The Clifford Space Geometry behind the Pioneer and Flyby Anomalies

Carlos Castro
Department of Physics
Texas Southern University, Houston, Texas. 77004

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Abstract

It is rigorously shown how the Extended Relativity Theory in Clifford spaces (C-spaces) can explain the variable radial dependence $a_p(r)$ of the Pioneer anomaly; its sign (pointing towards the sun); why planets $don't$ experience the anomalous acceleration and why the present day value of the Hubble scale R_H appears. It *is* the curvature-spin coupling of the planetary motions that hold the key. The $difference$ in the rate at which clocks tick in C-space translates into the C-space analog of Doppler shifts which may explain the anomalous redshifts in Cosmology, where objects which are *not* that far apart from each other exhibit very different redshifts. We conclude by showing how the empirical formula for the Flybys anomalies obtained by Anderson et al [10] can be derived within the framework of Clifford geometry.

Keywords: Extended Relativity in Clifford Spaces, Clifford Algebras, Pioneer and Flybys Anomaly.

1 Introduction : Weyl Geometry and Pioneer Anomaly

One of the unsolved problems in physics is what causes the apparent residual sunward acceleration of the Pioneer spacecraft [3] and why planets are not subjected to it. Many proposals have been presented by several authors, see [3] and references therein. Another unsolved problem which might be related to the Pioneer anomaly is what causes the unexpected change in acceleration for Earth flybys of spacecraft resulting in an unexpected energy increase [4]. The purpose of this work is to show how the Extended Relativity Theories in Clifford spaces C-spaces [12], [11] might solve satisfactory these problems.

It was recently argued [1] how Weyl's geometry and Mach's principle furnishes both the *magnitude* and *sign* (towards the sun) of the Pioneer anomalous acceleration firstly observed by Anderson et al. Weyl's Geometry can account for *both* the *origins* and the *value* of the observed vacuum energy density (dark energy). The source of dark energy was just the dilaton-like Jordan-Brans-Dicke scalar field ϕ that is required to implement Weyl invariance of the most *simple* of all possible actions. A *nonvanishing* value of the vacuum energy density of the order of $10^{-123} M^4_{Planck}$ was found consistent with observations. Weyl's geometry accounts also for the phantom scalar field in modern Cosmology in a very natural fashion.

The starting action was the Weyl-invariant Jordan-Brans-Dicke-like action involving the scalar ϕ field and the scalar Weyl curvature $\mathcal{R}_{Weyl}$

$$S[g_{\mu\nu}, A_\mu, \phi] \ = \ S[g'_{\mu\nu}, A'_\mu, \phi'] \ \Rightarrow$$

$$\frac{1}{16\pi} \int d^4x \ \sqrt{|g|} \ [\ \phi^2 \ \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) \ - \ \frac{1}{2} g^{\mu\nu} \ (D_\mu\phi)(D_\nu\phi) \ - \ V(\phi) \ ] \ =$$

$$\frac{1}{16\pi} \int d^4x \ \sqrt{|g'|} \ [\ (\phi')^2 \ \mathcal{R}'_{Weyl}(g'_{\mu\nu}, A'_\mu) \ - \ \frac{1}{2} g'^{\mu\nu} \ (D'_\mu\phi')(D'_\nu\phi') \ - \ V(\phi') \ ] \tag{1.1}$$

where under Wey scalings one has

$$\phi' \ = \ e^{-\Omega} \ \phi; \ \ g'_{\mu\nu} \ = \ e^{2\Omega} \ g_{\mu\nu}; \ \ \mathcal{R}'_{Weyl} \ = \ e^{-2\Omega} \ \mathcal{R}_{Weyl}; \ \ V(\phi') \ = \ e^{-4\Omega} \ V(\phi)$$

$$\sqrt{|g'|} \ = \ e^{4\Omega} \ \sqrt{|g|}; \ \ D'_\mu\phi' \ = \ e^{-\Omega} \ D_\mu\phi; \ \ A'_\mu \ = \ A_\mu \ - \partial_\mu\Omega. \tag{1.2}$$

The effective Newtonian coupling G is defined as $\phi^{-2} = G(\phi)$, it is spacetime dependent in general and has a Weyl weight equal to 2. Despite that one has *not* introduced any explicit dynamics to the A_μ field (there are no $F_{\mu\nu}F^{\mu\nu}$ terms in the action (1.1)) one still has the *constraint* equation obtained from the variation of the action w.r.t to the A^μ field and which leads to the pure-gauge configurations provided $\phi \neq 0$

$$\frac{\delta S}{\delta A^\mu} \ = \ 0 \ \Rightarrow 6 \ (\ 2 \ A_\mu\phi^2 \ - \ \partial_\mu(\phi^2) \ ) \ + \ \frac{1}{2} \ (\ 2 \ A_\mu\phi^2 \ - \ \partial_\mu(\phi)^2 \ ) \ =$$

$$-(6 + \frac{1}{2}) \ D_\mu \ \phi^2 \ = \ - \ 2 \ (6 + \frac{1}{2}) \ \phi \ D_\mu \ \phi \ = \ 0 \ \Rightarrow \ A_\mu \ = \ \partial_\mu \ log \ (\phi). \tag{1.3}$$

Hence, a variation of the action w.r.t the A_μ field leads to the pure gauge solutions (1.3) which is tantamount to saying that the scalar ϕ is Weyl-covariantly *constant* $D_\mu = 0$ in any gauge $D_\mu\phi = 0 \rightarrow e^{-\Omega} D_\mu\phi = D'_\mu\phi' = 0$ (for non-singular gauge functions $\Omega \neq \pm\infty$). Therefore, the scalar ϕ does not have true local dynamical degrees of freedom from the Weyl spacetime perspective. Since the gauge field is a total derivative, under a local gauge transformation with gauge function $\Omega = log \ \phi$, one can gauge away (locally) the gauge field and have $A'_\mu = 0$ in the new gauge. Globally, however, this may not be the case because there may be *topological* obstructions. Therefore, the last constraint equation (10) in the gauge $A'_\mu = 0$, forces $\partial_\mu \phi' = 0 \ \Rightarrow \phi' = \phi_o = constant$. Consequently $G' = \phi'^{-2}$ is also constrained to a constant G_N and one may set $G_N \ \phi_o^2 = 1$, where G_N is the observed Newtonian constant today.

The pure-gauge configurations leads to the Weyl integrability condition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$ when $A_\mu = \partial_\mu \Omega$, and means physically that if we parallel transport a vector under a closed loop, as we come back to the starting point, the *norm* of the vector has not changed; i.e, the rate at which a clock ticks does not change after being transported along a closed loop back to the initial point; and if we transport a clock from A to B along different paths, the clocks will tick at the same rate upon arrival at the same point B. This will ensure, for example, that the observed spectral lines of identical atoms will not change when the atoms arrive at the laboratory after taking different paths (histories) from their coincident starting point. If $F_{\mu\nu} \neq 0$ the Weyl geometry is no longer integrable.

With the Weyl-invariant action (1.1) at hand one found a realization of dark energy (the observed cosmological constant) as it was shown in [1]. The cosmological gauge A_μ in spherical coordinates is *defined* by

$$A_r \ = -\frac{1}{R_{Hubble}}; \ A_t \ = \ A_\varphi \ = \ A_\theta \ = \ 0. \tag{1.4}$$

and is associated with the present day Hubble scale $R_{Hubble} \sim 10^{28} \ cm$. The other gauge is the Einstein gauge

$$A'_\mu \ = \ 0 \ = \ A_\mu \ - \partial_\mu \Omega \ \Rightarrow \ A_r \ = \ - \frac{1}{R_H} \ = \partial_r \Omega \ \Rightarrow \ \Omega \ = \ - \frac{r}{R_H}. \tag{1.5}$$

From eq- (1.3) we learned that

$$A_\mu \ = \ \partial_\mu \ log \ \phi \ \Rightarrow \ A_r \ = \ - \frac{1}{R_H} \ \Rightarrow \phi \ = \ e^{-r/R_H} \ \phi_o. \tag{1.6}$$

such that the Newtonian couplings in the two different gauges "scale-frames of reference" are related as follows

$$\frac{\phi^2}{\phi_o^2} \ = \ \frac{G_N}{G(\phi)} \ \Rightarrow G(\phi) \ = \ G_N \ e^{2r/R_H}. \tag{1.7}$$

the effective Newtonian coupling in the cosmological gauge (cosmological "scale-frame of reference") *increases* with distance . In the Einstein gauge $A'_\mu = 0$, using the Weyl covariant constraint of eq-(1.3) stating that the scalar field ϕ is Weyl-covariantly constant (without true dynamics) and for non-singular gauge functions $\Omega \neq \pm\infty$, one can deduce that

$$D'_\mu \phi' \ = \ \partial_\mu \phi' \ - A'_\mu \phi' \ = \ \partial_\mu \phi' \ = \ e^{-\Omega} \ D_\mu \phi \ = 0 \ \Rightarrow \ \phi' = \phi_o. \tag{1.8}$$

