The Clifford Space Geometry of Conformal Gravity and $U(4) \times U(4)$ Yang-Mills Unification

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Abstract

It is shown how a Conformal Gravity and $U(4)\times U(4)$ Yang-Mills Grand Unification model in $four$ dimensions can be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4,C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Upon taking a real slice, and after symmetry breaking, it leads to ordinary Gravity and the Standard Model in four real dimensions. A brief conclusion about the Noncommutative star product deformations of this Grand Unified Theory of Gravity with the other forces of Nature is presented.

Keywords: C-space Gravity, Clifford Algebras, Grand Unification.

1 Introduction : The E_8 Geometry of $Cl(16)$ spaces

Not long ago, a Chern-Simons E_8 Gauge theory of Gravity [1] based on the octic E_8 invariant constructed in [2] was advanced as a unified field theory of a Lanczos-Lovelock Gravitational theory and a E_8 Generalized Yang-Mills (GYM) field theory. It was defined in the $15D$ boundary of a 16D bulk space. The Exceptional E_8 Geometry of the Clifford (16) ($Cl(16)$) Superspace Grand-Unification of Conformal Gravity and Yang-Mills was studied more recently, and in particular, it was discussed how an E_8 Yang-Mills in $8D$, after a sequence of symmetry breaking processes $E_8 \to E_7 \to E_6 \to SO(8,2)$, leads to a Conformal gravitational theory in $8D$ based on the conformal group $SO(8,2)$ in $8D$. Upon performing a Kaluza-Klein-Batakis [3] compactification on CP^2, involving a nontrivial *torsion*, leads to a Conformal Gravity-Yang-Mills unified theory based on the Standard Model group $SU(3) \times SU(2) \times U(1)$ in $4D$. Batakis [3] has shown that, contrary to the standard lore that it is not possible to obtain the Standard Model group from compactifications of $8D$ to $4D$, the inclusion of a nontrivial *torsion* in the internal $CP^2 = SU(3)/SU(2) \times U(1)$ space permits to do so.

Furthermore, it was shown [1] how a conformal (super) gravity and (super) Yang-Mills unified theory in any dimension can be embedded into a (super) Clifford-algebra-valued gauge field theory by choosing the appropriate orthogonal and symplectic Clifford group. The latter is required in order to introduce a graded exterior calculus in Superspace [15]. A candidate action for an Exceptional E_8 gauge theory of gravity in $8D$ was constructed in [1]. It is obtained by recasting the E_8 group as the semi-direct product of $GL(8, R)$ with a deformed Weyl-Heisenberg group associated with canonical-conjugate pairs of vectorial and antisymmetric tensorial generators of rank two and three. Other actions were proposed, like the *quartic* E_8 group-invariant action in $8D$ associated with the Chern-Simons E_8 gauge theory defined on the 7-dim boundary of a $8D$ bulk.

Grand-Unification models in $4D$ based on the exceptional E_8 Lie algebra have been known for sometime [4]. Both gauge bosons A^a_μ and left-handed (two-component) Weyl fermions are assigned to the adjoint 248-dim representation that coincides with the fundamental representation (a very special case for E_8). The Higgs bosons Φ are chosen from among the multiplets that couple to the symmetric product of two fermionic representations $\Psi^a_L C \Psi^b_L \Phi_{ab}$ (C is the charge conjugation matrix) such that $[\mathbf{248} \times \mathbf{248}]_S = \mathbf{1} + \mathbf{3875} + \mathbf{27000}$. Bars and Gunaydin [4] have argued that a physically relevant subspace in the symmetry *breaking* process of E_8 is $SO(16) \rightarrow SO(10) \times SU(4)$, where the 128 remaining massless fermions (after symmetry breaking) are assigned to the $(16, \bar{4})$ and $(\bar{1}6, 4)$ representations. $SU(4)$ serves as the family unification group (four fermion families plus four mirror fermion families of opposite chirality) and $SO(10)$ is the Yang-Mills GUT group.

This symmetry breaking channel occurs in the **135**-dim representation of $SO(16)$ that appears in the $SO(16)$ decomposition of the **3875**-dim representation of E_8 : $\mathbf{3875} = \mathbf{135} + \mathbf{1820} + \mathbf{1920}$. By giving a large v.e.v (vacuum expectation value) to the Higgs Φ_{ab} in the **135**-dim representation of $SO(16)$, corresponding to a symmetric traceless tensor of rank 2, *all* fermions and gauge bosons become super-heavy except for the adjoint representations of gauge bosons given in terms of the $SO(10) \times SU(4)$ decomposition as $(45, 1) + (1, 15)$. The spinor representations of the massless fermions is $128 = (16, \bar{4}) + (\bar{1}6, 4)$, leading to 4 fermion families plus their 4 mirror ones. In this process, only 120 fermions and 188 gauge bosons of the initial 248 have gained mass.

In $SO(10)$ GUT a right-handed massive neutrino (a $SU(5)$ singlet) is added to each Standard Model generation so that 16 (two-component) Weyl fermions can now be placed in the **16**-dim spinor representation of $SO(10)$ and, which in turn, can be decomposed in terms of $SU(5)$ representations as $\mathbf{16} = \mathbf{1} + \mathbf{5}^* + \mathbf{10}$ [8]. In the second stage of symmetry breaking, the fourth family of $\mathbf{5}^* + \mathbf{10}$; $\mathbf{5} + \mathbf{10}^*$ becomes heavy without affecting the remaining 3 families.

Later on [7] found that a Peccei-Quinn symmetry could be used to protect light fermions from acquiring super large massses. If this protection is to be maintained without destroying perturbative unification, *three* light families of fermion generations are singled out which is what is observed. In addition to the other three mirror families, several exotic fermions also remain light.

The other physically relevant symmetry breaking channel is $E_8 \rightarrow E_6 \times SU(3)$ with 3 fermion families (and their mirrors) assigned to the 27 ($\bar{2}7$) dim representation of E_6 :

$$248 \ = \ (1,8) + (78,1) \ + (27,3) \ + (\bar{2}7, \bar{3})$$

In this case, in addition to the 16 fermions assigned to the 16-dim dim spinor representation of $SO(10)$, there exist 11 exotic (two-component) Weyl fermions for each generation. The low energy phenomenology of superstring-inspired E_6 models has been studied intensively. New particles including new gauge bosons, massive neutrinos, exotic fermions, Higgs bosons and their superpartners, are expected to exist. See [9] for an extensive review and references about these superstring-inspired E_6 models. The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [5] under the assumption that there is a vacuum gluino condensate but this condensate is *not* accompanied by a dynamical generation of a mass gap in the pure E_8 gauge sector.

Exceptional, Jordan, Division and Clifford algebras are deeply related and essential tools in many aspects in Physics [10], [11], [12]. Ever since the discovery [13] that $11D$ supergravity, when dimensionally reduced to an n-dim torus led to maximal supergravity theories with hidden exceptional symmetries E_n for $n \leq 8$, it has prompted intensive research to explain the higher dimensional origins of these hidden exceptional E_n symmetries [14] . More recently, there has been a lot of interest in the infinite-dim hyperbolic Kac-Moody E_{10} and non-linearly realized E_{11} algebras arising in the asymptotic chaotic oscillatory solutions of Supergravity fields close to cosmological singularities [13], [14].

