The Riemann Hypothesis is a consequence of $\mathcal{CT}$-invariant Quantum Mechanics

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Abstract

The Riemann's hypothesis (RH) states that the nontrivial zeros of the Riemann zeta-function are of the form $s_n = 1/2 + i\lambda_n$. By constructing a continuous family of scaling-like operators involving the Gauss-Jacobi theta series and by invoking a novel $\mathcal{CT}$-invariant Quantum Mechanics, involving a judicious charge conjugation $\mathcal{C}$ and time reversal $\mathcal{T}$ operation, we show why the Riemann Hypothesis is true. An infinite family of theta series and their Mellin transform leads to the same conclusions.

1 Introduction

Riemann's outstanding hypothesis that the non-trivial complex zeros of the zeta-function $\zeta(s)$ must be of the form $s_n = 1/2 \pm i\lambda_n$, is one of most important open problems in pure mathematics. The zeta-function has a relation with the number of prime numbers less than a given quantity and the zeros of zeta are deeply connected with the distribution of primes [1]. References [2] are devoted to the mathematical properties of the zeta-function.

The RH has also been studied from the point of view of mathematics and physics [23], [4], [5], [6] among many others. We found recently a novel physical interpretation of the location of the nontrivial Riemann zeta zeros which corresponds to the presence of tachyonic-resonances/tachyonic-condensates in bosonic string theory. If there were zeros outside the critical line violating the RH these zeros do not correspond to poles of the string scattering amplitude [8]. The spectral properties of the λ_n's are associated with the random statistical fluctuations of the energy levels (quantum chaos) of a classical chaotic system [26]. Montgomery [9] has shown that the two-level correlation function of the distribution of the λ_n's coincides with the expression obtained by Dyson with the help of random matrices corresponding to a Gaussian unitary ensemble.

Wu and Sprung [10] have numerically shown that the lower lying non-trivial zeros can be related to the eigenvalues of a Hamiltonian whose potential has a $fractal$ shape and fractal dimension equal to $D = 1.5$. Wu and Sprung have made a very insightful and key remark pertaining the conundrum of constructing a one-dimensional integrable and time-reversal quantum Hamiltonian to model the imaginary parts of the zeros of zeta as an eigenvalue problem. This riddle of merging chaos with integrability is solved by choosing a fractal local potential that captures the chaotic dynamics inherent with the zeta zeros.

In [39] we *generalized* our previous strategy [3] to prove the RH based on extending the Wu and Sprung QM problem by invoking a judicious *superposition* of an infinite family of fractal Weierstrass functions parametrized by the prime numbers p in order to *improve* the expression for the fractal potential. A fractal SUSY QM model whose spectrum furnished the imaginary parts of the zeta zeros λ_n was studied in [3] , [39] based on a Hamiltonian operator that admits a factorization into two factors involving fractional derivative operators whose fractional (irrational) order is one-half of the fractal dimension of the fractal potential. A model of fractal spin has been constructed by Wellington da Cruz [22] in connection to the fractional quantum Hall effect based on the filling factors associated with the Farey fractions. The self-similarity properties of the Farey fractions are widely known to posses remarkable fractal properties [24]. For further details of the validity of the RH based on the Farey fractions and the Franel-Landau [25] shifts we refer to the literature on the zeta function.

In the key section below we start by reviewing our previous work [7] based on a family of scaling-like operators in one dimension involving the Gauss-Jacobi theta series (we also study the case of an infinite parameter family of theta series) where the inner product of their eigenfunctions $\Psi_s(t;l)$ is given by $(2/l)Z\ [\frac{2}{l}(2k-s^*-s)]$, where $Z(s)$ is the fundamental Riemann completed zeta function and $(l+4)/8 = k$. There is a one-to-one correspondence among the zeta zeros s_n (such that $Z[s_n] = 0$ and $\zeta(s_n) = 0$) with the eigenfunctions $\Psi_{s_n}(t;l)$ (of the latter scaling-like operators) which are orthogonal to the "ground" reference state $\Psi_{s_o}(t;l)$; where $s_o = \frac{1}{2} + i0$ is the center of symmetry of the location of the nontrivial zeta zeros . By invoking a novel $\mathcal{CT}$-invariant Quantum Mechanics, involving a judicious charge conjugation $\mathcal{C}$ and time reversal $\mathcal{T}$ operation, we show why the Riemann Hypothesis is true. The key reason why the Riemann hypohesis is true is due to the $\mathcal{CT}$ invariance and that the pseudo-norm $< \Psi_s|\mathcal{CT}|\Psi_s >$ *is not null*. Had the pseudo-norm $< \Psi_s|\mathcal{CT}|\Psi_s >$ been *null*, the RH would have been false.

2 $\mathcal{CT}$- symmetric Quantum Mechanics implies the Riemann Hypothesis

The essence of the proof of the RH relies in the construction of a $\mathcal{CT}$- symmetric Quantum Mechanics which is a novel generalization of the $\mathcal{PT}$-symmetric QM [36] and in establishing a one to one correspondence among the zeta zeros s_n with the states $\Psi_{s_n}(t)$ *orthogonal* to the *ground* (vacuum state) $\Psi_{s_o}(t)$ associated with the center of symmetry $s_o = \frac{1}{2} + i0$ of the non-trivial zeta zeros and corresponding to the fundamental Riemann function obeying the "duality" condition $Z(s) = Z(1-s)$. We shall begin with the construction of the Scaling

Operators related to the Gauss-Jacobi Theta series and the Riemann zeros [7] given by

$$D_1 = -\frac{d}{d\ln t} + \frac{dV}{d\ln t} + k. \tag{2.1}$$

such that its eigenvalues s are complex-valued, and its eigenfunctions are given by

$$\psi_s(t) = t^{-s+k} e^{V(t)}. \tag{2.2}$$

D_1 is not self-adjoint since it is an operator that does not admit an adjoint extension to the whole real line characterized by the *real* variable t. The parameter k is also real-valued. The eigenvalues of D_1 are complex valued numbers s. We also define the operator dual to D_1 as follows,

$$D_2 = \frac{d}{d\ln t} - \frac{dV(1/t)}{d\ln t} + k. \tag{2.3}$$

that is related to D_1 by the substitution $t \to 1/t$ and by noticing that

$$\frac{dV(1/t)}{d\ln(1/t)} = -\frac{dV(1/t)}{d\ln t}. \tag{2.4}$$

where $V(1/t)$ is not equal to $V(t)$.