Hence, the action (1.1) in the gauge $A'_\mu = 0 \Leftrightarrow \phi' = \phi_o = constant$ becomes

$$\frac{1}{16\pi} \int d^4x \sqrt{|g'|} \; [\; (\phi_o)^2 \; R_{Riemann}(g'_{\mu\nu}) \; - \; V(\phi_o) \;] \tag{1.9}$$

which is just the ordinary Einstein-Hilbert action with a *cosmological constant* Λ given by $2\Lambda \equiv G_N V(\phi_o)$ because $\phi_o^2 = 1/G_N$. The equations of motion associated with the action (1.9) are

$$R'_{\mu\nu} \; - \; \frac{1}{2} \; g'_{\mu\nu} \; R' \; + \; \Lambda \; g'_{\mu\nu} \; = \; 0. \tag{1.10}$$

and which admit the static spherically symmetric solutions corresponding to (Anti) de Sitter-Schwarzschild metrics

$$ds^2 \; = \; - \; (1 - \frac{2G_N M}{r} - \frac{\Lambda}{3} \; r^2) \; dt^2 + (1 - \frac{2G_N M}{r} - \frac{\Lambda}{3} \; r^2)^{-1} \; dr^2 + r^2 \; (sin^2\theta \; d\varphi^2 + d\theta^2 \;). \tag{1.11}$$

The metric solutions in the cosmological gauge $A_r = -\frac{1}{R_H}$ are simply obtained by a conformal transformation

$$g_{\mu\nu} \; = \; e^{-2\Omega} \; g'_{\mu\nu} \; \Rightarrow \; g_{tt} \; = \; e^{-2\Omega} \; g'_{tt} \; = \; - \; e^{2r/R_H} \; (1 - \frac{2G_N M}{r} \; - \; \frac{\Lambda}{3} \; r^2), \; etc \; .... \tag{1.12}$$

After this discussion we turn finally to the Pioneer anomaly. Upon expanding the exponential conformal factor of (1.12) in a power series yields

$$-g_{tt} \; = \; (\; 1 \; + \; \frac{2 \; r}{R_H} \; + \; \frac{1}{2} \; (\frac{2 \; r}{R_H})^2 + ...) \; (1 \; - \; \frac{2G_N M}{r} \; - \; \frac{\Lambda}{3} \; r^2) \; =$$

$$1 - \frac{2G_N M}{r} \; - \frac{\Lambda}{3} \; r^2 \; + \; \frac{2r}{R_H} \; - \frac{4G_N M}{R_H} \; - \frac{2 \; \Lambda \; r^3}{3 \; R_H} \; + \; .... \tag{1.13}$$

For scales $r << R_H$ corresponding to the Pioneer-Sun's distance one may neglect the higher order corrections in the expansion. From the g_{tt} component one can read-off the corrections to the Newtonian potential in natural units $c = 1$ from the Newtonian limit of Einstein's gravity : $-g_{tt} \sim 1 + 2V$ leading to

$$V_{effective}(r) \; = \; - \; \frac{G_N M}{r} \; - \; \frac{\Lambda}{6} \; r^2 \; + \; \frac{r}{R_H} \; - \; \frac{2G_N M}{R_H} \; - \; \frac{\Lambda \; r^3}{3 \; R_H} + ... \tag{1.14}$$

Therefore the acceleration (radial force per unit mass) acting on the Pioneer spacecraft after reinserting the speed of light c in its proper units and by setting $\Lambda = 3/R_H^2$ is given by

$$\frac{F_r}{m} \; = \; a \; = \; - \frac{\partial V_{eff}}{\partial r} \; = \; - \frac{G_N M}{r^2} \; - \; \frac{c^2}{R_H} \; (\; 1 \; - \; \frac{r}{r_H} \; - \; 3(\frac{r}{R_H})^2 \;) \; + \; .... \tag{1.15}$$

the leading correction to the Newtonian gravitational acceleration is $-c^2/R_H$, in this fashion one recovers the correct order of *magnitude* and *sign* (pointing *towards* the sun) of the Pioneer anomalous acceleration $a_P = -c^2/R_H = -8.98 \times 10^{-8} cm/sec^2$ when $r = 20$ **AU**. The experimental value [2] of the magnitude is $|a_P| = (8.74 \pm 1.33) \times 10^{-8} cm/sec^2$.

If one wanted to reproduce the *variable* $a_p(r)$ acceleration with distance one would have to choose a variable radial Weyl gauge field $A_r(r)$ such that $c^2 A_r(r) = a_p(r)$. In this case the scaling factor is

$$e^{-\int_{r_o}^{r} A_r(r)dr} = 1 - \int_{r_o}^{r} A_r(r)\ dr + \frac{1}{2} [\int_{r_o}^{r} A_r(r)\ dr\ ]^2 + ... \quad (1.16)$$

where the lower limit of the integral is the mean equatorial radius r_o of the sun. The leading relevant term in the effective potential (energy per unit mass) is now given by $-c^2 \int A_r(r)dr$, upon taking its (minus) derivative w.r.t r it gives the variable anomalous acceleration

$$c^2\ \partial_r \int_{o}^{r} A_r(r)\ dr = c^2\ A_r(r) = a_p(r). \quad (1.17)$$

the behavior of $A_r(r)$ must be such that as r reaches 20 **AU**, $c^2 A_r \rightarrow -c^2/R_H$.

However there were a series of unanswered questions :

1- Why planets revolving around the sun in elliptical orbits *don't* experience such anomalous acceleration ?.

2- Since the Weyl gauge field was pure gauge it does not have true physical degrees of freedom because it can be gauged to zero everywhere barring global topological obstructions. Hence, the anomaly would have been just *a gauge artifact*.

3- Why does the Hubble scale R_H appear ?

4- What is the source of the anomaly ?

It is the purpose of this work to solve these problems. In particular, we will see that it is *not* necessary to invoke the expansion of the Universe in order to explain why R_H appears. Nor is required to invoke dark mater, dark energy; Weyl-Brans-Dicke-Jordan theories of gravity [6], [1] ; scalar-tensor-vector modified theories of gravity [7], string theory, $f(R)$ theories of gravity [25] , etc... Satisfactory answers can be obtained directly from the Clifford space geometry of *spinning* objects, like our planets. It *is* the curvature-spin coupling of the planetary motions that hold the key. We conclude by showing how the empirical formula for the Flybys anomalies [10] can be derived within the framework of Clifford geometry.

2 The Extended Relativity Theory in Clifford Spaces

The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [12]. For a comprehensive review we refer to [11] . A natural generalization of the notion of a space-time interval in Minkowski space to C-space is

$$dX^2 \ = \ dX_0 \ dX^0 \ + \ dx_\mu \ dx^\mu \ + \ dx_{\mu\nu} \ dx^{\mu\nu} \ + \ ...... \tag{2.1}$$

The Clifford valued poly-vector is defined by

$$X \ = \ X^M \ E_M \ = \ X^0 \ \mathbf{1} + x^\mu \ \gamma_\mu + x^{\mu\nu} \ \gamma_\mu \wedge \gamma_\nu + ... \ x^{\mu_1\mu_2....\mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} .... \wedge \gamma_{\mu_D}. \tag{2.2}$$

denotes the position of a polyparticle in a manifold, called Clifford space or C-space. The series of terms in (2.2) terminates at a $finite$ value depending on the dimension D. A Clifford algebra $Cl(r,q)$ with $r+q=D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime

$$\frac{1}{2} \ \{\gamma^\mu, \gamma^\nu\} \ = \ \eta^{\mu\nu} \ \mathbf{1}. \tag{2.3}$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$. The multi-graded basis elements E_M of the Clifford-valued poly-vectors are

$$E_M \ \equiv \ \mathbf{1}, \quad \gamma^\mu, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \wedge ..... \wedge \gamma^{\mu_D}. \tag{2.4}$$

It is convenient to order the collective M indices as $\mu_1 < \mu_2 < \mu_3 < ...... < \mu_D$.

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x_{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x_{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetimr background. This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, ...D-2$. The $p = 0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}$.... have been *coined* in the string-brane literature [15] as the *holographic* areas, volumes, ...projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes.