Supersymmetric non-linear σ models of Kahler coset spaces $\frac{E_8}{SO(10)\times SU(3)\times U(1)}$; $\frac{E_7}{SU(5)}$; $\frac{E_6}{SO(10)\times U(1)}$ are known to contain three generations of quarks and leptons as (quasi) Nambu-Goldstone $superfields$ [6] (and references therein). The coset model based on $G = E_8$ gives rise to 3 left-handed generations assigned to the $\mathbf{16}$ multiplet of $SO(10)$, and 1 right-handed generation assigned to the $\mathbf{16}^*$ multiplet of $SO(10)$. The coset model based on $G = E_7$ gives rise to 3 generations of quarks and leptons assigned to the $\mathbf{5}^* + \mathbf{10}$ multiplets of $SU(5)$, and a Higgsino (the fermionic partner of the scalar Higgs) in the $\mathbf{5}$ representation of $SU(5)$.

The content of this work is to show why one does not need $Cl(16)$ nor E_8 to obtain a unification of gravity with the other forces in four dimensions. It can be attained in a simpler fashion as long as one works in C-spaces (Clifford spaces). A Conformal Gravity and $U(4) \times U(4)$ Yang-Mills Grand Unification model in $four$ dimensions can be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4,C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Upon taking a real slice and after symmetry breaking it leads to ordinary Gravity and the Standard Model in four real dimensions. A brief conclusion about the Noncommutative star product deformations of this Grand Unified Theory of Gravity with the other forces of Nature is presented.

2 Conformal Gravity and $U(4)\times U(4)$ Yang-Mills Unification from a Clifford Gauge Field Theory in C-spaces

A model of Emergent Gravity with the observed Cosmological Constant from a BF-Chern-Simons-Higgs Model was recently revisited [16] which allowed to show how a Conformal Gravity, Maxwell and $SU(2) \times SU(2) \times U(1) \times U(1)$ Yang-Mills Unification model in *four* dimensions can be attained from a Clifford Gauge Field Theory in a very natural and geometric fashion. In this work we will develop further the results of [16] to show how to construct a Complex Conformal Gravity-Maxwell and Yang-Mills Unification incorporating the full Standard Model in $4D$ based on a Clifford gauge field theory in in C-spaces (Clifford spaces) . Let $\eta_{ab} = (+,-,-,-)$, $\epsilon_{0123} = -\epsilon^{0123} = 1$, the Clifford $Cl(1,3)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \;\; \Gamma_5 = -i\,\Gamma_0\,\Gamma_1\,\Gamma_2\,\Gamma_3, \;\; (\Gamma_5)^2 = 1; \;\; \{\Gamma_5, \Gamma_a\} = 0; \quad (2.1)$$

$$\Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5; \quad \Gamma_{ab} = \frac{1}{2}\,(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)\,. \quad (2.2a)$$

$$\Gamma_{abc} = \epsilon_{abcd}\,\Gamma_5\,\Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5. \quad (2.2b)$$

$$\Gamma_a\,\Gamma_b = \Gamma_{ab} + \eta_{ab}, \quad \Gamma_{ab}\,\Gamma_5 = \frac{1}{2}\epsilon_{abcd}\,\Gamma^{cd}, \quad (2.2c)$$

$$\Gamma_{ab}\,\Gamma_c = \eta_{bc}\,\Gamma_a - \eta_{ac}\,\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2.2d)$$

$$\Gamma_c\,\Gamma_{ab} = \eta_{ac}\,\Gamma_b - \eta_{bc}\,\Gamma_a + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2.2e)$$

$$\Gamma_a\,\Gamma_b\,\Gamma_c = \eta_{ab}\,\Gamma_c + \eta_{bc}\,\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2.2f)$$

$$\Gamma^{ab}\,\Gamma_{cd} = \epsilon^{ab}{}_{cd}\,\Gamma_5 \; - 4\delta^{[a}_{[c}\,\Gamma^{b]}{}_{d]} \; - 2\delta^{ab}_{cd}. \quad (2.2g)$$

$$\delta^{ab}_{cd} = \frac{1}{2}\,(\delta^a_c\,\delta^b_d \; - \; \delta^a_d\,\delta^b_c\,). \quad (2.3)$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(1,3)$ algebra exists where the generators $\mathbf{1}, \Gamma_0, \Gamma_5, \Gamma_i\Gamma_5, i = 1,2,3$ are chosen to be Hermitian; while the generators $-i\,\Gamma_0 \equiv \Gamma_4$; $\Gamma_a,\; \Gamma_{ab}$ for $a,b = 1,2,3,4$ are chosen to be anti-Hermitian. For instance, the anti-Hermitian generators

Γ_k for $k = 1,2,3$ can be represented by 4×4 matrices, whose block diagonal entries are 0 and the 2×2 block off-diagonal entries are comprised of $\pm\sigma_k$, respectively, where σ_k, are the 3 Pauli's spin Hermitian 2×2 matrices obeying $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. The Hermitian generator Γ_0 has zeros in the main diagonal and $-\mathbf{1}_{2\times2}, -\mathbf{1}_{2\times2}$ in the off-diagonal block so that $-i\,\Gamma_0 = \Gamma_4$ is anti-Hermitian. The Hermitian Γ_5 chirality operator has $\mathbf{1}_{2\times2}, -\mathbf{1}_{2\times2}$ along its main diagonal and zeros in the off-diagonal block. The unit operator $\mathbf{1}_{4\times4}$ has 1 along the diagonal and zeros everywhere else.

Using eqs-(2.1-2.3) allows to write the $Cl(1,3)$ algebra-valued one-form as

$$\mathbf{A} \;=\; \left(i\; a_\mu\; \mathbf{1} + i\; b_\mu\; \Gamma_5 \;+\; e^a_\mu\; \Gamma_a \;+ i\; f^a_\mu\; \Gamma_a\; \Gamma_5 \;+\; \frac{1}{4}\omega^{ab}_\mu\; \Gamma_{ab} \right)\; dx^\mu. \quad (2.4)$$

The Clifford-valued anti-Hermitian gauge field A_μ transforms according to $A'_\mu \;=\; U^{-1}\; A_\mu\; U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The anti-Hermitian Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1}\; F\; U$. Decomposing the anti-Hermitian field strength in terms of the Clifford algebra anti-Hermitian generators gives

$$F_{\mu\nu} \;=\; i\; F^1_{\mu\nu}\; \mathbf{1} \;+ i\; F^5_{\mu\nu}\; \Gamma_5 \;+\; F^a_{\mu\nu}\; \Gamma_a \;+ i\; F^{a5}_{\mu\nu}\; \Gamma_a\; \Gamma_5 \;+\; \frac{1}{4}F^{ab}_{\mu\nu}\; \Gamma_{ab}. \quad (2.5)$$

where $F = \frac{1}{2}\; F_{\mu\nu}\; dx^\mu \wedge dx^\nu$. The field-strength components are given by

$$F^1_{\mu\nu} \;= \partial_\mu a_\nu - \partial_\nu a_\mu \quad (2.6a)$$

$$F^5_{\mu\nu} \;=\; \partial_\mu b_\nu - \partial_\nu b_\mu + 2e^a_\mu f_{\nu a} - 2e^a_\nu f_{\mu a} \quad (2.6b)$$

$$F^a_{\mu\nu} \;=\; \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2f^a_\mu b_\nu - 2f^a_\nu b_\mu \quad (2.6c)$$

$$F^{a5}_{\mu\nu} \;=\; \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2e^a_\mu b_\nu - 2e^a_\nu b_\mu \quad (2.6d)$$

$$F^{ab}_{\mu\nu} \;=\; \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{\,b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \longleftrightarrow \nu. \quad (2.6e)$$

