_4, E_6, E_7, E_8. The exceptional group G_2 is the automorphsim group of the Octonions. Recently, the Chern-Simons Lagrangians corresponding to the large N limit of Exceptional Jordan Matrix models by Smolin and Ohwashi [67] were presented in [68] that may describe the nonperturbative behaviour of a bosonic M, F theory.

4.3 The $E_8 \times E_8$ Yang-Mills from a $Cl(16)$ Gauge Theory

It is well known among the experts that the E_8 algebra admits the $SO(16)$ decomposition $\mathbf{248} \rightarrow \mathbf{120} \oplus \mathbf{128}$. The E_8 admits also a $SL(8, R)$ decomposition. Due to the triality property , the $SO(8)$ admits the vector $\mathbf{8}_v$ and spinor representations $\mathbf{8}_s, \mathbf{8}_c$. After a triality rotation, the $SO(16)$ vector and spinor representations decompose as

$$\mathbf{16} \rightarrow \mathbf{8}_s \oplus \mathbf{8}_c.$$

$$\mathbf{128}_s \rightarrow \mathbf{8}_v \oplus \mathbf{56}_v \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v.$$

$$\mathbf{128}_c \rightarrow \mathbf{8}_s \oplus \mathbf{56}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_c. \qquad (4-3)$$

To connect with (real) Clifford algebras, i.e. how to fit E_8 into a Clifford structure , start with the 248-dim E_8 = 120-dim bivector adjoint of D_8 + 128-dim D_8 chiral-spinor and so embed E_8 in the Clifford algebra $Cl(16)$, with graded structure

$$1\ 16\ 120\ 560\ 1820\ 4368\ 8008\ 11440\ 12870$$

$$11440\ 8008\ 4368\ 1820\ 560\ 120\ 16\ 1. \qquad (4-4)$$

and total dimension $2^{16} = 65,536 = (128+128)(128+128)$.

From the modulo 8 periodicity of Clifford algebras one has $Cl(16) = Cl(2 \times 8) = Cl(8) \otimes Cl(8)$, meaning that the $2^{16} = 256 \times 256$ $Cl(16)$ matrices can be obtained by replacing each single one of the *entries* of the $2^8 = 256 = 16 \times 16$ $Cl(8)$ matrices by 16×16 matrices. In particular, $120 = 1 \times 28 + 8 \times 8 + 28 \times 1$ and $128 = 8 \times 8 + 8 \times 8$, hence the 248-dim E_8 algebra decomposes into a 120+128 dim structure such that E_8 can be represented indeed within a tensor product of Cl(8) algebras. At the E_8 Lie algebra level, the E_8 gauge connection decomposes into the $SO(16)$ vector $I, J = 1, 2, ...16$ and spinor $A = 1, 2, ...128$ indices as follows

$$\mathcal{A}_\mu = \mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A$$

where X_{IJ}, Y_A are the E_8 generators. The Clifford algebra structure behind $\mathcal{A}^{IJ} X_{IJ}$ and $\mathcal{A}^A Y_A$ is

$$\mathcal{A}_\mu^{IJ} X_{IJ} = A_\mu^{i_1 i_2}(\gamma_{i_1 i_2} \otimes 1) + A_\mu^{j_1 j_2}(1 \otimes \gamma_{j_1 j_2}) + A_\mu^{ij}(\gamma_i \otimes \gamma_j) \qquad (4-5)$$

The decomposition in (4-5) yields the $28 + 28 + 64 = 120$-dim vector representation of $SO(16)$, and the decomposition

$$\mathcal{A}_\mu^A Y_A = A_\mu^{i_1}(\gamma_{i_1} \otimes 1) + A_\mu^{i_1 i_2 i_3}(\gamma_{i_1 i_2 i_3} \otimes 1) + A_\mu^{i_1 i_2 .... i_5}(\gamma_{i_1 i_2 .... i_5} \otimes 1) + A_\mu^{i_1 i_2 i_3 ..... i_7}(\gamma_{i_1 i_2 i_3 .... i_7} \otimes 1). \qquad (4-6)$$

yields the $8+56+56+8 = 128$-dim spinor representation of $SO(16)$. Therefore, a $E_8 \times E_8$ Yang-Mills theory can naturally be embedded into a gauge theory in Clifford-spaces based on the $Cl(16)$ group. The details of this will be presented in future work.

The global structure of physics could be described by many copies of E_8, or equivalently by a very large Clifford algebra so that the $Cl(8)$ building blocks are consistently connected with each other so that their 8-dim vector spaces fit together to form a large E_8 lattice. We may recall that the conventional von Neuman Hyperfinite II1 factor is roughly an infinite-dimensional version of the spinor representation of Complex Clifford algebras, which have periodicity 2 and so are like an infinite limit of what Baez calls "... the fermionic Fock space over $C^(2n)$ ..." and then generalize it to the case of Real Clifford Algebras with periodicity 8 so that one gets is an infinite limit of a tensor product of a lot of copies of 256-dim Cl(8).

Each $Cl(8)$ factor would describe physics locally in the neighborhood of a given space-time point, and all the $Cl(8)$ factors in the generalized Hyperfinite II1 factor (roughly an infinite tensor product) would be linked together to form (at the next higher energy level above our quaternionic 4-dim physical spacetime plus 4-dim CP^2 internal symmetry space) a higher-energy real/octonionic 8-dim spacetime as described in [50]. When one takes quantum superpositions in the many-worlds quantum theory, quantum loops/graphs of higher and higher order appear, whose description involves the prime numbers and which may be closely related to the p-adic geometry.

This completes our outline about the gist of Smith's work [50] that emphasize the crucial importance of $Cl(8, R)$ and $D = 8$ to correctly describe the Standard Model and Gravity in 4-dim; in conjunction to explaining how the Exceptional gauge structures appear within the context of Jordan algebras. As stated earlier, a Clifford-space extension of Smith's $D = 8$ Lagrangian (4-2) is provided by eqs-(3-31, 3-32, 3-34) and deserves further investigations.

This concludes our work. One needs to study the full quantum theory and solve the questions about ghosts, unitarity, renormalization, anomalies, asymptotic freedom....of these Clifford (tensorial) space extensions of the Standard Model and whether or not to slide the 10 Tev scale onto the Planck scale if no experimental signals of these new particles are seen at the 10 Tev scale. These are very difficult questions.

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