If we take the differential dX and compute the scalar product among two polyvectors $< dX^\dagger dX >_{scalar}$ [13] , [14] , [16] we obtain the C-space extension of the particles proper time in Minkowski space. The symbol $X^\dagger$ denotes the *reversion* operation and involves *reversing* the order of all the basis γ^μ elements in the expansion of X . It is the analog of the transpose (Hermitian) conjugation $(\gamma^\mu \wedge \gamma^\nu)^\dagger = \gamma^\nu \wedge \gamma^\mu$, etc... Therefore, the inner product can be rewritten as the scalar part of the geometric product as $< X^\dagger X >_{scalar}$. The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$< X'^\dagger \ X' >_{scalar} \ = \ < (R^{-1})^\dagger \ X^\dagger \ R^\dagger \ R \ X \ R^{-1} >_{scalar} =$$

$$< R \ X^\dagger \ X \ R^{-1} >_{scalar} \ = \ < X^\dagger \ X >_{scalar} =$$

$$(X^0)^2 \ + \ \Lambda^{2D-2} \ (x_\mu x^\mu) \ + \ \Lambda^{2D-4} \ (x_{\mu\nu} x^{\mu\nu}) \ + \ .... \ + \ (x_{\mu_1\mu_2.....\mu_D})(x^{\mu_1\mu_2.....\mu_D}) \tag{2.5}$$

we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to match units. The Planck scale can be set to unity for convenience.

This condition $R^\dagger = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R in eq-(2.5) because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p})^\dagger = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} ..... \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p} \tag{2.6}$$

Hence only those terms that *change* sign (under the reversal operation) are permitted in the exponential defining $R = exp[\theta^A E_A]$. For example, in $D = 4$, in order to satisfy the condition $R^\dagger = R^{-1}$, one must have from the behavior under the reversal operation expressed in eq-(2.6) that

$$R \ = \ exp \ [\ \theta^{\mu_1\mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} + \theta^{\mu_1\mu_2\mu_3} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3} \ ]. \tag{2.7}$$

such that

$$R^\dagger \ = \ exp \ [\ \theta^{\mu_1\mu_2} (\gamma_{\mu_1} \wedge \gamma_{\mu_2})^\dagger + \theta^{\mu_1\mu_2\mu_3} (\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3})^\dagger \ ] =$$

$$exp \ [\ - \theta^{\mu_1\mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} - \theta^{\mu_1\mu_2\mu_3} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3} \ ] = R^{-1}. \tag{2.8}$$

These transformations are the analog of Lorentz transformations in C-spaces which transform a poly-vector X into another poly-vector X' given by $X' = RXR^{-1}$. The theta parameters $\theta^{\mu_1\mu_2}, \theta^{\mu_1\mu_2\mu_3}$ are the C-space version of the Lorentz rotations/boosts parameters. The ordinary Lorentz rotation/boosts involves only the $\theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2}$ terms, because the Lorentz algebra generator can be represented as $\mathcal{M}^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$. The $\theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$ are the C-space corrections to the ordinary Lorentz transformations when $D = 4$.

The above transformations are *active transformations* since the transformed Clifford number X' (polyvector) is different from the "original" Clifford number X. Considering the transformations of components we have $X' = X'^M E_M =$

$L^M{}_N\, X^N E_M = RXR^{-1}$, from which we can deduce that the basis poly-vectors transform as $L^M{}_N E_M = RE_N R^{-1}$ so that

$$L^M{}_N \;=\; \langle E^M\, RE_N R^{-1}\rangle_{scalar} \;\equiv\; < E^M\; E'_N >_{scalar} . \tag{2.9}$$

For example, in $D = 4$ an ordinary boost with parameter $\theta^t_{x^2}$ along the x^2 direction is tantamount of a "rotation" with an imaginary angle along the $x^1 - x^2$ plane where x^1 denotes the time coordinate and x^2, x^3, x^4 are the spatial coordinates. In C-space one must have as well a "rotation" along the $x^1 - x^{12}$ directions with generalized boost parameter $\theta^t_{12} = \theta^1_{12}$. Hence one has the generalized C-space transformations

$$(t)' \;=\; L^t_M(\theta^{t1};\theta^{t12})(X^M) \;=\; L^t_t\; t \;+ L^t_x\; x \;+ L^t_{12}\; x^{12}. \tag{2.10a}$$

$$(x)' \;=\; L^x_M(\theta^{t1};\theta^{t12})(X^M) \;=\; L^x_t\; t \;+ L^x_x\; x \;+ L^x_{12}\; x^{12}. \tag{2.10b}$$

$$(x^{12})' \;=\; L^{x^{12}}_M(\theta^{t1};\theta^{t12})(X^M) \;=\; L^{x^{12}}_t\; t \;+ L^{x^{12}}_x\; x \;+ L^{x^{12}}_{12}\; x^{12}. \tag{2.10c}$$

notice the presence of the *extra* terms containing the area coordinates x^{12} in the transformations for the t, x variables, which are not present in the standard Lorentz transformations. Also, there is an extra dependence on the boost parameter $\theta^t_{12} = \theta^1_{12}$ in the generalized Lorentz matrices L^M_N. In the more general case, when there are more non-vanishing *theta* parameters , the indices M of the X^M coordinates must be *restricted* to those directions in C-space which involve the t, x^1, x^{12}, x^{123}..... directions as required by the C-space poly-particle dynamics.

The C-space invariant proper time associated with a polyparticle motion is then :

$$< dX^\dagger dX >_{scalar} \;=\; d\Sigma^2 \;=\; dX_0\, dX^0 + \Lambda^{2D-2} dx_\mu dx^\mu + \Lambda^{2D-4} dx_{\mu\nu} dx^{\mu\nu} + .. \tag{2.11}$$

Here we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops,..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkowski space interval:

$$ds^2 = c^2 dt^2 - dx_i dx^i. \tag{2.12}$$

A similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [12].

The author [13] has shown why the derivatives of the area-bivector coordinates $(dx^{\mu\nu}/ds)$ with respect to the ordinary spacetime proper time parameter $s = c\tau \neq ct$ (where $s \neq \Sigma$) can be identified with the spin $S^{\mu\nu}$ (per unit mass) and such that the poly-geodesic equation of a poly-particle leads to the terms of the Papapetrou equation coupling the curvature Riemann tensor to the spin

$R^{\rho}_{\mu_1\mu_2\mu_3}\ S^{\mu_1\mu_2}\ (dx^{\mu_3}/ds)$. The introduction of generalized gravity in *curved* C-spaces involves area, volume, hypervolume metrics and leads to a higher derivative Gravity with Torsion. Area metrics were first introduced by Cartan long ago. A thorough discussion of superluminal behavior in ordinary spacetime while *not* being superluminal in C-space can be found in [11] and why there is no Einstein-Podolski-Rosen paradox in Clifford spaces can be seen in [18]. The analog of photons in C-space are *tensionless* branes. See [11] for further details about the Extended Relativity Theory in curved Clifford spaces and Grand Unification [21], [22]. References about Clifford algebras can be found in [17].

3 The Pioneer Anomaly from dynamics in curved C-spaces

Having reviewed very briefly the basic tenets of the Extended Relativity in C-spaces, and after pointing out the following key remarks : (i) the Clifford scalar component of the polyvector $X^0 \neq x^o = ct$; (ii) the Clifford-scalar components of the C-space metric $g_{00} \neq g_{tt}$; (iii) $\Sigma = \xi$ is the C-space proper time variable which is *not* equal to the proper time variable of ordinary Relativity : $\xi \neq s = c\tau$; (iv) The area-bivector coordinates $x^{\mu\nu}$ are *not* a higher dimensional version of Euler angles; one may begin by writing the poly-geodesic equation in (curved) C-spaces

$$\frac{d^2 X^M}{d\xi^2} \ + \ \Gamma^M_{LN}\ \frac{dX^L}{d\xi}\ \frac{d^2 X^N}{d\xi} \ = \ 0 \ \Rightarrow$$

$$\frac{d^2 x^r}{d\xi^2} \ + \ \Gamma^r_{\mu\nu}\ \frac{dx^\mu}{d\xi}\ \frac{dx^\nu}{d\xi} \ = \ - \ \Gamma^r_{00}\ \frac{dX^0}{d\xi}\ \frac{dX^0}{d\xi} \ -$$

$$\Gamma^r_{[\mu\nu]\ \lambda}\ \frac{dx^{\mu\nu}}{d\xi}\ \frac{dx^\lambda}{d\xi} \ - \ \Gamma^r_{[\mu\nu]\ [\lambda\sigma]}\ \frac{dx^{\mu\nu}}{d\xi}\ \frac{dx^{\lambda\sigma}}{d\xi} \ - \ .... \qquad (3.1)$$