The eigenfunctions of the D_2 operator are $\Psi_s(\frac{1}{t})$ (with eigenvalue s) which can be shown to be equal to $\Psi_{1-s}(t)$ when $l = 4(2k-1)$ [7] resulting from the properties of the Gauss-Jacobi theta series under the $x \to 1/x$ transformations. Since $V(t)$ can be chosen arbitrarily, we choose it to be related to the Bernoulli string spectral counting function, given by the Jacobi theta series,

$$e^{2V(t)} = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t^l} = 2\omega(t^l) + 1. \tag{2.5}$$

This choice is justified in part by the fact that Jacobi's theta series ω has a deep connection to the integral representations of the Riemann zeta-function [28].

Latter arguments will rely also on the following related function defined by Gauss,

$$G(1/x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} = 2\omega(1/x) + 1. \tag{2.6}$$

where $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. The Gauss-Jacobi series obeys the relation

$$G(\frac{1}{x}) \;=\; \sqrt{x}\; G(x). \tag{2.7}$$

Then, our V is such that $e^{2V(t)} = G(t^l)$. We defined x as t^l. We call $G(x)$ the Gauss-Jacobi theta series (GJ).

Defining $H_A = D_2 D_1$ and $H_B = D_1 D_2$ we were able to show in [7], due to the relation $\Psi_s(1/t) = \Psi_{1-s}(t)$ based on the properties of the Gauss-Jacobi series, $G(\frac{1}{x}) \;=\; \sqrt{x}\; G(x)$ that

$$H_A\ \Psi_s(t) = s(1-s)\Psi_s(t). \quad H_B\ \Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t}). \tag{2.8}$$

Therefore, despite that H_A, H_B are not Hermitian they have the same spectrum $s(1-s)$ which is real-valued only in the critical line and in the real line. Eq-(2.8) is the one-dimensional version of the eigenfunctions of the two-dimensional hyperbolic Laplacian given in terms of the Eisenstein's series. Had H_A, H_B been Hermitian one would have had an immediate proof of the RH. Hermitian operators have a real spectrum, hence if $s(1-s)$ is real this means that $1-s = c\ s^*$, for a real valued c, and $1-s^* = c\ s$. Subtracting :

$$1-s-(1-s^*) = -(s-s^*) = -c\ (s-s^*)\ \Rightarrow\ (s-s^*)\ (1-c) = 0. \tag{2.9}$$

If $c \neq 1$ then one has $s - s^* = 0 \Rightarrow s = real$. And if $c = 1$ then $s - s^* \neq 0$ such that the Imaginary part of s is not zero. Therefore, the condition $1-s = c\ s^*$ for $c = 1$ leads immediately to $s = \frac{1}{2} + i\lambda$. From eq-(2.8) resulting from properties of the Gauss-Jacobi series $G(\frac{1}{x}) = \sqrt{x}\ G(x)$ it follows that under the "time reversal " $\mathcal{T}$ operation $t \rightarrow \frac{1}{t}$ the eigenfunctions $\Psi_s(t)$ behave as

$$\mathcal{T}\ \Psi_s(t)\ =\ \Psi_s(\frac{1}{t})\ =\ \Psi_{1-s}(t). \tag{2.10}$$

and the Hamiltonian operators $H_A = D_2 D_1,\ H_B = D_1 D_2$ transform as

$$\mathcal{T}\ H_B\ \mathcal{T}^{-1}\ =\ H_A, \quad \mathcal{T}\ H_A\ \mathcal{T}^{-1}\ =\ H_B. \tag{2.11}$$

To prove the relations (2.11) one must recur to the properties of the discrete charge and time reversal operations

$$\mathcal{C}^2 = 1\ \Rightarrow \mathcal{C}^{-1} = \mathcal{C}; \quad \mathcal{T}^2 = 1\ \Rightarrow \mathcal{T}^{-1} = \mathcal{T}. \tag{2.12}$$

where the charge conjugation operation is defined by $\mathcal{C}\Psi_s(t) = (\Psi_s(t))^* = \Psi_{s^*}(t)$; i.e. it is defined by taking the complex conjugate of s since the variable t and parameter k appearing in the definition of the eigenfunctions in eq-(2.2) are real. Therefore, upon using eqs-(2.8, 2.12) and by writing

$$\mathcal{T}\ H_B\ \mathcal{T}^{-1}\ \Psi_s(t)\ =\ \mathcal{T}\ H_B\ \mathcal{T}\ \Psi_s(t)\ =\ \mathcal{T}\ H_B\ \Psi_s(\frac{1}{t})\ =$$

$$s(1-s)\ \mathcal{T}\ \Psi_s(\frac{1}{t})\ =\ s(1-s)\ \Psi_s(t)\ =\ H_A\ \Psi_s(t)\ \Rightarrow \tag{2.13a}$$

$$\mathcal{T}\ H_B\ \mathcal{T}^{-1}\ =\ H_A\ \Leftrightarrow\ \mathcal{T}\ H_A\ \mathcal{T}^{-1}\ =\ H_B \tag{2.13b}$$

which follows from (2.13a) since the variable s parametrizing the eigenfunctions $\Psi_s(t)$ span a *continuum* of values. The nontrivial zeta zeros s_n corresponding to $\Psi_{s_n}(t)$ are a discrete subset of the continuum of states $\Psi_s(t)$.