A Clifford-algebra-valued dimensionless anti-Hermitian scalar field $\Phi(x^\mu) = \Phi^A(x^\mu)\; \Gamma_A$ belonging to a section of the Clifford bundle in $D = 4$ can be expanded as

$$\Phi \;=\; i\; \phi^{(1)}\; \mathbf{1} \;+\; \phi^a\; \Gamma_a \;+\; \phi^{ab}\; \Gamma_{ab} \;+ i\; \phi^{a5}\; \Gamma_a\; \Gamma_5 \;+ i\; \phi^{(5)}\; \Gamma_5 \quad (2.7)$$

so that the covariant exterior differential is

$$d_A\; \Phi \;=\; (d_A\; \Phi^C)\; \Gamma_C \;=\; \big(\; \partial_\mu\; \Phi^C \;+\; \mathcal{A}^A_\mu\; \Phi^B\; f_{AB}^{\;\;C}\; \big)\; \Gamma_C\; dx^\mu\; .. \quad (2.8)$$

where

$$[\mathcal{A}_\mu,\; \Phi] \;=\; \mathcal{A}^A_\mu\; \Phi^B\; [\Gamma_A,\; \Gamma_B] \;=\; \mathcal{A}^A_\mu\; \Phi^B\; f_{AB}^{\;\;C}\; \Gamma_C\; . \quad (2.9)$$

The first term in the action is

$$I_1 \;=\; \int_{M_4} d^4x \;\; \epsilon^{\mu\nu\rho\sigma} \; < \; \Phi^A \; F^B_{\mu\nu} \; F^C_{\rho\sigma} \; \Gamma_A \; \Gamma_B \; \Gamma_C \; >_0 \; . \tag{2.10}$$

where the operation $< \, ... >_0$ denotes taking the *scalar* part of the Clifford geometric product of $\Gamma_A \; \Gamma_B \; \Gamma_C$. The scalar part of the Clifford geometric product of the gammas is for example

$$< \; \Gamma_a \; \Gamma_b \; > \;\; = \delta_{ab}, \quad < \; \Gamma_{a_1 a_2} \; \Gamma_{b_1 b_2} \; > \;\; = \delta_{a_1 b_1} \; \delta_{a_2 b_2} - \delta_{a_1 b_2} \; \delta_{a_2 b_1}$$

$$< \; \Gamma_{a_1} \; \Gamma_{a_2} \; \Gamma_{a_3} \; > \;\; = \;\; 0, \quad < \; \Gamma_{a_1 a_2 a_3} \; \Gamma_{b_1 b_2 b_3} \; > \;\; = \;\; \delta_{a_1 b_1} \; \delta_{a_2 b_2} \; \delta_{a_3 b_3} \;\; \pm \;\; ......$$

$$< \; \Gamma_{a_1} \; \Gamma_{a_2} \; \Gamma_{a_3} \; \Gamma_{a_4} \; > \;\; = \;\; \delta_{a_1 a_2} \; \delta_{a_3 a_4} \;\; - \;\; \delta_{a_1 a_3} \; \delta_{a_2 a_4} \;\; + \;\; \delta_{a_2 a_3} \; \delta_{a_1 a_4}, \;\; etc \; ...... \tag{2.11}$$

The integrand of (2.10) is comprised of terms like

$$F^{ab} \wedge F^{cd} \; \phi^{(5)} \; \epsilon_{abcd}; \quad F^{(1)} \wedge F^{(5)} \; \phi^{(5)}; \quad F^a \wedge F^{a5} \; \phi^{(5)};$$

$$2 \; F^a_{\;b} \wedge F^b_{\;a} \; \phi^{(1)}; \quad F^{(1)} \wedge F^{(1)} \; \phi^{(1)}; \quad F^{(5)} \wedge F^{(5)} \; \phi^{(1)};$$

$$F^{(1)} \wedge F^{ab} \; \phi_{ab}; \quad F^{(1)} \wedge F^{a5} \; \phi_{a5}; \quad F^{(1)} \wedge F^a \; \phi_a;$$

$$F^a \wedge F_a \; \phi^{(1)}; \quad F^{a5} \wedge F_{a5} \; \phi^{(1)}; \quad F^{ab} \wedge F^c \; (\eta_{bc}\phi_a - \eta_{ac}\phi_b);$$

$$F^{ab} \wedge F^c \; \phi^{5d} \; \epsilon_{abcd}; \quad F^a \wedge F^{b5} \; \phi^{cd} \; \epsilon_{abcd}; \; .... \tag{2.12}$$

The numerical factors and signs of each one of the above terms is determined from the relations in eqs-(2.1-2.2). Due to the fact that $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{\rho\sigma\mu\nu}$ the terms like

$$F^a_{\;b} \wedge F^{bc} \; \phi_{ac} \;\; = \;\; F^{bc} \wedge F^a_{\;b} \; \phi_{ac} \;\; = \;\; F^{cb} \wedge F_b^{\;a} \; \phi_{ac} \;\; =$$

$$F^c_{\;b} \wedge F^{ba} \; \phi_{ac} \;\; = \;\; - \, F^a_{\;b} \wedge F^{bc} \; \phi_{ac} \; \Rightarrow \; F^a_{\;b} \wedge F^{bc} \; \phi_{ac} \;\; = \;\; 0$$

$$F^a \wedge F^b \; \phi_{ab} \;\; = \;\; 0; \quad F^{a5} \wedge F^{b5} \; \phi_{ab} \;\; = \;\; 0; \; F^{a5} \wedge F^{b5} \; \phi^{cd} \; \epsilon_{abcd} \;\; = 0, \; .... \tag{2.13}$$

vanish. Thus the action (2.10) is a generalization of the McDowell-Mansouri-Chamseddine-West action. The Clifford-algebra generalization of the Chern-Simons-like terms [16] are

$$I_2 \;\; = \;\; \int_{M_4} \; < \;\; \Phi^E \;\; d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \;\; \Gamma_{[E} \; \Gamma_A \; \Gamma_B \; \Gamma_C \; \Gamma_{D]} \;\; >_0 \;\; =$$

$$\int_{M_4} \; \Big(\; \phi^{(5)} \;\; d\phi^a \wedge d\phi^b \wedge d\phi^c \wedge d\phi^d \;\; \epsilon_{abcd} \;\; - \;\; \phi^a \;\; d\phi^{(5)} \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \;\; \epsilon_{abcd} \;\; + \;\; .... \Big). \tag{2.14}$$

where $d\phi^A$ is the covariant exterior differential $(\partial_\mu \phi^A + [\mathbf{A}_\mu, \phi]^A) dx^\mu$. The Clifford-algebra generalization of the Higgs-like potential is given by

$$I_3 \;\; = \;\; - \int_{M_5} \; < \;\; d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \wedge d\Phi^E \;\; \Gamma_{[A} \; \Gamma_B \; \Gamma_C \; \Gamma_D \; \Gamma_{E]} \;\; >_0 \;\; V(\Phi) \;\; =$$

$$- \int_{M_5} d\Phi^5 \wedge d\Phi^a \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \ \epsilon_{abcd} \ V(\Phi) \ + \ .... \qquad (2.15)$$

where

$$V(\Phi) \ = \ \kappa \ \left(\ \Phi_A \ \Phi^A \ - \mathbf{v}^2 \ \right)^2 \qquad (2.16a)$$

and

$$\Phi_A \ \Phi^A \ = \ \phi^{(1)} \ \phi_{(1)} + \phi^a \ \phi_a + \phi^{ab} \ \phi_{ab} + \phi^{a5} \ \phi_{a5} + \phi^{(5)} \ \phi_{(5)}. \qquad (2.16b)$$