In [11] we have shown that the leading contributions of the generalized connection in C-space is $\Gamma^r_{[\mu\nu]\ \lambda}(\mathbf{X}) \sim R^r_{[\mu\nu]\ \lambda}(x^\mu)$ such that

$$\Gamma^r_{[\mu\nu]\ \lambda}\ \frac{dx^{\mu\nu}}{d\xi}\ \frac{dx^\lambda}{d\xi} \ = \ R^r_{[\mu\nu]\ \lambda}\ \frac{dx^{\mu\nu}}{d\xi}\ \frac{dx^\lambda}{d\xi}. \qquad (3.2)$$

$$\Gamma^r_{[\mu\nu]\ [\lambda\sigma]}\ \frac{dx^{\mu\nu}}{d\xi}\ \frac{dx^{\lambda\sigma}}{d\xi} \ = \ R^r_{\mu\nu\tau}\ T^\tau_{\lambda\sigma}\ \frac{dx^{\mu\nu}}{d\xi}\ \frac{dx^{\lambda\sigma}}{d\xi}. \qquad (3.3)$$

where $T^\tau_{\lambda\sigma}$ are torsion terms. Once again we must emphasize that the C-space proper time ξ is *not* the same as the ordinary spacetime proper time, $(d\xi)^2 \neq c^2(d\tau)^2 = dx_\mu dx^\mu$. The normalization condition of the polyvector valued velocities in C-space is given by

$$1 \; = \; g_{00} \; (\frac{dX^0}{d\xi})^2 \; + \; g_{\mu\nu} \frac{dx^\mu}{d\xi} \frac{dx^\nu}{d\xi} \; + \; g_{\mu_1\mu_2} \; g_{\nu_1\nu_2} \; \frac{dx^{\mu_1\nu_1}}{d\xi} \frac{dx^{\mu_2\nu_2}}{d\xi} \; + \; .... \quad (3.4)$$

In order to match units one must introduce in (3.4) powers of a length scale parameter l . For example, if X^0 and ξ are taken to be *dimensionless* , then the powers of $\frac{dx^{\mu_1\nu_1}}{d\xi} \frac{dx^{\mu_2\nu_2}}{d\xi}$ must be accompanied by a factor of $(1/l)^4$. Powers of $\frac{dx^{\mu_1\nu_1\rho_1}}{d\xi} \frac{dx^{\mu_2\nu_2\rho_2}}{d\xi}$ require factors of $(1/l)^6$, etc..... From eq-(3.4) one learns that

$$g_{00} \; (\frac{dX^0}{d\xi})^2 \; = \; 1 \; - \; g_{\mu\nu} \; \frac{dx^\mu}{d\xi} \frac{dx^\nu}{d\xi} \; - \; g_{\mu_1\mu_2} \; g_{\nu_1\nu_2} \; \frac{dx^{\mu_1\nu_1}}{d\xi} \frac{dx^{\mu_2\nu_2}}{d\xi} \; -$$

$$g_{\mu_1\mu_2} \; g_{\nu_1\nu_2} \; g_{\rho_1\rho_2} \; \frac{dx^{\mu_1\nu_1\rho_1}}{d\xi} \frac{dx^{\mu_2\nu_2\rho_2}}{d\xi} - g_{\mu_1\mu_2} \; g_{\nu_1\nu_2} \; g_{\rho_1\rho_2} \; g_{\sigma_1\sigma_2} \frac{dx^{\mu_1\nu_1\rho_1\sigma_1}}{d\xi} \frac{dx^{\mu_2\nu_2\rho_2\sigma_2}}{d\xi} \quad (3.5)$$

A suitable anti-symmetrization of indices in the products $g_{\mu_1\mu_2} g_{\nu_1\nu_2}$ and $g_{\mu_1\mu_2} g_{\nu_1\nu_2} g_{\rho_1\rho_2}$, ... must be made above. The values of $g_{00} \; (\frac{dX^0}{d\xi})^2$ in the left hand side of eq-(3.5) for the planetary case *differ*, in general, from the values in the spacecraft case.

The anomalous radial acceleration of Pioneer is

$$a_p(r) \; = \; - c^2 \; \Gamma^r_{00}(r) \; (\frac{dX^0}{d\xi})^2 \; - \; c^2 \; R^r_{[\mu\nu] \; \lambda} \; \frac{dx^{\mu\nu}}{d\xi} \; \frac{dx^\lambda}{d\xi} \; + \; .... \quad (3.6)$$

where $X^0(\xi), x^{\mu\nu}(\xi), ...$ are the Pioneer components of the polyvector $\mathbf{X}(\xi)$ "worldline" through C-space. In the case of Pioneer, the curvature-area-bivector velocities (curvature-spin) coupling contribution given by the terms in the r.h.s of (3.6) are *negligible*, for this reason it experiences an overall anomalous acceleration. Strictly speaking, the spacecraft is not truly point-like and can naturally spin around an axis. However, the magnitude of its spin and the size of the spacecraft (a few meters in size) are *no* match for the extremely small curvature terms that are coupled to its spin. If the spinning angular velocity of the spacecraft were to be *extremely* large, it could compensate for the extremely small curvature factors, but this is not the case. Therefore, one may *neglect* the curvature-spin terms and the higher order grade polyvector components of Pioneer, so that eq-(3.6) becomes

$$a_p(r) \; \simeq \; - c^2 \; \Gamma^r_{00}(r) \; (\frac{dX^0}{d\xi})^2 \; = \; c^2 \; A^r \; (r) \; g_{00}(r) \; (\frac{dX^0}{d\xi})^2. \quad (3.7)$$

where the connection (gauge field) $A^r(r)$ (*not* to be confused with the Weyl field !) is the defined from the relations

$$A^r \; (r) \; g_{00} \; = \; - \; \Gamma^r_{00} \; = \; - \; g^{rr} \; \partial_r \; (g_{00}) \; \Rightarrow$$

$$A^r \;=\; -\; g^{rr}\; \partial_r\; log \mid g_{00}(r) \mid \;=\; g^{rr}\; A_r(r) \Rightarrow A_r \;=\; -\; \partial_r\; log \mid g_{00}(r) \mid \quad (3.8)$$

so the anomalous radial acceleration of Pioneer can be recast as

$$a_p(r) \simeq c^2\; A^r(r)\; (\; 1 - z^2_{pioneer}(r)\;). \quad (3.9a)$$

where

$$z^2_{ploneer}(\xi) \;\equiv\; \frac{1}{l^2}\; (\frac{ds_{pioneer}}{d\xi})^2. \quad (3.9b)$$

resulting from the normalization condition of the generalized velocities in C-space

$$1 \;=\; g_{00}\; (\frac{dX^0_{pioneer}}{d\xi})^2 + l^{-2}\; g_{\mu\nu}\; \frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi} + l^{-4}\; g_{\mu_1\mu_2}\; g_{\nu_1\nu_2}\; \frac{dx^{\mu_1\nu_1}}{d\xi}\frac{dx^{\mu_2\nu_2}}{d\xi} + ... =$$

$$g_{00}\; (\frac{dX^0_{pioneer}}{d\xi})^2 \;+\; l^{-2}\; (\frac{ds}{d\xi})^2 \;+\; ......\; \sim\; g_{00}\; (\frac{dX^0_{pioneer}}{d\xi})^2 \;+\; z^2_{pioneer} \quad (9c)$$

after neglecting the bivector, trivector, ... contributions.

The planetary dynamics in C-spaces $differ$ from the Pioneer case because of their *spinning* degrees of freedom. Planets will not exhibit the anomalous acceleration if there is a *cancellation* mechanism in the leading terms of the form

$$-\; c^2\; \Gamma^r_{00}(r)\; (\frac{dX^0_{planets}}{d\xi})^2 \;-\; c^2\; R^r_{[\mu\nu]\;\lambda}\; \frac{dx^{\mu\nu}}{d\xi}\; \frac{dx^\lambda}{d\xi} \;\simeq\; 0. \quad (3.10)$$

The r.h.s of (3.10) does not strictly need to be *zero* but it should be much *smaller* than $any\ of\ the\ values$ of the Pioneer's anomalous acceleration $a_p(r) = c^2 A^r(r)$ observed along its history; otherwise the anomalous effects on the Earth and other planets would have been observed by now. It is understood in eq-(3.10) that $r = r(\xi)$ is the radial coordinate of the planets as a function of the ξ proper time in C-space.