We will show next that if the RH is true, the H_A and H_B operators are invariant under the $\mathcal{CT}$ operation. Afterwards we will prove that the *converse* is also true, namely that if the H_A and H_B operators are invariant under the $\mathcal{CT}$ operation, the RH is true. Let us begin with the first part of the proof which requires to show that if the RH is true one must have

$$\mathcal{CT}\ H_A\ [\mathcal{CT}]^{-1}\ =\ H_A,\quad \mathcal{CT}\ H_B\ [\mathcal{CT}]^{-1}\ =\ H_B. \tag{2.14}$$

which is equivalent to having vanishing commutators

$$[\ \mathcal{CT}, H_A\ ]\ =\ 0,\quad [\ \mathcal{CT}, H_B\ ]\ =\ 0. \tag{2.15}$$

and having simultaneous eigenfunctions $\Psi_s(t),\ \Psi_s(\frac{1}{t})$ of H_A, $\mathcal{CT}$ and H_B, $\mathcal{CT}$, respectively, where the "charge" conjugation operation $\mathcal{C}$ (that takes $s \rightarrow s^*$) is defined by

$$\mathcal{C}\ \Psi_s(t)\ =\ (\Psi_s(t))^*\ =\ \Psi_{s^*}(t),\quad \mathcal{C}\ \Psi_s(\frac{1}{t})\ =\ (\Psi_s(\frac{1}{t}))^*\ =\ \Psi_{s^*}(\frac{1}{t}). \tag{2.16}$$

To prove that H_A and H_B are invariant under the $\mathcal{CT}$ operation (if the RH is true) we must show first that the eigenfunctions $\Psi_s(t)$ and $\Psi_s(\frac{1}{t}) = \Psi_{1-s}(t)$ are $\mathcal{CT}$ and $\mathcal{TC}$ invariant if $1 - s = s^* \Leftrightarrow 1 - s^* = s \Leftrightarrow s = \frac{1}{2} + i\lambda$, i.e. if the RH is true, then the eigenfunctions $\Psi_s(t)$ and $\Psi_s(\frac{1}{t}) = \Psi_{1-s}(t)$ are *invariant* under the $\mathcal{CT}$ operation.

Hence, if $\mathcal{CT}\Psi_s(t)\ =\ \Psi_s(t)$, the eigenfunction is invariant under the action of $\mathcal{CT}$ such that

$$\mathcal{CT}\ \Psi_s(t)\ =\ \mathcal{C}\ \Psi_s(\frac{1}{t})\ =\ \mathcal{C}\ \Psi_{1-s}(t)\ =\ \Psi_{1-s^*}(t) =\ \Psi_s(t)\ \textit{because}\ 1 - s^* = s. \tag{2.17}$$

If $\mathcal{CT}\Psi_s(\frac{1}{t})\ =\ \Psi_s(\frac{1}{t})$, the eigenfunction is invariant under the action of $\mathcal{CT}$ such that

$$\mathcal{CT}\ \Psi_s(\frac{1}{t})\ =\ \mathcal{C}\ \Psi_s(t)\ =\ \Psi_{s^*}(t)\ =\ \Psi_{1-s}(t)\ =\ \Psi_s(\frac{1}{t})\ \textit{because}\ 1 - s = s^*. \tag{2.18}$$

and similarly

$$\mathcal{TC}\ \Psi_s(t)\ =\ \mathcal{T}\ \Psi_{s^*}(t)\ =\ \Psi_{s^*}(\frac{1}{t})\ =\ \Psi_{1-s^*}(t) =\ \Psi_s(t)\ \textit{because}\ 1 - s^* = s. \tag{2.19}$$

$$\mathcal{TC}\ \Psi_s(\frac{1}{t})\ =\ \mathcal{T}\ \Psi_{s^*}(\frac{1}{t})\ =\ \Psi_{s^*}(t)\ =\ \Psi_{1-s}(t)\ =\ \Psi_s(\frac{1}{t})\ \textit{because}\ 1 - s = s^*. \tag{2.20}$$

from eqs- (2.17-2.20) we conclude that if the RH is true, it follows that $1 - s = s^* \Leftrightarrow\ 1 - s^* = s\ \Leftrightarrow s = \frac{1}{2} + i\lambda$, and which implies the $\mathcal{CT}$ *invariance* of the eigenfunctions. (The trivial zeros live in the real line at the location of the negative even integers).

Using the properties $\mathcal{C}^{-1} = \mathcal{C},\ \mathcal{T}^{-1} = \mathcal{T}$ and after having shown the $\mathcal{CT} = \mathcal{TC}$ invariance of the eigenfunctions, if the RH is true, one can write the action on $\Psi_s(t)$ of

$$\mathcal{CT}\ H_A\ [\mathcal{CT}]^{-1}\ \Psi_s(t)\ =\ \mathcal{CT}\ H_A\ \mathcal{TC}\ \Psi_s(t)\ =\ \mathcal{CT}\ H_A\ \Psi_s(t)\ =\ \mathcal{CT}\ s(1-s)\ \Psi_s(t)\ =$$

$$s(1-s)\ \mathcal{CT}\ \Psi_s(t)\ =\ s(1-s)\ \Psi_s(t)\ =\ H_A\ \Psi_s(t). \tag{2.21}$$

where the action of $\mathcal{CT}\ s(1-s)\ \Psi_s(t)$ is assumed to be linear $s(1-s)\ \mathcal{CT}\Psi_s(t)$. If the action was anti-linear one would have $s^*(1-s^*)\ \mathcal{CT}\Psi_s(t)$ instead. If the RH is true then $s^*(1-s^*) = s(1-s)$ and there is no distinction between the linear and anti-linear actions. Since the above eq-(2.21) is valid for *a continuum* of values of s (the nontrivial zeta zeros s_n corresponding to $\Psi_{s_n}(t)$ are a discrete subset of the continuum of states $\Psi_s(t)$) parametrizing the eigenfunctions $\Psi_s(t)$ and given by $s = \frac{1}{2} + i\lambda$, one learns from (2.21) that