Vacuum solutions can be found of the form

$$< \phi^{(5)} > \ = \ \mathbf{v}; \ \ < \phi^{(1)} > \ = \ < \phi^a > \ = \ < \phi^{ab} > \ = \ < \phi^{a5} > \ = \ 0. \qquad (2.17)$$

A variation of $I_1 + I_2 + I_3$ given by eqs-(2.19,2.14, 2.15) w.r.t ϕ^5, and taking into account the v.e.v of eq-(2.17) which minimize the potential (2.16a) solely *after the variation* w.r.t the scalar fields is taken place, allows to eliminate the scalars on-shell leading to

$$I_1 + I_2 + I_3 \ = \ \frac{4}{5} \ \mathbf{v} \int_M d^4x \ \Big(\ F^{ab} \wedge F^{cd} \ \epsilon_{abcd} \ + \ F^{(1)} \wedge F^{(5)} \ + \ F^a \wedge F^{a5} \Big) \ =$$

$$\frac{4}{5} \ \mathbf{v} \int_M d^4x \ \Big(\ F^{ab}_{\mu\nu} \ F^{cd}_{\rho\sigma} \ \epsilon_{abcd} \ + \ F^{(1)}_{\mu\nu} \ F^{(5)}_{\rho\sigma} \ + \ F^a_{\mu\nu} \ F^{a5}_{\rho\sigma} \ \Big) \ \epsilon^{\mu\nu\rho\sigma}. \qquad (2.18)$$

where Einstein's summation convention over repeated indices is implied. Despite that one has chosen the v.e.v conditions (2.17) on the scalars, one must not forget the equations which result from their variations. Hence, performing a variation of $I_1 + I_2 + I_3$ w.r.t the remaining scalars $\phi^1, \phi^a, \phi^{ab}, \phi^{a5}$, and taking into account the v.e.v of eq-(2.17) which minimize the potential (2.16a), yields

$$2 \ F^a_{\ b} \wedge F^b_{\ a} \ + \ F^{(1)} \wedge F^{(1)} \ + \ F^{(5)} \wedge F^{(5)} \ + \ F^a \wedge F_a \ + \ F^{a5} \wedge F_{a5} \ = \ 0. \qquad (2.19a)$$

$$F^{(1)} \wedge F^a \ + \ F^{ab} \wedge F^c \ \eta_{bc} \ = \ 0. \qquad (2.19b)$$

$$F^{(1)} \wedge F_{ab} \ + \ F^c \wedge F^{d5} \ \epsilon_{abcd} \ = \ 0. \qquad (2.19c)$$

$$F^{(1)} \wedge F_{a5} \ + \ F^{bc} \wedge F^d \ \epsilon_{abcd} \ = \ 0. \qquad (2.19d)$$

From eqs-(2.19) one can infer that $F^1 = F^a = 0, \ a = 1,2,3,4$ are solutions compatible with eqs-(2.19b, 2.19c, 2.19d), while the non-zero values F^{ab}, F^5, F^{a5} will be constrained to obey

$$2 \ F^a_{\ b} \wedge F^b_{\ a} \ + \ F^{(5)} \wedge F^{(5)} \ + \ F^{a5} \wedge F_{a5} \ = \ 0. \qquad (2.19e)$$

Therefore, when $F^1 = F^a = 0$ the action (2.18) will then reduce to

$$S = \frac{4}{5}\ \mathbf{v} \int_M d^4x \quad (\ F^{ab}_{\mu\nu}\ F^{cd}_{\rho\sigma}\ \epsilon_{abcd} \ ) \ \epsilon^{\mu\nu\rho\sigma}. \tag{2.20}$$

A solution to the the zero torsion condition $F^a = 0$ can be simply found by setting $f^a_\mu = 0$ in eq-(2.6c), and which in turn, furnishes the Levi-Civita spin connection $\omega^{ab}_\mu(e^a_\mu)$ in terms of the tetrad e^a_μ. Upon doing so, the field strength F^{ab} in eq-(2.6e) when $f^a_\mu = 0$ and $\omega^{ab}_\mu(e^a_\mu)$ becomes then $F^{ab} = R^{ab}(\omega^{ab}_\mu) + 4e^a \wedge e^b$, where $R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ is the standard expression for the Lorentz-curvature two-form in terms of the Levi-Civita spin connection. Finally, the action (2.20) becomes the Macdowell-Mansouri-Chamseddine-West action [17], [18]

$$S \ = \ \frac{4}{5}\ \mathbf{v} \int \ d^4x \ (\ R^{ab} \ + 4 \ e^a \wedge e^b \ ) \ \wedge \ (\ R^{cd} \ + 4 \ e^c \wedge e^d \ ) \ \epsilon_{abcd}. \tag{2.21}$$

comprised of the Gauss-Bonnet term $R \wedge R$; the Einstein-Hilbert term $R \wedge e \wedge e$, and the cosmological constant term $e \wedge e \wedge e \wedge e$.

In order to have the proper dimensions of $(length)^{-2}$ in the above curvature $R + e \wedge e$ terms, one has to introduce the suitable length scale parameter l in the terms $\frac{1}{l^2} e \wedge e$. A vacuum solution to a theory based on the action (2.21) is (Anti) de Sitter space $R^{ab} + \frac{4}{l^2} e^a \wedge e^b = 0 \Rightarrow R^{ab} = -\frac{4}{l^2} e^a \wedge e^b$. The (Anti) de Sitter throat size can be set to be equal to the length scale l. If we wish to recover the same results as those found in [16] obtained after the elimination of the v.e.v and consistent with the correct value of the observed vacuum energy density one requires to set $l \sim R_H$ where R_H is the Hubble scale. A value of $l = L_{Planck} = L_P$ would yield a huge cosmological constant. The (Anti) de Sitter throat size can be set to the Hubble scale due to the key presence of the numerical factor $< \phi^5 >= v$ in (2.20) which implies that the gravitational constant $G = L^2_{Planck}$ (in natural units of $\hbar = c = 1$) and the vacuum energy density ρ are fixed in terms of the throat-size of the (Anti) de Sitter space l and $|v|$ as

$$\frac{8}{5}\ \frac{1}{l^2}\ |v| \ \sim \ \frac{1}{16\pi G} \ = \ \frac{1}{16\pi L_P^2}; \quad |\rho| \ \sim \ \frac{4}{5}\ \frac{1}{l^4}\ |v|. \tag{2.22a}$$