If X^0 and ξ are taken to be *dimensionless*, the term $-c^2\Gamma^r_{00}(r)\; (\frac{dX^0}{d\xi})^2$ has already the right dimensions of acceleration because Γ^r_{00} has dimensions of $(length)^{-1}$. However, one must scale the other terms by a factor of $(1/length)^2$ as follows

$$-c^2\; R^r_{[\mu\nu]\;\lambda}\; \frac{dx^{\mu\nu}}{d\xi}\; \frac{dx^\lambda}{d\xi} \;\times\; \frac{1}{l^2} \;=\; -\; c^2\; R^r_{[\mu\nu]\;\lambda}\; \frac{dx^{\mu\nu}}{ds}\; \frac{dx^\lambda}{ds} \;\times\; \frac{1}{l^2}\; (\frac{ds}{d\xi})^2. \quad (3.11)$$

in order to have the proper units of acceleration since the curvature has $(length)^{-2}$. The standard proper time $s = c\tau \;\sim ct$ in the standard non-relativistic limit

For the Schwarzschild solution the *relevant* components of the curvature tensor that couple to the spin tensor are

$$R^r_{r\phi\phi} \;\sim\; -\; \frac{2GM}{c^2\; r}, \quad R^r_{rtr} \;\sim\; -\; \frac{2GM}{c^2\; r^3}, \quad .... \;. \quad (3.12)$$

where we kept the leading order terms of the curvature tensor. Therefore when one takes the C-space proper time parameter ξ to be dimensionless, the curvature coupling to the area-bivector velocity $(dx^{r\phi}/d\xi) \equiv S^{r\phi}/m = (\omega_{spin}\ \rho_{earth}/c)$, and after introducing a length scale parameter l to match units, is given by

$$-R^{r}_{[r\phi]\phi}\ \frac{dx^{r\phi}}{d\xi}\ \frac{d\phi}{d\xi}\ \times \frac{1}{l^2} \ = \ -R^{r}_{[r\phi]\phi}\ \frac{dx^{r\phi}}{ds}\ \frac{d\phi}{ds}\ \times \frac{1}{l^2}\ (\frac{ds}{d\xi})^2 \ =$$

$$\frac{2GM}{c^2\ r}\ (\frac{\omega_{spin}\ \rho}{c})\ \frac{\omega_{spin}}{c} \ = \ \frac{2GM}{c^2\ r}\ \frac{S^{r\phi}}{mc}\ \frac{\omega_{spin}}{c}\ \times \frac{1}{l^2}\ (\frac{ds}{d\xi})^2. \tag{3.13}$$

the value of the term

$$\frac{2GM}{c^2\ r}\ \frac{dx^{r\phi}}{ds}\ \frac{d\phi}{ds} \ = \ \frac{2GM}{c^2\ r}\ (\frac{\omega_{spin}\ \rho}{c})\ \frac{\omega_{spin}}{c} \ = \ \frac{2GM}{c^2\ r}\ (\frac{\omega_{spin}\ \rho}{c})^2\ \frac{1}{\rho}$$

corresponding to the numerical parameters of the Earth's motion given by : $\omega_{spin} = (2\pi/24 \times 3600)\ sec^{-1}$, the Schwarzschild radius of the sun $\frac{2GM_{sun}}{c^2} \sim 3\ Kms$; the mean equatorial radius of the Earth $\rho_{earth} \sim\ 6.4 \times 10^3\ Kms$; the mean Earth-Sun distance $r_o = 1$ **AU** $\sim 1.49 \times 10^8\ Kms$, gives a very interesting number indeed, the inverse of the Hubble scale $(1/R_H)$ and which lends credence to our proposal to explain the anomaly in terms of the geometry of Clifford spaces. Hence, by plugging the numerical values corresponding to the Earth's motion one gets

$$\frac{2GM}{c^2\ r}\ (\frac{\omega_{spin}\ \rho}{c})^2\ \frac{1}{\rho} \ = \ 7.571 \times 10^{-24}\ Km^{-1} \ = \ \frac{1}{1.32 \times 10^{28}\ cm} \ \simeq \ \frac{1}{R_H} \tag{3.14}$$

The age of the universe is between 12 and 18 Gyr, therefore this value for the Hubble scale R_H in (3.14) falls exactly in the range

$$1.13 \times 10^{28}\ cm\ <\ 1.32 \times 10^{28}\ cm\ <\ 1.69 \times 10^{28}\ cm. \tag{3.15}$$

therefore, one arrives at the very interesting numerical result in eq-(3.14) which is very close to the value of $1/R_H$ after substituting the numerical values corresponding to the spinning motion of the Earth around its axis. After multiplying eq-(3.14) by c^2 leads to an acceleration very close to c^2/R_H (modulo the key factor $z^2_{earth}(\xi)$ in the r.h.s of (3.13)). Acceleration which is due to the coupling of the Earth's *spin* to the Riemann curvature tensor $R^r_{r\phi\phi} = -(2GM_{sun}/c^2 r)$, at the location of the mean Earth-Sun distance $r_o = 1$ **AU**. This is an interesting numerical coincidence that warrants further investigation. Because the Hubble scale today $R_H = c/H(today)$ is not the same as in the very distant past, unless the Hubble parameter $H(t) = constant$ and the speed of light remains constant with time, if one is to maintain the same type of numerical relation (3.14) among the spinning angular velocity of the Earth, its radius, its distance from the sun, the Newtonian coupling, ... one would have concluded that at least one of those parameters, like G or c, would have to change accordingly with the expansion of the Universe as Dirac-Eddington suggested long ago. An increase in the Earth's radius, with the expansion of the Universe, would have lead to the exploding planetary hypothesis, [23] that we will not go into it.

There are two other numerical coincidences that deserve to be mentioned. The numerical magnitude of the value $a_p(r)$ at the location $r = 1$ **AU** is approximately [2]

$$(\frac{1.16}{8.94} \times 10^{-6}) \times (8.94 \times 10^{-8}) \frac{cm}{sec^2} = 1.298 \times 10^{-7} \frac{c^2}{R_H} \qquad (3.16a)$$

It turns out that the number 1.298×10^{-7} in (3.16a) is very close to the number

$$(137.036 \times 20)^{-2} = 1.33 \times 10^{-7} \sim 1.298 \times 10^{-7}. \qquad (3.16b)$$

where 137.036 is the inverse fine structure constant (at the scale of the Bohr radius) and 20 **AU** is approximately the location where the magnitude of $a_p(r)$ attains its maximum and which is also very close to the mean Uranus-Sun distance 19.22 **AU**. Another numerical coincidence has been pointed out to us by Smith [20]. The acceleration produced by the Sun's gravitational attraction on a test body at a distance $d = 137.036 \times 20$ **AU** is also very close c^2/R_H

$$\frac{GM_{sun}}{d^2} = \frac{1}{2} \frac{2GM}{c^2} \frac{c^2}{d^2} =$$
$$\frac{8.94 \times 10^{20} \times 1.5 \times 10^5}{(137 \times 20 \times 1.49 \times 10^{13})^2} \frac{cm}{sec^2} \sim \frac{c^2}{R_H} \qquad (3.17)$$

From eqs- (3.16, 3.17) one finds the interesting scaling relations

$$a_p(r = 1\mathbf{AU}) \sim (137.036 \times 20)^{-2} a_p(r = 20\mathbf{AU}) \sim (137.036 \times 20)^{-4} \frac{GM_{sun}}{(1 \mathbf{AU})^2}. \qquad (3.18)$$

Are these results in eqs-(3.14, 3.16, 3.17, 3.18) mere *irrelevant* numerical *coincidences* or is it design ? In the Hydrogen atom, we know how the Rydberg scale, the Bohr radius and the classical electron radius scale among themselves in powers of $(e^2/\hbar c)^{-1} = 137.036$. The Conformal group $SO(4,2)$ in four dimensions is the largest known symmetry group of the Hydrogen atom. Long ago, Wyler [19], based on wave equations in bounded complex homogeneous domains, has shown that the Conformal Group $SO(4,2)$ is one of the groups whose Greens' functions (associated to the conformally invariant wave equations) yields the numerical value for the fine structure constant (at the scale of the Bohr radius). The fine structure appears as a numerical coefficient in the Greens' function that is given explicitly in terms of the ratios of geometrical measures in those complex domains. Unfortunately, these results by Wyler were dismissed as senseless numerology; nevertheless to this day no one, to my knowledge, has provided a rigorous physical argument against the results by Wyler.

After this brief detour, we proceed with the other components of the curvature and spin. In the non-relativistic regime $s = c\tau \sim ct$, the temporal coordinate is $x^4 = ct$ so $x^{r4} = x^{rt}$ of dimensions $length)^2$. Therefore

$$-R^r_{[rt]t} \frac{dx^{rt}}{ds} \frac{cdt}{ds} \times \frac{1}{l^2} (\frac{ds}{d\xi})^2 = \frac{2GM}{c^2 \ r^3} \rho \times \frac{1}{l^2} (\frac{ds}{d\xi})^2. \qquad (3.19)$$

after substituting $\frac{dx^{rt}}{ds} = \rho$ and setting $cdt/ds \simeq 1$ in the standard non-relativistic limit. The second term of (3.19) is

$$\frac{2GM}{c^2 \ r^3} \rho_{planets} \times \frac{(ds/d\xi)^2_{planets}}{l^2} = \frac{2GM}{c^2 \ r^3} \rho_{planets} \ z^2_{planets}(\xi). \qquad (3.20)$$

where the velocities expressing the *rate* of change of the proper time $s = c\tau$ w.r.t the C-space proper time ξ is defined by

$$z^2_{planets}(\xi) \equiv \frac{1}{l^2} (\frac{ds_{planets}}{d\xi})^2. \qquad (3.21)$$