$$\mathcal{CT}\ H_A\ [\mathcal{CT}]^{-1}\ =\ H_A. \tag{2.22}$$

and identical results follow for

$$\mathcal{CT}\ H_B\ [\mathcal{CT}]^{-1}\ =\ H_B. \tag{2.23}$$

by acting on $\Psi_s(\frac{1}{t})$. To sum up, we have proved from eqs-(2.21, 2.22, 2.23) that the Hamiltonians H_A, H_B are *invariant* under the $\mathcal{CT}$ operation given by eqs-(2.14, 2.15) as a direct result of the $\mathcal{CT}$ invariance of the eigenfunctions, and which in turn, follows if the RH is true. We need to prove now the *converse*. In the second part of our derivation we will not assume that the RH is true, but instead will assume that the Hamiltonians H_A, H_B are *invariant* under the $\mathcal{CT}$ operation, and from there we will prove the RH.

The invariance of the H_A, H_B operators under $\mathcal{CT}$ implies the vanishing commutators $[H_A, \mathcal{CT}] = [H_B, \mathcal{CT}] = 0$ as expressed by eqs-(2.14, 2.15). When the operators H_A, H_B commute with $\mathcal{CT}$, there exits new eigenfunctions $\Psi_s^{\mathcal{CT}}(t)$ of the H_A operator with eigenvalues $s^*(1-s^*)$. Let us focus only in the H_A operator since similar results follow for the H_B operator. Defining

$$\mid \Psi_s^{\mathcal{CT}}(t)\ >\ \equiv\ \mathcal{CT} \mid \Psi_s(t)\ > . \tag{2.24}$$

one can see that it is also an eigenfunction of H_A with eigenvalue $s^*(1-s^*)$:

$$H_A \mid \Psi_s^{\mathcal{CT}}(t)\ >\ =\ H_A\ \mathcal{CT} \mid \Psi_s(t)\ >\ =\ H_A \mid \Psi_{1-s^*}(t)\ >\ =$$

$$s^*(1-s^*) \mid \Psi_{1-s^*}(t)\ >\ =\ s^*(1-s^*)\ \mathcal{CT} \mid \Psi_s(t)\ >\ =\ (E_s)^* \mid \Psi_s^{\mathcal{CT}}(t)\ > . \tag{2.25}$$

where we have defined $(E_s)^* = s^*(1-s^*)$. Given

$$[H_A, \mathcal{CT}]\ =\ 0\ \Rightarrow\ < \Psi_s \mid [H_A, \mathcal{CT}] \mid \Psi_s >\ =\ 0\ \Rightarrow$$

$$< \Psi_s \mid H_A\ \mathcal{CT} \mid \Psi_s >\ -\ < \Psi_s \mid \mathcal{CT}\ H_A \mid \Psi_s >\ =$$

$$(E_s)^* \ < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ - \ E_s \ < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = $$
$$(E^*_s \ - \ E_s) \ < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = \ 0. \tag{2.26}$$

From (2.26) one has two cases to consider.

• Case A : If the pseudo-norm is null

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = \ 0 \ \Rightarrow \ (E_s \ - \ E^*_s) \ \neq \ 0 \tag{2.27}$$

then the *complex* eigenvalues $E_s = s(1-s)$ and $E^*_s = s^*(1-s^*)$ are *complex conjugates* of each other. In this case the RH would be false and there are quartets of non-trivial Riemann zeta zeros given by $s_n, \ 1 - s_n, \ s^*_n, \ 1 - s^*_n$.

• Case B : If the pseudo-norm is *not* null :

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ \neq \ 0 \ \Rightarrow \ (E_s \ - \ E^*_s) \ = \ 0 \tag{2.28}$$

then the eigenvalues are *real* given by $E_s = s(1-s) = E^*_s = s^*(1-s^*)$ and which implies that $s = real$ (location of the trivial zeta zeros) and/or $s = \frac{1}{2} + i\lambda$ (location of the non-trivial zeta zeros). In this case the RH would be true and the non-trivial Riemann zeta zeros are given by $s_n = \frac{1}{2} + i\lambda_n$ and $1 - s_n = s^*_n = \frac{1}{2} - i\lambda_n$. We are going to prove next why Case A does and cannot occur, therefore the RH is true because we are left with case B.

As stated in the begining of this section, the essence of the proof relies in establishing a one to one correspondence among the zeta zeros s_n with the states $\Psi_{s_n}(t)$ *orthogonal* to the *ground* (vacuum state) $\Psi_{s_o}(t)$ associated with the center of symmetry $s_o = \frac{1}{2} + i0$ of the non-trivial zeta zeros and corresponding to the fundamental Riemann function obeying the "duality" condition $Z(s) = Z(1-s)$. The inner products $< \Psi_{s_o}(t) \mid \Psi_{s_n}(t) > \ = \ Z[s_n] = 0$ fix the location of the nontrivial zeta zeros s_n since $Z[s]$ is proportional to $\zeta(s)$ as we show next.

We have to consider a family of D_1 operators, each characterized by two real numbers k and l which can be chosen arbitrarily. The measure of integration $d\ln t$ is scale invariant. Let us mention that D_1 is also invariant under scale transformations of t and $F = e^V$ since $dV/(d\ln t) = d\ln F/(d\ln t)$. In [31] only one operator D_1 is introduced with the number $k = 0$ and a different (from ours) definition of F.