Eliminating the vacuum expectation value (vev) value v from eq-(2.22a) yields a geometric mean relationship among the three scales:

$$\frac{1}{32\pi\ l^2}\ \frac{1}{L_P^2} \ \sim \ |\rho|. \tag{2.22c}$$

By setting the throat-size of the (Anti) de Sitter space $l = R_H$, to coincide precisely with the Hubble radius $R_H \sim 10^{61}\ L_P$, the relation (2.22c) furnishes the correct order of magnitude for the observed vacuum energy density [16]

$$|\rho| \ \sim \ \frac{1}{32\pi}\ \frac{1}{R_H^2}\ \frac{1}{L_P^2} \ \sim \ (\frac{L_P}{R_H})^2\ \frac{1}{L_P^4} \ \sim \ 10^{-122}\ (M_{Planck})^4. \tag{2.22d}$$

A value of $l = L_p$ would yield a huge vacuum energy density (cosmological constant). The (Anti) de Sitter throat size must be of the order of the Hubble scale. The reason one can obtain the correct numerical value of the cosmological constant is due to the *key* presence of the numerical factor $< \phi^5 > \ = v$ in (2.21) and whose value is *not* of the order of unity because it would have led to $l \sim L_P$, and in turn, to a huge cosmological constant. The value of v is of the order of $(R_H/L_P)^2 \sim 10^{122}$.

One should emphasize that our results in this section are based on a very *different* action (2.10) (plus the terms in eqs-(2.14,2.15)) than the invariant gravitational action studied by Chameseddine [26] based on the constrained gauge group $U(2,2)$ broken down to $U(1,1) \times U(1,1)$. In general, our action (2.10) is comprised of many *more* terms displayed by in eq-(2.12) than the action chosen by Chamseddine

$$I \ = \ \int_M Tr \ (\Gamma_5 \ F \ \wedge \ F).$$

Secondly, our procedure furnishes the correct value of the cosmological constant via the key presence of the v.e.v $< \phi^5 >= v$ in all the terms of the action (2.21). Thirdly, by invoking the equations of motion (2.19) resulting from a *variation* of $I_1 + I_2 + I_3$ w.r.t the scalar components of Φ^A, one does *not* need to impose by *hand* the zero torsion constraints as done by [26]. The condition $F^a = 0$ results from solving eqs-(2.19).

To sum up, ordinary gravity with the correct value of the cosmological constant emerges from a very specific vacuum solution. Furthermore, there are *many* other vacuum solutions of the more fundamental action associated with the expressions $I_1 + I_2 + I_3$ of eqs-(2.10, 2.14. 2.15) and involving *all* of the terms in eq-(2.12). For example, for *constant* field configurations Φ^A, the inclusion of all the gauge field strengths in eq-(2.12) *contain* the Euler type terms $F^{ab} \wedge F^{cd} \epsilon_{abcd}$; theta type terms $F^1 \wedge F^1$; $F^5 \wedge F^5$ corresponding to the Maxwell a_μ and Weyl dilatation b_μ fields, respectively; Pontryagin type terms $F^a_{\ b} \wedge F^b_{\ a}$; torsion squared terms $F^a \wedge F^a$, etc ... all in one stroke.

At this stage we may provide the relation of the action (2.21) to the Conformal Gravity action based in gauging the conformal group $SO(4,2) \sim SU(2,2)$ in $4D$. The operators of the Conformal algebra can be written in terms of the Clifford algebra generators as [25]

$$P_a \ = \ \frac{1}{2} \Gamma_a \ (1 \ - \ \Gamma_5); \quad K_a \ = \ \frac{1}{2} \Gamma_a \ (1 \ + \ \Gamma_5); \quad D \ = \ - \ \frac{1}{2} \ \Gamma_5, \ \ L_{ab} \ = \ \frac{1}{2} \ \Gamma_{ab}. \tag{2.23}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra $SO(4,2) \sim SU(2,2)$ generators (2.23), after straightforward algebra using $(\Gamma_a)^2 = -1$ for $a = 1,2,3,4$; $(\Gamma_5)^2 = 1$; $\{\Gamma_a, \Gamma_5\} = 0$; the explicit evaluation of the commutators yields

$$[P_a, \ D] \ = \ P_a; \quad [K_a, \ D] \ = \ - \ K_a; \quad [P_a, \ K_b] \ = \ - \ 2 g_{ab} \ D \ + \ 2 \ L_{ab}$$

$$[P_a,\ P_b] \ = \ 0; \quad [K_a, K_b] \ = \ 0; .... \tag{2.24}$$

which is consistent with the $SU(2,2) \sim SO(4,2)$ commutation relations. Notice that the K_a, P_a generators in (2.23) are both comprised of anti-Hermitian Γ_a and Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively, and the dilation D operator is Hermitian.

Having established this, a real-valued tetrad V_μ^a field and its real-valued partner $\tilde{V}_\mu^a$ can be defined in terms of the real-valued gauge fields e_μ^a, f_μ^a, as follows

$$e_\mu^a \ - \ f_\mu^a \ = \ V_\mu^a; \quad e_\mu^a \ + \ f_\mu^a \ = \ \tilde{V}_\mu^a. \tag{2.25}$$

such that

$$e_\mu^a \ \Gamma_a \ + \ f_\mu^a \ \Gamma_a\Gamma_5 \ = \ V_\mu^a \ P_a \ + \ \widetilde{V}_\mu^a \ K_a. \tag{2.26}$$

The components of the torsion and conformal-boost curvature two-forms of conformal gravity are given respectively by the linear combinations of eqs-(2.6c, 2.6d)

$$F_{\mu\nu}^a \ - \ F_{\mu\nu}^{a5} \ = \ \widetilde{F}_{\mu\nu}^a[P]; \quad F_{\mu\nu}^a \ + \ F_{\mu\nu}^{a5} \ = \ \widetilde{F}_{\mu\nu}^a[K] \ \Rightarrow$$

$$F_{\mu\nu}^a \ \Gamma_a \ + \ F_{\mu\nu}^{a5} \ \Gamma_a \ \Gamma_5 \ = \ \widetilde{F}_{\mu\nu}^a[P] \ P_a \ + \ \widetilde{F}_{\mu\nu}^a[K] \ K_a. \tag{2.27}$$

The components of the curvature two-form corresponding to the Weyl dilation generator are $F_{\mu\nu}^5$ (2.6b). The Lorentz curvature two-form is *contained* in $F_{\mu\nu}^{ab} \ dx^\mu \wedge dx^\nu$ (2.6e) and the Maxwell curvature two-form is $F_{\mu\nu}^1 \ dx^\mu \wedge dx^\nu$ (2.6a).

To sum up, the real-valued tetrad gauge field V_μ^a (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\widetilde{V}_\mu^a$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e_\mu^a \ \pm \ f_\mu^a$ associated with the $\Gamma_a, \ \Gamma_a \ \Gamma_5$ generators of the Clifford algebra $Cl(1,3)$ of the tangent space of spacetime $\mathcal{M}^4$ after performing a Wick rotation $-i \ \Gamma_0 = \Gamma_4$.