The cancellation condition(3.10) must be supplemented with the normalization of the polyvector components of the velocities in C-space

$$1 = g_{00} (\frac{dX^0_{planets}}{d\xi})^2 + l^{-2} g_{\mu\nu} \frac{dx^\mu}{d\xi} \frac{dx^\nu}{d\xi} + l^{-4} g_{\mu_1\mu_2} g_{\nu_1\nu_2} \frac{dx^{\mu_1\nu_1}}{d\xi} \frac{dx^{\mu_2\nu_2}}{d\xi} + \ldots\ldots\ldots =$$

$$g_{00} (\frac{dX^0_{planets}}{d\xi})^2 + l^{-2} (\frac{ds}{d\xi})^2 + l^{-4} (\frac{ds}{d\xi})^2 \frac{dx^{\mu\nu}}{ds} \frac{dx_{\mu\nu}}{ds} + \ldots\ldots =$$

$$g_{00} (\frac{dX^0_{planets}}{d\xi})^2 + z^2_{planets} - \frac{\rho^2}{l^2} z^2_{planets} + (\frac{\omega\rho}{c})^2 z^2_{planets} + \ldots\ldots\ldots \qquad (3.22a)$$

where the *dominant* contribution from the $(dx^{\mu\nu}/d\xi)(dx_{\mu\nu}/d\xi)$ terms is

$$g_{rr} \ g_{tt} \frac{dx^{rt}}{d\xi} \frac{dx^{rt}}{d\xi} = g_{rr} \ g_{tt} \frac{dx^{rt}}{ds} \frac{dx^{rt}}{ds} (\frac{ds}{d\xi})^2 = - \ l^2 \ \rho^2 \ z^2_{planets}(r). \qquad (3.22b)$$

since $g_{rr} \ g_{tt} = -1$ for the Schwarzschild solution and $(dx^{rt}/ds) \ (dx_{rt}/ds) = \rho^2$, where ρ is the radius of the planets. We shall neglect the contribution from the higher grade polyvectors.

The velocities $z^2(\xi)$ are explicit functions of the C-space affine proper time parameter ξ associated with each one of the planetary "worldlines" in C-spaces. If the dynamical system is integrable one can rewrite these velocities in terms of the r-coordinates $z^2(\xi)$ as $z^2(\xi(r)) = \tilde{z}^2(r)$ in the same way that one can eliminate the coordinate time parameter in the ordinary falling motion of a test particle towards the Earth from a height h yielding the velocity-height relationship $v^2(h) = 2gh$. The latter relation can be obtained from the conservation of energy relation $-mgh + \frac{1}{2}mv^2 = 0$ and/or by eliminating t from the two equations $v = gt$ and $h = \frac{1}{2}gt^2$. Therefore, in the same fashion, one can rewrite eq-(3.22b) above in terms of the r-coordinates of the planets furnishing an expression which is a function of r.

Notice that one is *not* violating the *equivalence* principle in C-space, despite that it *is* violated in ordinary spacetime. Planets do *not* experience the anomalous acceleration, while the Pioneer spacecraft does, because of the spinning degrees of freedom of the *extended* planetary objects, compared to the *pointlike* spacecraft. The spacecraft as a rigid body can spin about any axis but due to its very small size compared to the size of the planets its curvature-spin coupling is *negligible* compared to those of the planets, unless the spacecraft spins at an incredible hight rate, which is not the case. Both the planets and Pioneer follow poly-geodesics in C-space, which for the case of Pioneer do *not appear* as geodesic motion in ordinary spacetime. Its acceleration in ordinary spacetime is

$$a_{pioneer}(r) \ = \ c^2 A^r(r) \ (\ 1 \ - \ z^2_{pioneer}(r) \ ) \ = \ -c^2 \ \Gamma^r_{00}(r) \ (\frac{dX^0_{pioneer}}{d\xi})^2(\xi(r)) \ \Rightarrow$$

$$c^2 A^r(r) \ = \ \frac{a_{pioneer}(r)}{1 \ - \ z^2_{pioneer}(r)}. \qquad (3.23)$$

The cancellation condition which yields a *zero* anomalous net acceleration of the planets leads to the relationship

$$- \ c^2 \ \Gamma^r_{00}(r) \ (\frac{dX^0_{planets}}{d\xi})^2 \ = \ [\ c^2 \ A^r(r) \ ] \ [g_{00}(r) \ (\frac{dX^0_{planets}}{d\xi})^2] \ =$$

$$\left[\frac{a_{pioneer}(r)}{1 - z^2_{pioneer}(r)} \right] \ \left[\ 1 - z^2_{planets}(\ 1 - \frac{\rho^2}{l^2} \ + (\frac{\omega\rho}{c})^2 \ ) \ \right] \ =$$

$$-c^2 \ \left[\ (\frac{2GM_{sun}}{c^2 \ r^3}) \ \rho \ + \ (\frac{2GM_{sun}}{c^2 \ r}) \ (\frac{\omega\rho}{c})^2 \ \frac{1}{\rho} \ \right] \ z^2_{planets}(r). \qquad (3.24a)$$

From this last equation one finds the following relationship between the functional forms of $z^2_{planets}(r)$ and $z^2_{pioneer}(r) = z^2_p(r)$

$$z^2_{planets}(r) \ =$$

$$\left[\ 1 \ - \ (\frac{\rho}{l})^2 \ + \ (\frac{\omega\rho}{c})^2 \ + \ (\frac{2GM_{sun}}{c^2 \ r}) \ (\frac{z^2_p(r) - 1}{a_p(r)}) \ [\ (\frac{\rho}{r}) \ (\frac{c^2}{r}) \ + \ (\frac{\omega\rho}{c})^2 \ (\frac{c^2}{\rho}) \ ] \ \right]^{-1}. \qquad (3.24b)$$

ρ is the mean equatorial radius of the planet; ω is the spin angular velocity about its axis; r is its distance to the Sun. The value of the fundamental length scale l parameter in C-spaces appearing above (3.24) must be such that $z^2_{planets}(r) > 0$. A "tachyonic" like behavior would occur when $z^2 < 0$ which is the analog of $m^2 < 0$. A physical criteria how to choose the scale l in (3.24b) is based in setting the scale l as one which is *larger* than the radius of gyration of the planets. By radius of gyration $l_{planets}$ of each planet one means a scale $l_{planets}$ such that $\omega_{planets}\ l_{planets}\ = c$. Therefore the value of l in eq-(3.24b) must be such that $l_{planets} \leq l$. When the saturation limit $l_{planets} = l$ is attained, for each one of the planets, the second and third terms in the r.h.s of (3.24) cancel out and one is left with

$$z^2_{planets}(r) \ =$$

$$\left[1 \ + \ (\frac{2GM_{sun}}{c^2\ r})\ (\frac{z^2_{pioneer}(r) - 1}{a_p(r)})\ [\ (\frac{\rho}{r})\ (\frac{c^2}{r})\ +\ (\frac{\omega\rho}{c})^2\ (\frac{c^2}{\rho})\ ]\ \right]^{-1}. \quad (3.24c)$$

therefore, eq-(3.24c) states that the rate at which proper time $z = l^{-1}(ds/d\xi)$ flows with respect to the C-space proper time ξ for the Pioneer spacecraft is *not* the same as the rate of flow for the particular planet, despite that the Pioneer spacecraft happens to be at the very *same* orbital location r as the planet is from the Sun. This $difference$ in the rate at which clocks tick in C-space translates into the C-space analog of Doppler shifts. This fact should be explored further in connection to the anomalous redshifts in Cosmology, where objects which are *not* that far apart from each other exhibit very different redshifts [23]. This phenomenon also has precursors in theories based on dilation and conformal symmetry as analyzed long ago by [24].

An immediate question comes to mind when one looks at (3.24) establishing a constraint relation among the velocities $z^2_{planets}(r)$ and $z^2_{pioneer}(r)$. Why the C-space motion of Pioneer, determined by the values of $z^2_{pioneer}(r)$, is *related* to the C-space motion of the planets determined by the values $z^2_{planets}(r)$? The answer lies in Mach's principle. Motion, the inertia of an object, only has meaning when it is referred relative to other objects. The origins of the constraint relation (3.24) among $z^2_{planets}(r)$ and $z^2_{pioneer}(r)$ arises only when the cancellation mechanism (3.10) occurs by which the planets don't experience the anomalous acceleration that Pioneer does. If one removes the cancellation mechanism (3.10), planets would experience an acceleration and the very *particular* constraint relation (3.24) between the Pioneer and planetary C-space motion would *not* have risen.