We define the inner product as follows,

$$\langle f|g\rangle = \int_0^\infty f^* g \frac{dt}{t}.$$

Based on this definition the inner product of two eigenfunctions of D_1 is

$$\begin{aligned} \langle \psi_{s_1}|\psi_{s_2}\rangle \ &= \int_0^\infty e^{2V} t^{-s_{12}+2k-1} dt \\ &= \frac{2}{l} Z\left[\frac{2}{l}(2k - s_{12})\right], \end{aligned} \tag{2.29}$$

where we have denoted $s_{12} = s_1^* + s_2 = x_1 + x_2 + i(y_2 - y_1)$ and used the expressions for the Gauss-Jacobi theta function and the definition of the fundamental Riemann function $Z[s]$ resulting from the Mellin transform as shown below in eq-(2.31).

We notice that

$$\langle \psi_{s_1} | \psi_{s_2} \rangle = \langle \psi_{s_o} | \psi_s \rangle, . \tag{2.30}$$

thus, the inner product of ψ_{s_1} and ψ_{s_2} is equivalent to the inner product of ψ_{s_o} and ψ_s, where $s_o = 1/2 + i0$ and $s = s_{12} - 1/2$. The integral is evaluated by introducing a change of variables $t^l = x$ (which gives $dt/t = (1/l)dx/x$) and using the result provided by the Gauss-Jacobi Theta given in Karatsuba and Voronin's book [2]. The fundamental Riemann function $Z[s]$ in eq-(2.29) can be expressed in terms of the Jacobi theta series, $\omega(x)$ defined by eqs-(2.5, 2.6) as

$$\begin{aligned}\int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 x} x^{s/2-1} dx &= \\ &\int_0^\infty x^{s/2-1}\omega(x) dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty \left[x^{s/2-1} + x^{(1-s)/2-1}\right] \omega(x) dx \\ &= Z(s) = Z(1-s),\end{aligned} \tag{2.31}$$

where the fundamental Riemann (completed zeta) function is

$$Z(s) \equiv \pi^{-s/2}\ \Gamma(\frac{s}{2})\ \zeta(s), . \tag{2.32}$$

which obeys the functional relation $Z(s) = Z(1-s)$.

Since the right-hand side of (2.31) is defined for all s this expression gives the analytic continuation of the function $Z(s)$ to the entire complex s-plane [2]. In this sense the fourth "=" in (2.31) is not a genuine equality. Such an analytic continuation transforms this expression into the inner product, defined by (2.29).

A recently published report by Elizalde, Moretti and Zerbini [11] (containing comments about the first version of our paper [7]) considers in detail the consequences of the analytic continuation implied by equation (2.31). One of the consequences is that equation (2.29) loses the meaning of being a scalar product. Arguments by Elizalde *et al.* [11] show that the construction of a genuine inner product is impossible.

Therefore from now on we will loosely speak of a "scalar product" realizing that we do not have a scalar product as such. The crucial problem is whether there are zeros outside the critical line (but still inside the critical strip) and not the interpretation of equation (2.29) as a genuine inner product. Despite this, we still rather loosely refer to this mapping as a scalar product. The states still have a real norm squared, which however need not to be positive-definite.

Here we must emphasize that our arguments do not rely on the validity of the zeta-function regularization procedure [12], which precludes a rigorous interpretation of the right hand side of (2.31) as a scalar product. Instead, we can simply replace the expression "scalar product of ψ_{s_1} and ψ_{s_2}" by the map S of complex numbers defined as

$$\begin{aligned} S: \quad \mathcal{C}\otimes\mathcal{C} \quad &\rightarrow \quad \mathcal{C} \\ (s_1,s_2) \quad &\mapsto \quad S(s_1,s_2)=\frac{2}{l}Z(as+b), \end{aligned} \quad . \tag{2.33}$$

where $s = s_1^* + s_2 - 1/2$ and $a = -2/l; b = (4k-1)/l$. In other words, our arguments do not rely on an evaluation of the integral $\langle\psi_{s_1}|\psi_{s_2}\rangle$, but only on the mapping $S(s_1,s_2)$, defined as the finite part of the integral (2.29). The kernel of the map $S(s_1,s_2) = \frac{2}{l}Z(as+b)$ is given by the values of s such that $Z(as+b)=0$, where $\langle\psi_{s_1}|\psi_{s_2}\rangle = \langle\psi_{s_o}|\psi_s\rangle$ and $s_o = 1/2 + i0$. Notice that $2b+a = 4(2k-1)/l$. We only need to study the "orthogonality" (and symmetry) conditions with respect to the "vacuum" state s_o to prove why $a+2b=1$. By symmetries of the "orthogonal" states to the "vacuum" we mean always the symmetries of the kernel of the S map.

The "inner" products are trivially divergent due to the contribution of the $n=0$ term (the zero modes) of the GJ theta series in the integral (2.29). From now on, we denote for "inner" product in (2.29) as *the finite part* of the integrals by simply removing the trivial infinity. We shall see in the next paragraphs, that this "additive" regularization is in fact compatible with the symmetries of the problem.

For example, the $n=0$ (zero modes) term yields a divergence in the integral (2.29), when one evaluates the inner product of the states $< \Psi_s(t)|\Psi_{1-s^*}(t) >$ (at the end of this work we will see why we select this particular pair of states), given by the limits $lim_{t\rightarrow\infty}\ t^{2k-1}/2k-1 \rightarrow \infty$ when $2k-1>0$; $lim_{t\rightarrow 0}\ -t^{2k-1}/2k-1 \rightarrow \infty$ when $2k-1<0$; and $lim_{t\rightarrow\infty}\ 2ln[t] \rightarrow \infty$ when $2k-1=0$. Therefore, *the finite part* of the integral (2.29) will be our definition of the inner product, and after recurring to the Mellin transform of the Jacobi series $\omega(x)$ (2.31), yields the sought after result $\frac{2}{l}Z[\ \frac{2}{l}(2k-(s_1^*+s_2))\ ]$ in the r.h.s of (2.29) (where $l \neq \pm\infty$). A thorough discussion of the regularization of the integrals (2.29) can be found in H.M. Edwards book [2] (chapter 10) [1]. Another regularization of the integrals (2.29) could be obtained by recurring to a complex contour encircling $t=0$ and $t=\infty$. Complex contours have been used recently to define inner products in $\mathcal{PT}$-invariant QM in a Krein Space by [37]. At the end of this work we propose another family of theta series where *no* regularization is needed in the construction of the inner products.