A different basis given fully in terms of anti-Hermitian generators of the form

$$\mathcal{P}_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ - i \ \Gamma_5); \quad \mathcal{K}_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ + i \ \Gamma_5); \quad \mathcal{D} \ = \ \frac{i}{2} \ \Gamma_5, \ \ L_{ab} \ = \ \frac{1}{2} \ \Gamma_{ab}. \tag{2.28}$$

leads to a *different* algebra $SO(6) \sim SU(4)$ and whose commutators *differ* from those in (2.24)

$$[\mathcal{P}_a, \ \mathcal{D}] \ = \ \mathcal{K}_a; \quad [\mathcal{K}_a, \ \mathcal{D}] \ = \ - \ \mathcal{P}_a; \quad [\mathcal{P}_a, \ \mathcal{K}_b] \ = \ - \ 2g_{ab} \ \mathcal{D}$$

$$[\mathcal{P}_a, \ \mathcal{P}_b] \ = [\mathcal{K}_a, \ \mathcal{K}_b] \ = \ \frac{1}{2}\Gamma_{ab} \ = \ L_{ab}; \ .... \tag{2.29}$$

The anti-Hermitian generators $\mathcal{P}_a, \mathcal{K}_a, \mathcal{D}, L_{ab}$ are associated to the $SO(6) \sim SU(4)$ algebra and which can be explicitly established from the one-to-one correspondence

$$\mathcal{P}_a \;=\; \frac{1}{2}\Gamma_a \; (1 \;-i\;\Gamma_5) \;\longleftrightarrow\; -\;\Sigma_{a5}; \quad \mathcal{K}_a \;=\; \frac{1}{2}\Gamma_a \; (1 \;+i\;\Gamma_5) \;\longleftrightarrow\; \Sigma_{a6}$$

$$\mathcal{D} \;=\; \frac{i}{2}\;\Gamma_5 \;\longleftrightarrow\; \Sigma_{56}; \quad L_{ab} \;=\; \frac{1}{2}\;\Gamma_{ab} \;\longleftrightarrow\; \Sigma_{ab} \qquad (2.30)$$

The $SO(6)$ Lie algebra in $6D$ associated to the anti-Hermitian generators Σ_{AB} $(A, B = 1, 2, ..., 6)$ is defined by the commutators

$$[\Sigma_{AB},\; \Sigma_{CD}] \;=\; g_{BC}\;\Sigma_{AD} \;-\; g_{AC}\;\Sigma_{BD} \;-\; g_{BD}\;\Sigma_{AC} \;+\; g_{AD}\;\Sigma_{BC}. \quad (2.31)$$

where g_{AB} is a diagonal $6D$ metric with signature $(-, -, -, -, -, -)$. One can verify that the realization (2.28) and correspondence (2.30) is consistent with the $SO(6) \sim SU(4)$ commutation relations (2.29). The extra $U(1)$ Abelian generator in $U(4) = U(1) \times SU(4)$ is associated with the unit $\mathbf{1}$ generator.

In general the unitary *compact* group $U(p+q; C)$ is related to the *noncompact* unitary group $U(p, q; C)$ by the Weyl unitary trick [20] mapping the anti-Hermitian generators of the compact group $U(p + q; C)$ to the anti-Hermitian *and* Hermitian generators of the noncompact group $U(p, q; C)$ as follows : The $(p+q)\times(p+q)$ $U(p+q; C)$ complex matrix generator is comprised of the diagonal blocks of $p\times p$ and $q\times q$ complex anti-Hermitian matrices $M_{11}^\dagger = -M_{11}$; $M_{22}^\dagger = -M_{22}$, respectively. The off-diagonal blocks are comprised of the $q \times p$ complex matrix M_{12} and the $p \times q$ complex matrix $-M_{12}^\dagger$, i.e. the off-diagonal blocks are the anti-Hermitian complex conjugates of each other. In this fashion the $(p + q) \times (p + q)$ $U(p + q; C)$ complex matrix generator $\mathbf{M}$ is anti-Hemitian $\mathbf{M}^\dagger = -\mathbf{M}$ such that upon an exponentiation $U(t) = e^{t\mathbf{M}}$ it generates a unitary group element obeying the condition $U^\dagger(t) = U^{-1}(t)$ for $t = real$. This is what occurs in the $U(4)$ case.

In order to retrieve the noncompact $U(2, 2; C)$ case, the Weyl unitary trick requires leaving M_{11}, M_{22} intact but performing a Wick rotation of the off-diagonal block matrices $i\; M_{12}$ and $-i\; M_{12}^\dagger$. In this fashion, M_{11}, M_{22} still retain their anti-Hermitian character, while the off-diagonal blocks are now $Hermitian$ complex conjugates of each-other. This is precisely what occurs in the realization of the Conformal group generators in (2.23). For example, P_a, K_a both contain anti-Hermitian Γ_a and Hermitian pieces $\Gamma_a\Gamma_5$. Notice now that despite the name "unitary" group $U(2, 2; C)$, the exponentiation of the P_a and K_a generators does not furnish a truly unitary matrix obeying $U^\dagger = U^{-1}$. The complex extension of $U(p+q, C)$ is $Gl(p+q; C)$. Since the algebras $U(p+q; C), U(p, q; C)$ differ only by the Weyl unitary trick, they both have identical complex extensions $GL(p + q; C)$ [20]. The technical problem with the general linear groups like $GL(N, R)$ is that (its covering) admits $infinite$-dimensional spinorial representations but *not* finite-dimensional ones. For a thorough discussion of the physics of infinite-component fields and the perturbative renormalization property of metric affine theories of gravity based on (the covering of) $GL(4, R)$ we refer to [21].

At the beginning of this section we had the anti-Hermitian generators Γ_a obeying $(\Gamma_a)^2 = -\mathbf{1}$ for $a = 1, 2, 3, 4$ (*no* summation over the a indices is implied) and where Γ_4 was defined by a Wick rotation as $\Gamma_4 = -i\,\Gamma_0$. The group $U(2,2)$ consists of the 4×4 complex matrices which preserve the *sesquilinear* symmetric metric $g_{\alpha\beta}$ associated to the following quadratic form in C^4

$$< u, u > \;=\; \bar{u}^\alpha\, g_{\alpha\beta}\, u^\beta \;=\; \bar{u}^1 u^1 \;+\; \bar{u}^2 u^2 \;-\; \bar{u}^3 u^3 \;-\; \bar{u}^4 u^4. \qquad (2.32)$$

obeying the *sesquilinear* conditions

$$< \lambda\, v, u > \;=\; \bar{\lambda} < v, u >; \quad < v, \lambda\, u > \;=\; \lambda < v, u > . \qquad (2.33)$$

where λ is a complex parameter and the bar operation denotes complex conjugation. The metric $g_{\alpha\beta}$ can be chosen to be given precisely by the chirality $(\Gamma_5)_{\alpha\beta}$ 4×4 matrix representation whose entries are $\mathbf{1}_{2\times2},\; -\; \mathbf{1}_{2\times2}$ along the main diagonal blocks, respectively, and 0 along the off-diagonal blocks. The $U(2,2) = U(1) \times SU(2,2)$ metric-preserving group transformations are generated by the generators given explicitly in (2.23) and by the unit operator.