We proceed next to determine the functional form of $g_{00}(r)$ based on the relations

$$A^r(r) \ = \ - \ g^{rr}\ \partial_r\ log\ |g_{00}(r)| \ < \ 0; \quad a_{pioneer} \ < \ 0. \quad (3.25)$$

$$g_{00}(r) \ > \ 0; \ \ \partial_r\ g_{00}(r) \ < \ 0; \ \ g^{rr} \ = \ - \ (1 - \frac{2GM_{sun}}{c^2 r}) \ < 0, \ \ for\ r > \frac{2GM}{c^2}. \quad (3.26)$$

and

$$\frac{1 - z^2_{pioneer}(r)}{a_{pioneer}(r)} \ = \ \frac{1}{c^2\ A^r(r)} \ = \ - \ \frac{1}{c^2\ g^{rr}\ \partial_r\ log\ |(g_{00}(r))|}. \quad (3.27)$$

from the above equation one obtains

$$z^2_{pioneer}(r) \ = \ 1 \ + \ \frac{a_{pioneer}(r)}{c^2 \ g^{rr} \ \partial_r \ log \ |g_{00}(r)|} \ < \ 1 \qquad (3.28)$$

where $z^2_{pioneer}(r) \ < \ 1$ due to the conditions in eqs-(3.5) when the bivector, trivector, ... higher grade components are neglected, and which imply that the second term in the r.h.s of eq- (3.28) is *negative* as it should because $a_p(r) < 0$: it points *towards* the Sun. Hence, the negative sign of $a_{pioneer}(r)$ is consistent with the condition $z^2_{pioneer} \ < 1$ derived from the normalization of the C-space poly-vector-valued velocities (3.5) and after neglecting the higher grade contributions due to its negligible size compared to the planets.

Finally we are in a position to determine the functional form of $g_{00}(r)$ from the results in eqs-(3.28) in terms of the variable values of $z^2_{pioneer}(r) < 1$. It is given by the exponential of the following integral

$$g_{00}(r) \ \equiv \ \Phi(r) \ = \ \Phi_o \ exp \ \left[\ \int_{r_o}^{r} \ - \ g_{rr}(r) \ \frac{a_{pioneer}(r)}{1 - z^2_{pioneer}(r)} \ \ dr \ \right]. \qquad (3.29)$$

since $a_p(r) \ < \ 0, \ 1 - z^2_{pioneer} \ > \ 0$ and $- \ g_{rr}(r) \ = \ (1 \ - \ \frac{2GM}{c^2 \ r})^{-1} \ > \ 0$ for $r \ > \ (2GM/c^2) \ \sim \ 3 \ Kms$, the Schwarzchild radius of the sun, the sign of the exponential is *negative*. Thus $g_{00}(r) \ = \ \Phi(r)$ is a *decreasing* function of r from the value of $\Phi_o \ > \ 1$ at $r \ = \ r_o \ > \ 3Kms$ to the asymptotic value of $g_{00}(r = \infty) = \Phi(r = \infty) = 1$ and which means that when the upper limit of the integral is set to $r = \infty$, its value is $log(1/\Phi_o)$. Therefore, the value Φ_o is fixed in terms of the integral from r_o to $r = \infty$ where r_o is equal to the mean equatorial radius of the Sun $r_o = r_{sun} = 6.961 \times 10^5 \ Kms = 4.67 \times 10^{-3}$ **AU**.

The functional form of $z^2_{pioneer} = \frac{1}{l^2}(\frac{ds}{d\xi})^2$ for a hyperbolic trajectory can be simplified considerably if one assumes a purely *radial* (poly) geodesic trajectory defined by

$$z^2_{pioneer} \ \equiv \ \frac{g_{tt}}{l^2} \ (\frac{cdt}{d\xi})^2 \ + \ \frac{g_{rr}}{l^2} \ (\frac{dr}{d\xi})^2; \quad g_{rr} < 0, \ g_{tt} > 0. \qquad (3.30)$$

and

$$\frac{c^2}{l^2} \ \frac{d^2 r}{d\xi^2} \ + \ \frac{c^2}{l^2} \ \Gamma^r_{\mu\nu} \ \frac{dx^\mu}{d\xi} \ \frac{dx^\nu}{d\xi} \ = \ a_{pioneer}(r(\xi)). \qquad (3.31a)$$

$$\frac{c^2}{l^2} \ \frac{d^2 (ct)}{d\xi^2} \ + \ \frac{c^2}{l^2} \ \Gamma^t_{\mu\nu} \ \frac{dx^\mu}{d\xi} \ \frac{dx^\nu}{d\xi} \ = \ 0. \qquad (3.31b)$$

The solutions to eqs-(3.31) determine the functional form of $z^2_{pioneer}$ in eq-(3.30) which is to be used directly inside the integrand of eq-(3.29) and that yields the sought-after expression for $g_{00}(r) \ = \ \Phi(r)$, given in terms of the *empirically* known function $a_{pioneer}(r)$ and $z^2_{pioneer}$.

Rigorously speaking, we must start firstly with the analog of the Einstein-Hilbert action plus polyvector-valued matter (scalars, vectors, antisymmetric tensors ...) in C-spaces and after solving the field equations, upon invoking suitable boundary and initial conditions, one must *verify* whether or not the expression we have found in eq-(3.29) for one of the components of the C-space metric $g_{00}(r) = \Phi(r)$, and whose functional form is *fixed* in terms of the *empirical* graph of the anomalous Pioneer acceleration $a_p(r)$ found by [2], corresponds indeed *to a solution* to the field equations in a *curved* C-space. This is a much more ambitious task because the C-space scalar curvature $\mathcal{R}(G_{MN})$ is given by sums of powers of the ordinary Riemannian curvature plus sums of powers of Torsion terms [11]. It is a higher derivative gravity.

To sum up, the Extended Relativity Theory in (Clifford) C-spaces furnishes an anomalous Pioneer acceleration $a_p(r)$ obeying eq-(3.29) which shares *all* the features of the observed Pioneer anomaly : magnitude and sign, for *all* values of r. It is important to emphasize that so far we have assumed that the Schwarzschild solutions g_{tt}, g_{rr} obeying $g_{tt}\ g_{rr} = -1$ are the ones which are to be used in *all* of the above equations. However, there is *caveat* due to the fact that one expects the solutions to the extended gravitational field equations in C-spaces to be given by *deviations* from the Schwarzschild solutions, $\tilde{g}_{tt}, \tilde{g}_{rr}$. For this reason one expects the values of $z^2_{pioneer}$ defined by eq-(3.30), the solutions to eqs-(3.31a, 3.31b) and the expression for $g_{00}(r) = \Phi(r)$ of eq-(3.29) to *change* accordingly.

Deviations from the Schwarzschild solutions to tackle the Pioneer anomaly based on $f(R)$ theories of gravity can be found in [25]. Moffat et al [7] have found fits of the graph $a_p(r)$ based on solutions to scalar-vector-tensor modified theories of gravity. However, they did not explain why planets don't experience the anomalous acceleration. The curve fit by [7] relied in writing the modified Newtonian acceleration in terms of a scale-dependent gravitational coupling as $-(G(r)M/r^2)$ where the coupling function $G(r)$ was of the form $G(r) = G_o + \Delta G(r)$. The variation piece $\Delta G(r)$ had two terms : (i) a Yukawa-like piece involving a *modulated* amplitude $G_o\ f(r)\ (1 + \frac{r}{\gamma(r)})$ times the decaying exponential : $-\ G_o\ f(r)\ (1 + \frac{r}{\gamma(r)})\ exp(-r/\mu(r))$, and (ii) : the amplitude term $G_o\ f(r)$ itself. There were 3 input functions : $f(r),\ \mu(r),\ \gamma(r)$ in the data fitting procedure by [7]. In our case, we have shown that only two functions $z^2_{pioneer}$ and $g_{00}(r) = \Phi(r)$ are required in eq-(3.29).

Another important point we wish to address here is that the C-space metric component $g_{00}(r) = \Phi(r)$ may provide a Clifford-algebraic interpretation of the dilaton field; while the *dual* component to $g_{00}(r)$ is the (axial) pseudo-scalar component of the C-space metric G_{MN} where M, N are the highest grade polyvector elements, the ones associated with the directions $x_{[1234]}\gamma^{1234}$ in C-space. Thus, the piece of the metric $G_{[1234]\ [1234]}$ could have an interpretation in terms of the axion field. In this way one would have provided a nice Clifford-geometric formulation of the axion and dilaton which are among the dark matter candidates, along the gravitino, neutralino, and other supersymmetric particles, etc...

Some important remarks are in order :

- $\Phi(r)$ is *not* the BDJ scalar of the introduction, $\Phi(r) = g_{00}(r)$ is *dimensionless*, whereas the BDJ scalar field has mass dimensions. The connection $A^r = -g^{rr}\partial_r \ log \ |g_{00}(r)|$ is *not* the Weyl connection.