We can easily show that if a and b are such that $2b+a=1$, then the symmetries of all the states ψ_s orthogonal to the "vacuum" state are preserved

[1]We thank M. Rios for reminding us

by any map S, equation (2.33), which leads to $\frac{2}{l}Z(as+b)$. In fact, if the state associated with the complex number $s = x+iy$ is orthogonal to the "vacuum" state and the "scalar product" is given by $\frac{2}{l}Z(as+b) = \frac{2}{l}Z(s')$, then the Riemann zeta-function has zeros at $s' = x'+iy'$, s'^*, $1-s'$ and $1-s'^*$. If we equate $as+b = s'$, then $as^*+b = s'^*$. Now, $1-s'$ will be equal to $a(1-s)+b$, and $1-s'^*$ will be equal to $a(1-s^*)+b$, if, and only if, $2b+a = 1$. Therefore, all the states ψ_s orthogonal to the "vacuum" state, parameterized by the complex number $1/2+i0$, will then have the same symmetry properties with respect to the critical line as the nontrivial zeros of zeta.

Notice that our choice of $a = -2/l$ and $b = (4k-1)/l$ is compatible with this symmetry if k and l are related by $l = 4(2k-1)$. Conversely, if we assume that the orthogonal states to the "vacuum" state have the same symmetries of $Z(s)$, then a and b must be constrained to obey $2b+a = 1$. It is clear that a map with arbitrary values of a and b does not preserve the above symmetries and for this reason we have now that $s' = as+b = a(s-1/2)+1/2$

Therefore, concluding, the inner product $\langle\Psi_{s_1}|\Psi_{s_2}\rangle$ is equal to $\langle\Psi_{s_o}|\Psi_s\rangle = \frac{2}{l}\ Z[a(s-1/2)+1/2] = \frac{2}{l}Z(s')$ where $s = s_1^*+s_2-1/2$. For example, if we set the particular value $l = -2$, then $k = \frac{1}{4}$, $a = 1$, $b = 0$, and consequently $s' = s$ in this case such that the position of the zeros $(s_n)' = s_n$ have a one to one correspondence with the location of the orthogonal states Ψ_{s_n} to the reference state Ψ_{s_o} and we can finally write $<\Psi_{s_o}|\Psi_{s_n}> = -Z[s_n] = 0$ as announced.

The inner product of two states is

$$\langle\Psi_{s_1}|\Psi_{s_2}\rangle \ = \ (\frac{2}{l})\ Z\ [\ \frac{2}{l}(2k-(s_1^*+s_2))\ ]. \tag{2.34}$$

and one finds that the states Ψ_s when s lies in the critical line $s = \frac{1}{2}+i\lambda$ have equal norm

$$<\Psi_s\ |\Psi_s> \ = \ \frac{2}{l}\ Z\ [\frac{1}{2}]\ \neq\ 0, \quad (l \neq \pm\infty). \tag{2.35}$$

In general, the norm of the states is not proportional to $Z[\frac{1}{2}]$. Only when $s = \frac{1}{2}+i\lambda$.

It is very important to emphasize that having a discrete family of orthogonal states $\Psi_{s_n}(t)$ to the reference ground state $\Psi_{s_o}(t)$ ($s_o = \frac{1}{2}+i0$) does *not* mean that these states are orthogonal among themselves. For example, when $l = -2; k = \frac{1}{4}$, one has $<\Psi_{s_o}|\Psi_{s_n}> = -Z[s_n] = 0; <\Psi_{s_o}|\Psi_{s_m}> = -Z[s_m] = 0$ but $<\Psi_{s_m}|\Psi_{s_n}> = -Z[s_m^*+s_n-\frac{1}{2}] \neq 0$. The procedure how to construct a discrete ortho-normal basis of states was outlined in [39].

Since $Z[\frac{1}{2}] < 0$ in order to have a positive definite norm one requires to choose $l < 0$. $Z(s)$ is real-valued along the critical line because when $s = \frac{1}{2}+i\lambda \Rightarrow\ 1-s = s^*$ then as a result of the functional equation one must have $Z(s) = Z(1-s) = Z(s^*) = (Z(s))^*$ which implies that $Z(1/2+i\lambda) = real$.

After this lengthy discussion, we are ready now to study cases A and B in eqs-(2.27, 2.28) respectively. From the explicit form of eq-(2.34) depicting the inner product of two arbitrary states , by choosing for example that $l = -2 \Rightarrow k = \frac{1}{4}$, one concludes that the pseudo-norm

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = \ < \Psi_s \mid\mid \Psi_{1-s^*} > \ = \ -Z[\ -(\frac{1}{2} - (s^* + 1 - s^*))\ ] \ =$$

$$-Z[\frac{1}{2}] \ \neq 0. \tag{2.36}$$

and consequently case **A** of eq-(2.27) is ruled out and case **B** of eq-(2.28) stands. Concluding, since the pseudo-norm (2.36) is *not null* this implies that the eigenvalues E_s, E_s^* obey eq-(2.28) and are *real*-valued $E_s = s(1-s) = E_s^* = s^*(1-s^*)$ which means that the Riemann Hypothesis *is true*.