The Lie algebra $SU(2,2) \sim SO(4,2)$ corresponds to the conformal group in $4D$. The special unitary group $SU(p+q;C)$ in addition to being sesquilinear metric-preserving is also volume-preserving. One can view $g_{\alpha\beta}$ as a spin-space metric since the complex vector components u^α can be interpreted as spinors; spinors are the left/right ideal elements of the Clifford algebra $Cl(4,C)$ and can be visualized as the respective columns and rows of a 4×4 complex matrix.

The group $U(4)$ consists of the 4×4 complex matrices which preserve the *sesquilinear* symmetric metric $g_{\alpha\beta}$ associated to the following quadratic form in C^4

$$< u, u > \;=\; \bar{u}^\alpha\, g_{\alpha\beta}\, u^\beta \;=\; \bar{u}^1 u^1 \;+\; \bar{u}^2 u^2 \;+\; \bar{u}^3 u^3 \;+\; \bar{u}^4 u^4. \qquad (2.34)$$

The metric $g_{\alpha\beta}$ is now chosen to be given by the unit $\mathbf{1}_{\alpha\beta}$ diagonal 4×4 matrix. The $U(4) = U(1) \times SU(4)$ metric-preserving group transformations are generated by the $15 + 1$ anti-Hermitian generators $\Sigma_{AB},\; i\,\mathbf{1}$ given in (2.28).

In the most general case one has the following isomorphisms of Lie algebras [20]

$$SO(5,1) \sim SU^*(4) \sim SL(2,H); \quad SO^*(6) \sim SU(3,1);$$

$$SO(4,2) \sim SU(2,2); \quad SO(3,3) \sim SL(4,R); \quad SO(6) \sim SU(4). \qquad (2.35)$$

where the asterisks in $SU^*(4), SO^*(6)$ denote the *noncompact* versions of the compact groups $SU(4), SO(6)$ and $SL(2,H)$ is the special linear Mobius algebra over the field of quaternions H. All these algebras are related to each other via the Weyl unitary trick, therefore they admit an specific realization in terms of the $Cl(4,C)$ generators.

It is well known among the experts that $U(4)$ can also be realized in terms of $SO(8)$ generators as follows : Given the Weyl-Heisenberg "superalgebra" involving the N fermionic creation and annihilation (oscillators) operators

$$\{a_i, a_j^\dagger\} \ = \ \delta_{ij}, \quad \{a_i, a_j\} \ = \ 0, \ \{a_i^\dagger, a_j^\dagger\} \ = \ 0; \quad i,j \ = \ 1,2,3,.....\ N. \quad (2.36)$$

one can find a realization of the $U(N)$ algebra bilinear in the oscillators as ${E_i}^j \ = \ a_i^\dagger \ a_j$ and such that the commutators

$$[{E_i}^j, \ {E_k}^l] \ = \ a_i^\dagger \ a_j \ a_k^\dagger \ a_l \ - \ a_k^\dagger \ a_l \ a_i^\dagger \ a_j \ =$$

$$a_i^\dagger \ (\delta_{jk} - a_k^\dagger \ a_j) \ a_l \ - \ a_k^\dagger \ (\delta_{li} - a_i^\dagger \ a_l) \ a_j \ = \ a_i^\dagger \ (\delta_{jk}) \ a_l \ - \ a_k^\dagger \ (\delta_{li}) \ a_j \ =$$

$$\delta_k^j \ {E_i}^l - \ \delta_i^l \ {E_k}^j. \quad (2.37)$$

reproduce the commutators of the Lie algebra $U(N)$ since

$$-a_i^\dagger \ a_k^\dagger \ a_j \ a_l \ + \ a_k^\dagger \ a_i^\dagger \ a_l \ a_j \ = \ - \ a_k^\dagger \ a_i^\dagger \ a_l \ a_j \ + \ a_k^\dagger \ a_i^\dagger \ a_l \ a_j \ = \ 0. \quad (2.38)$$

due to the anti-commutation relations (2.36) yielding a double negative sign $(-)(-) = +$ in (2.38). Furthermore, one also has an explicit realization of the Clifford algebra $Cl(2N)$ Hermitian generators by defining the even-number and odd-number generators as

$$\Gamma_{2j} \ = \ \frac{1}{2} \ (a_j \ + \ a_j^\dagger); \quad \Gamma_{2j-1} \ = \ \frac{1}{2i} \ (a_j \ - \ a_j^\dagger). \quad (2.39)$$

The Hermitian generators of the $SO(2N)$ algebra are defined as usual $\Sigma_{mn} = \frac{i}{2}[\Gamma_m, \Gamma_n]$ where $m,n \ = \ 1,2,....2N$. Therefore, the $U(4), SO(8), Cl(8)$ algebras admit an explicit realization in terms of the fermionic Weyl-Heisenberg oscillators $a_i, a_j^\dagger$ for $i,j \ = \ 1,2,3,4$. $U(4)$ is a subalgebra of $SO(8)$ which is a subalgebra of $Cl(8)$. The Conformal algebra in $8D$ is $SO(8,2)$ and also admits an explicit realization in terms of the $Cl(8)$ generators similarly as the realization of $SO(4,2) \sim SU(2,2)$ in terms of the $Cl(4,C)$ generators displayed in (2.23). The compact version of $SO(8,2)$ is $SO(10)$ which is a GUT group candidate. $U(5)$, $SO(10)$, $Cl(10)$ admit a realization in terms of the fermionic Weyl-Heisenberg oscillators $a_i, a_j^\dagger$ for $i,j = 1,2,3,4,5$.

The group $U(5)$ played a key role in the construction of the deformed Born's reciprocal complex gravitational theory in $4D$ [22] with a Hermitian complex metric $g_{(\mu\nu)} + ig_{[\mu\nu]}$ where the anti-symmetric component can be identified with the Kalb-Ramond field $B_{\mu\nu}$ in string theory. Born's reciprocal relativity in flat spacetimes is based on the principle of a *maximal* speed limit (speed of light) and a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) and where coordinates and momenta are unified on a single footing. In particular, a *deformed* Born's complex reciprocal general relativity theory in curved spacetimes (without the need to introduce star products) was constructed as a local gauge theory of the *deformed* version of the original Quaplectic group proposed by [23] that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* coordinates and momenta generators $[Z_a, Z_b] \neq 0$.

The Hermitian metric is complex-valued with symmetric and nonsymmetric components and there are *two* different complex-valued Hermitian Ricci tensors $\mathcal{R}_{\mu\nu}, \mathcal{S}_{\mu\nu}$. The deformed Born's reciprocal gravitational action linear in the Ricci scalars $\mathcal{R}, \mathcal{S}$ with Torsion-squared terms and BF terms is the most natural action. Nonsymmetric metrics were first considered by Einstein [24] in an attempt to unify Gravity with Electromagnetism.