• By coupling Φ to fermionic matter, like massive neutrinos in the sun, of the form $\Phi(r) \ \mathcal{L}_{matter}$ in the most general Lagrangian, the solar neutrinos *become* a source of the metric component in C-space $g_{00}(r) = \Phi(r)$. Thus, a flux of Solar massive neutrinos might be a natural source of $g_{00}(r) = \Phi(r)$ which is an intrinsic manifestation of the Pioneer anomaly $a_p(r)$ via eq-(3.29) ; i.e. the distribution of matter determines the C-space geometry, and in turn, the C-space geometry indicates matter (Pioneer and planets) how to move in C-space.

• It is warranted to find solutions to the field equations associated to the most general Lagrangian in C-space involving the C-space curvature $\mathcal{R}(G_{MN})$, that contains sums of powers of the ordinary Riemannian curvature and torsion terms, and the C-space polyvector-valued matter fields (scalars, vectors, anti-symmetric tensors of rank two, rank three, ...). Having found solutions it is when one can verify whether or not the expression for $g_{00}(r)$ given by eq-(3.29) is consistent with the solutions found for $g_{00}(r)$ via the field equations in C-space. This project warrants further investigation and is very relevant because it is desirable to derive the functional form of $a_{pioneer}(r)$ from first principles. Such theory in C-space is a generalization of the scalar-vector-tensor theories of modified gravity [7], [5].

To sum up , the cancellation between the two terms of eq-(3.10) corresponding to the motion of the *spinning* planets throughout C-space is the reason why planets do not experience an anomaly. In the case of Pioneer, the curvature-spin coupling contribution given by the second term in (3.10) is *negligible*, for this reason it experiences an overall anomalous acceleration. Strictly speaking, the spacecraft is not truly point-like and can naturally spin around an axis. However, the magnitude of its spin and the size of the spacecraft (a few meters in size) are *no* match for the *extremely small* curvature terms that are coupled to its spin. Of course, if the spinning angular velocity of the spacecraft were to be *extremely* large, it could compensate for the extremely small curvature factors, but this is not the case.

We conclude with a discussion about the Flybys anomalies. An explanation why there is an an apparent increase in the speed of an object due to the spinning degrees of freedom and based on the geometry in C-spaces goes as follows. The momentum of the probe (spacecraft) p^μ is just one component of the polyvector-valued momentum

$$\mathbf{P} \ = \ \pi \ \mathbf{1} \ + \ p^\mu \ \gamma_\mu \ + \ p^{\mu\nu} \ \gamma_\mu \wedge \gamma_\nu \ + ... \tag{3.32}$$

where as usual, a momentum scale parameter κ must be included in the expansion (3.32) in order to match units. We take $\mathbf{P}$ and π to be *dimensionless*. If one focus in just the translation and spinning pieces $p^\mu, p^{\mu\nu}$, the effective momentum of the probe is

$$\mathbf{P} \ = \ \frac{1}{\kappa} \ p^\mu \ \gamma_\mu \ + \ \frac{1}{\kappa^2} \ p^{\mu\nu} \ \gamma_{\mu\nu}. \tag{3.33}$$

where $\gamma_{\mu\nu} = \gamma_\mu \wedge \gamma$ (we omit factors of 1/2 for simplicity.) The magnitude-squared of $\mathbf{P}$ is given by the scalar part of the Clifford geometric product

$$|\mathbf{P}|^2 \;=\; \frac{1}{\kappa^2}\; p^\mu\; p_\mu \;+\; \frac{1}{\kappa^4}\; p^{\mu\nu}\; p_{\mu\nu}. \tag{3.34}$$

resulting from the scalar contractions $\gamma^\mu\gamma_\mu$ and $\gamma^{\mu\nu}\gamma_{\mu\nu}$, respectively. Since the area-momentum is related to the spin [13] $p^{\mu\nu} \leftrightarrow m^2c^2\; S^{\mu\nu}$; after factoring out the $p^\mu\; p_\mu = m^2c^2$ term and taking the square root of (3.34) one has

$$|\mathbf{P}| \;=\; \frac{mc}{\kappa}\; \sqrt{\;1\;+\;(\frac{mc}{\kappa})^2\; S^2}\;\sim\;\frac{mc}{\kappa}\;(\;1\;+\;\frac{1}{2}\;(\frac{mc}{\kappa})^2\; S^2 + ...) \tag{3.35}$$

Upon setting the dimensionless $|\mathbf{P}|$ quantity equal to $(mV_{eff}/mc) = m(v + \delta v)/mc$; where $V_{eff} = v + \delta v$ is the effective velocity resulting from the translational plus spinning degrees of freedom; choosing the κ parameter to obey $\frac{mc}{\kappa} = \frac{v}{c}$; straightforward algebra yields a *positive* (an *increase* in velocity) fractional change of the velocity

$$(\frac{\delta v}{v})_{probe}\;\sim\;\frac{1}{2}\;(\frac{v^2}{c^2})_{probe}\; S^2_{probe} \tag{3.36}$$

The problem now is to relate the values in the r.h.s of (3.36) to the translational and spinning degrees of freedom of the Earth when the probe flybys past it. The empirical formula proposed by [10] for the Flyby anomaly, in terms of the spin angular velocity ω and radius of the Earth ρ, is

$$(\frac{\delta v}{v})_{flyby}\;=\;2\;\frac{\omega\;\rho}{c}\;\delta\; cos\;\phi\;=\;2\;\frac{\omega\;\rho}{c}\;(\;cos\;\phi_{in}\;-\;cos\;\phi_{out}) \tag{3.37}$$

where ϕ_{in}, ϕ_{out} are the inbound and outbound equatorial angle of the spacecraft. In order to study the empirical flyby equation (3.37) within the context of C-space, one needs to study the full scattering problem of the Earth-probe system. For instance, by writing the energy-momentum conservation laws (assuming elastic scattering) in C-space involving both the poly-vectors $\mathbf{P}_{probe}$ and $\mathbf{P}_{earth}$; the net poly-momentum $\mathbf{P}_{probe} + \mathbf{P}_{earth} = constant$ is conserved during the flyby process. In this way one could argue that the gain of the probe's poly-momentum $(\delta P)_{probe} > 0$ is *correlated* to a relative loss in the Earth's value $(\delta P)_{earth} < 0$; i.e. the gain in the velocity by the spacecraft is due to an exchange with the spin-motion of the earth, as eq-(3.37) indicates.

To show why this can work, one needs to take the Clifford geometric product $(\mathbf{P}_{probe}+\mathbf{P}_{earth})\bullet(\mathbf{P}_{probe}+\mathbf{P}_{earth})$, upon doing so one is going to have *couplings* of the form $2\;\kappa^{-3}\;(p^\mu)_{probe}\;(P^{\nu\sigma})_{earth}\;\gamma_{\mu\nu\sigma}$ which bears similarities with (3.37) in the components of $(P^{r\phi})_{earth} = M_{earth}\;\omega\;\rho$. The presence of the cosine factors (3.37) can be understood in $D = 3$ by noticing that $\gamma_{\mu\nu\sigma} \sim \epsilon_{\mu\nu\sigma}\;\mathbf{1}$ inducing an *inner product* structure as follows

$$2\;\kappa^{-3}\;(p^\mu)_{probe}\;(P^{\nu\sigma})_{earth}\;\epsilon_{\mu\nu\sigma}\;=\;2\;\kappa^{-1}\;(p^\mu)_{probe}\;(J_\mu)_{earth}\;=$$

$$2\ \kappa^{-1}\ |p|_{probe}\ |J|_{earth}\ cos(\alpha). \qquad (3.38).$$

where $(J_\mu)_{earth} \equiv \kappa^{-2}\ (P^{\nu\sigma})_{earth}\ \epsilon_{\mu\nu\sigma}$. If this above coupling (3.38) is the main contribution to the flyby anomaly, one can attribute the change $\kappa^{-1}\ \delta|p|_{probe}$ to the latter coupling giving

$$\delta\ |p|_{probe}\ =\ 2\ |p|_{probe}\ |J|_{earth}\ cos(\alpha)\ \Rightarrow\ \frac{\delta\ |p|_{probe}}{|p|_{probe}}\ =\ 2\ |J|_{earth}\ cos(\alpha)\ \ (3.39).$$

The magnitude $|J| = dx^{r\phi}/ds \sim \omega\rho/c$. Comparing these latter values for the ingoing and outgoing trajectories, before and after the scattering, one has

$$(\frac{\delta\ |p|_{probe}}{|p|_{probe}})_{in} - (\frac{\delta\ |p|_{probe}}{|p|_{probe}})_{out}\ =\ 2\ [\ |J|_{in}\ cos(\alpha)_{in} - |J|_{out}\ cos(\alpha)_{out}\ ].\ \ (3.40)$$

Therefore, eq-(3.40) does have the same functional form as the empirical formula (3.37), since $|J|_{earth}$ is a dimensionless quantity involving the spin of the earth, $(\omega\rho/c)$ and when one has small mass probes compared to the Earth's mass, one has $|J|_{in} \sim |J|_{out}$. This approach to the flyby anomalies will be the subject of further investigations.

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