The results of eq-(2.36) and conclusions remain the same for other choices of the parameters l, k so far as l, k are constrained to obey the condition $l = 4(2k-1) \Leftrightarrow a + 2b = 1$ imposed from the symmetry considerations since the orthogonal states $\Psi_{s_n}(t)$ to the reference state $\Psi_{s_o}(t)$ must obey the same symmetry conditions with respect to the critical line and real line as the non-trivial zeta zeros :

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = \ < \Psi_s \mid\mid \Psi_{1-s^*} > \ = \ \frac{2}{l}\ Z[\ \frac{2}{l}(2k - (s^* + 1 - s^*))\ ] \ =$$

$$\frac{2}{l}\ Z[\frac{2}{l}(2k-1)] \ = \frac{2}{l}\ Z[\frac{1}{2}] \ \neq 0, \quad (l \neq \pm\infty). \tag{2.37}$$

as a result of $l = 4(2k-1)$. Therefore, if $\mathcal{CT}$ invariance holds and the pseudo-norm $< \Psi_s | \mathcal{CT} | \Psi_s >$ *is not null*, the RH is true. Since de La Vallee-Poussin-Hadamard's theorem rules out zeta zeros at the boundaries of the critical strip $s = 0 + i\lambda,\ s = 1 + i\lambda$, in our above discussion one may restrict the domain of values of s to lie inside the critical strip $0 < Real\ s < 1$.

At the end of this work we will have an infinite family of H_A, H_B operators associated with an infinite family of potentials $V_{jm}(t)$ corresponding to an infinite family of theta series with the advantage that *no* regularization of the inner products is necessary. Another salient feature is that the the pseudo-norm $< \Psi_s^{jm} |\ \mathcal{CT}\ | \Psi_s^{jm} >$ is *not* null (see below) as result that the zeta function $\zeta(s)$ has *no* zeros at $s = \frac{1}{2} - 2m,\ m = 1, 2, 3, ..., \infty$. The relevance of the behavior of $\zeta(\frac{1}{2} - 2m) \neq 0,\ m = 1, 2, 3, ..., \infty$ is that it automatically avoids looking at the behavior of zeta at $s = 1/2$. Armitage [15] has found a zeta function $\zeta_L(s)$ defined over the *algebraic* number field L that has a zero at $s = 1/2$ and presumably satisfies the RH . This finding would not be compatible with the result of eq-(2.37) and which was based on a regularized inner product. Therefore, a well defined inner product that leads to the result (see below) $< \Psi_s^{jm} |\ \mathcal{CT}\ | \Psi_s^{jm} > \ \sim\ \zeta(\frac{1}{2} - 2m) \neq 0,\ m = 1, 2, 3, ..., \infty$ is no longer in variance with the behaviour of the zeta function $\zeta_L(s)$ defined over the *algebraic* number field L that has a zero at $s = 1/2$ [15].

Identical results follow if we had defined a new family of potentials $V_{2j}(t)$ in terms of a *weighted* theta series $\Theta_{2j}(t)$ and whose Mellin transform yields the infinite family of extended zeta functions of Keating [40] and their associated completed zeta functions as shown by Coffey [41]. The Hermite polynomials weighted theta series associated to $2j = even$ degree polynomials are defined by

$$e^{2V_{2j}(t)} \;=\; \Theta_{2j}(t) \;\equiv\; \sum_{n=-\infty}^{n=\infty} (8\pi)^{-j}\; H_{2j}(n\sqrt{2\pi t})\; e^{-\pi n^2 t}. \tag{2.38}$$

and are related to the potentials $V_{2j}(t)$ which appear in the definitions of the differential operators (2.1, 2.2). The weighted theta series obeys the relation

$$\frac{(-1)^j}{\sqrt{t}}\; \Theta_{2j}(\frac{1}{t}) \;=\; \Theta_{2j}(t). \tag{2.39}$$

Only when $j = even$ in (2.39) one can implement $\mathcal{CT}$ invariance to the new family of Hamiltonians H_A, H_B associated with the potentials $V_{2j}(t)$ of (2.38) because $H_A\Psi_s(t) = s(1-s)\Psi(t)$ and $H_B\Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t})$ would only be valid when $j = even$ as a result of the relations (2.1, 2.2, 2.3) and (2.38, 2.39).

The Mellin transform based on the weighted $\Theta_{2j}(t)$ [41] requires once again to *extract* the zero mode $n = 0$ contribution of $\Theta_{2j}(t)$ (to *regularize* the divergent integrals) in order to arrive at

$$\int_0^\infty \frac{1}{2}[\,\Theta_{2j}(t) - (8\pi)^{-2j}\; H_{2j}(0)\,]\; t^{s/2-1}\; dt \;=\; P_j(s)\,\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s),\; Re\; s > 0. \tag{2.40}$$

in the definition of the (regularized) inner products of eigen-states (2.29) associated to the new potentials (2.38). The polynomial pre-factor in front of the completed Riemann zeta $Z(s) = \pi^{-s/2}\;\Gamma(\frac{s}{2})\;\zeta(s)$ is given in terms of a terminating Hypergeometric series [41]

$$P_j(s) = (8\pi)^{-j}(-1)^j\frac{(2j)!}{j!}\;{}_2F_1(-j, \frac{s}{2}; \frac{1}{2}; 2). \tag{2.41}$$

The orthogonal states $\Psi_{s_n}(t)$ to the ground state $\Psi_{s_o}(t)$ ($s_o = \frac{1}{2} + i0$) will now be enlarged to include the nontrivial zeta zeros and the zeros of the polynomial $P_j(s)$.

The polynomial $P_j(s)$ has simple zeros on the critical line $Re\; s = \frac{1}{2}$, obeys the functional relation $P_j(s) = (-1)^j P_j(1-s)$ and in particular $P_j(s = \frac{1}{2}) = 0$ when $j = odd$ [41]. It is only when $j = even$ that $P_j(s = \frac{1}{2}) \neq 0$ and when we can implement $\mathcal{CT}$ invariance resulting from the relation (2.39) and which is consistent with the results of eqs-(2.36, 2.37). Consequently, if one demands $\mathcal{CT}$ invariance one arrives at the same conclusions as before when $j = even$; i.e. the eigenvalues $E_s = s(1-s)$ are real in the $j = even$ case due to the condition $P_j(s = \frac{1}{2}) \neq 0$. Thus, the RH is consistent with $\mathcal{CT}$ invariance. In the $j = odd$ case, one cannot implement $\mathcal{CT}$ invariance and $P_j(s = \frac{1}{2}) = 0$.

Finally, we propose another family of theta series where *no* regularization is needed in the construction of the inner products. There is a two-parameter family of theta series $\Theta_{2j,2m}(t)$ that in principle could yield well defined inner products *without* the need to extract the zero mode $n = 0$ divergent contribution. Given

$$e^{2V_{2j,2m}(t)} \;=\; \Theta_{2j,2m}(t) \;\equiv\; \sum_{n=-\infty}^{n=\infty} n^{2m}\; H_{2j}(n\sqrt{2\pi t})\; e^{-\pi n^2 t}. \qquad (2.42)$$

when $m \neq 0$, the zero mode $n = 0$ does *not* contribute to the sum and the Mellin transform of $\Theta_{2j,2m}(t)$, after exploiting the symmetry of the even-degree Hermite polynomials, is [40], [41]

$$\int_0^\infty \;[\; 2 \sum_{n=1}^{n=\infty} n^{2m}\; H_{2j}(n\sqrt{2\pi t})\; e^{-\pi n^2 t}\;]\; t^{s/2-1}\; dt \;=$$

$$2\; (8\pi)^j\; P_j(s)\; \pi^{-s/2}\; \Gamma(\frac{s}{2})\; \zeta(s-2m); \quad Re\ s\; >\; 1+2m, \quad m = 1, 2, ... \quad (2.43)$$

The problem is to find the analytical continuation of the Mellin transform (2.43) for *all* values of s in the complex plane and to verify that $\Theta_{2j,2m}(\frac{1}{t})$ has the same behaviour as $\Theta_{2j}(\frac{1}{t})$ in eq-(2.39) compatible with the $\mathcal{CT}$ invariance when $j = even$. The analytical continuation of $\zeta(s)$ was found by Remann in his celebrated paper. A Poisson re-summation formula should lead to a relation similar to (2.39). If these two conditions are met we would have at our disposal a well defined inner product of the states $\Psi_s(t)$ (without the need to regularize it by extracting out the zero $n = 0$ mode of the theta series). In particular the inner product of the states $\Psi_s(t)$ with the *shifted* "ground" state $\Psi_{\frac{1}{2}+2m}(t),\ m = 1, 2, ...$ corresponding to the potentials in (2.42), by recurring to the result (2.43) and following similar steps as in (2.29) is

$$<\; \Psi_{\frac{1}{2}+2m}(t) \mid \Psi_s(t)\; > \;=\; -\; 2\; (8\pi)^j\; P_j(s+2m)\; \pi^{-(s+2m)/2}\; \Gamma(\frac{s+2m}{2})\; \zeta(s). \qquad (2.44)$$

this result requires *fixing* uniquely the values $l = -2; k = \frac{1}{4}$. The nontrivial zeta zeros s_n would correspond to the states $\Psi_{s_n}(t)$ orthogonal to the shifted "ground" state $\Psi_{\frac{1}{2}+2m}(t)$.

It remains to prove when $l = -2, k = \frac{1}{4}$ and $s_{12} = s_1^* + s_2 = s_1^* + (1 - s_1^*) = 1$ that

$$<\Psi_s \mid \mathcal{CT} \mid \Psi_s> \;=\; <\Psi_s \mid\mid \Psi_{1-s^*}> \;=$$

$$\int_0^\infty \;[\; 2 \sum_{n=1}^{n=\infty} n^{2m}\; H_{2j}(n\sqrt{2\pi t})\; e^{-\pi n^2 t}\;]\; t^{\frac{2(-s_{12}+2k)}{2l}-1}\; dt \;=$$

$$-\; 2\; (8\pi)^j\; P_j(s = \frac{1}{2})\; \pi^{-1/4}\; \Gamma(\frac{1}{4})\; \zeta(\frac{1}{2} - 2m)\; \neq 0; \quad j = even,\ m = 1, 2, 3, ... \qquad (2.45)$$

Hence, one would arrive at a definite solid conclusion based on a well defined inner product : the RH is true because $\zeta(\frac{1}{2} - 2m) \neq 0$ when $m = 1, 2, ...$

and $P_j(\frac{1}{2}) \neq 0$ when $j = even$. This finding can be inferred from the nonzero pseudo-norm $< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ \neq 0$ in (2.45) and upon following our previous arguments as in (2.36, 2.37) that rule out case **A** , single out case **B**, and that leads to $E_s = s(1-s) = real \ \Rightarrow s = \frac{1}{2} + i\lambda$ (and/or s = real). Consequently the RH is true if, and only if, $\mathcal{CT}$ invariance holds. The key reason why the Riemann hypohesis is true is due to the $\mathcal{CT}$ invariance and that the pseudo-norm $< \Psi_s|\mathcal{CT}|\Psi_s >$ *is not null*. Had the pseudo-norm $< \Psi_s|\mathcal{CT}|\Psi_s >$ been *null*, the RH would have been false. It remains to be seen whether our procedure is valid to prove the grand-Riemann Hypothesis associated to the L-functions.

For applications of the the Quantum Mellin transform see [42]. For a recent interesting study of the Riemann zeta function and the construction of a pseudo-differential operator related to the zeta function, its quantization and many physical applications, see [38] . For related topics to the Riemann zeta function, number theory, fractals, supersymmetry, strings, random matrix models, fractal statistics, ...we refer to [4], [5], [6], [8], [9], [13], [14], [16], [17], [18], [19], [20], [21], [22], [24], [23], [25], [26] J, [27], [28], [29], [30], [31], [32], [33], [34], [41], [43].

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