After this thorough discussion about unitary groups, we shall explain next how both algebras $U(2,2)$ and $U(4)$ can be encoded, separately, in the $Cl(4,C) \oplus Cl(4,C)$ decomposition of the complex Clifford algebra $Cl(5,C)$ after selecting the appropriate basis of generators displayed in (2.23) and (2.28). Complex Clifford algebras have a periodicity of 2 given by $Cl(N+2;C) = Cl(N;C) \otimes M(2,C)$ where $M(2,C)$ is the 2×2 matrix algebra over the complex numbers C. Therefore, in even dimensions $Cl(2m;C) = M(2^m, C)$ is the $2^m \times 2^m$ matrix algebra over the complex numbers C; and in odd dimensions $Cl(2m+1;C) = Cl(2m;C) \oplus Cl(2m,C)$ so that $Cl(5,C) = Cl(4,C) \oplus Cl(4,C)$. Real Clifford algebras have a periodicity of 8 : $Cl(N+8;R) = Cl(N) \otimes M(16,R)$ where $M(16;R)$ is the 16×16 matrix algebra over the reals R. To sum up, $Cl(5,C) = Cl(4,C) \oplus Cl(4,C) = M(4,C) \oplus M(4,C)$. The first Clifford algebra $Cl(4,C)$ factor will carry the Conformal Gravitational and Maxwell degrees of freedom associated with $U(2,2)$. The second Clifford algebra $Cl(4,C)$ factor will carry the $U(4)$ Yang-Mills degrees of freedom.

$U(2,2)$ admits the compact subgroup $U(2) \times U(2) = SU(2) \times SU(2) \times U(1) \times U(1)$ after symmetry breaking. The groups $U(2,2), U(4)$ are not large enough to accommodate the Standard Model Group $SU(3) \times SU(2) \times U(1)$ as its maximally compact subgroup. The GUT groups $SU(5), SU(2) \times SU(2) \times SU(4)$ are large enough to achieve this goal. In general, the group $SU(m+n)$ has $SU(m) \times SU(n) \times U(1)$ for compact subgroups. For this reason, the second step needed to be able to generate the minimal extension of the Standard Model group $SU(3)_c \times SU(2)_L \times U(1)_Y$ involves the inclusion of the extra components of a poly-vector valued field A_M in C-spaces (Clifford spaces).

Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have ben studied in [19], [25] where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate

$$\mathbf{X} \ = \ \varphi \, \mathbf{1} \ + \ x_\mu \ \gamma^\mu \ + \ x_{\mu_1\mu_2} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \ + \ x_{\mu_1\mu_2\mu_3} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \ + \ .... \quad (2.40)$$

In order to match dimensions in each term of (2.40) a length scale parameter must be suitably introduced. In [25] we introduced the Planck scale as the expansion parameter in (2.40). The scalar component φ of the spacetime poly-vector valued coordinate $\mathbf{X}$ was interpreted by [27] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

$\mathcal{A}_M(\mathbf{X}) = A^I_M(\mathbf{X}) \ \Gamma_I$ is a poly-vector valued gauge field whose gauge group is based on the Clifford algebra $Cl(5,C) = Cl(4,C) \oplus CL(4,C)$ spanned by

$16+16$ generators. The expansion of the poly-vector $\mathcal{A}^I_M$ is also of the form

$$\mathcal{A}^I_M \ = \ \Phi^I \, \mathbf{1} + A^I_\mu \, \gamma^\mu + A^I_{\mu_1\mu_2} \, \gamma^{\mu_1} \wedge \gamma^{\mu_2} + A^I_{\mu_1\mu_2\mu_3} \, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} + ... \quad (2.41)$$

In order to match dimensions in each term of (2.41) another length scale parameter must be suitably introduced. For example, since $A_{\mu\nu\rho}$ has dimensions of $(length)^{-3}$ and A_μ has dimensions of $(length)^{-1}$ one needs to introduce another length parameter in order to match dimensions. This length parameter does not need to coincide with the Planck scale. The Clifford-algebra-valued gauge field $\mathcal{A}^I_\mu(x^\mu)\Gamma_I$ in ordinary spacetime is naturally embedded into a far richer object $\mathcal{A}^I_M(\mathbf{X})$ in C-spaces. The advantage of recurring to C-spaces associated with the $4D$ spacetime manifold is that one can have a (complex) Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills unification in a very geometric fashion.

To briefly illustrate how it can be attained, let us write in $4D$ the several components of the C-space poly-vector valued $Cl(5, C)$ gauge field $\mathbf{A}(\mathbf{X})$ as

$$A^I_0 = \Phi^I; \ \ \mathcal{A}^I_\mu; \ \ \mathcal{A}^I_{\mu\nu}; \ \ \mathcal{A}^I_{\mu\nu\rho} \ = \ \epsilon_{\mu\nu\rho\sigma} \, \widetilde{\mathcal{A}}^I_\sigma; \ \ \mathcal{A}^I_{\mu\nu\rho\sigma} \ = \ \epsilon_{\mu\nu\rho\sigma} \, \widetilde{\Phi}^I. \quad (2.42)$$

where Φ^I and $\widetilde{\Phi}^I$ correspond to the scalar (pseudo-scalars) components of the poly-vector gauge field. Let us freeze all the degrees of freedom of the poly-vector C-space coordinate $\mathbf{X}$ in $\mathbf{A}(\mathbf{X})$ except those of the ordinary spacetime vector coordinates x^μ. As we have shown in this section, Conformal Gravity and Maxwell are encoded in the components of $\mathcal{A}^A_\mu \, \Gamma_A$ where Γ_A span the 16 basis elements of the $Cl(4, C)$ algebra. The antisymmetric tensorial gauge field of rank three $\mathcal{A}^A_{\mu\nu\rho}$ is dual to the vector $\widetilde{\mathcal{A}}^A_\sigma$ and has 4 independent spacetime components ($\sigma = 1, 2, 3, 4$), the same number as the vector gauge field $\mathcal{A}^I_\mu$. Therefore, there is another copy of the Conformal Gravity-Maxwell multiplet based on the algebra $U(2, 2)$ encoded in the field $\widetilde{\mathcal{A}}^A_\sigma$.

In order to accommodate the Standard Model Group $SU(3)_c \times SU(2)_L \times U(1)_Y$ one must not forget that there is an *additional* $U(4)$ group associated to the *second* factor algebra $Cl(4, C)$ in the decomposition of $Cl(5, C) = CL(4, C) \oplus Cl(4, C)$. Hence, the basis of 32 generators of $Cl(5, C)$ given by Γ_I ($I = 1, 2, 3....., 32$) appearing in $\mathcal{A}^I_\mu \, \Gamma_I$, and *in the dual* to the rank 3 anti-symmetric tensor in C-space $\mathcal{A}^I_{\mu\nu\rho} \, \Gamma_I = \epsilon_{\mu\nu\rho\sigma} \, \widetilde{A}^I_\sigma \, \Gamma_I$ will provide another copy of the Conformal Gravitational-Maxwell multiplet (based on the algebra $U(2, 2)$) and of the $U(4)$ Yang-Mills multiplet.

To conclude, the combination of the fields $\mathcal{A}^I_\mu \, \Gamma_I$ and $\widetilde{A}^I_\mu \, \Gamma_I$, when Γ_I are the 32 generators of the (complex) $Cl(5, C)$ algebra, by *doubling* the number of $Cl(4, C)$ degrees of freedom in the internal group space and doubling the number of degrees of freedom in spacetime, will yield two copies of a Conformal Gravity-Maxwell-like multiplet which can be assembled into a Complex Gravity-Maxwell-like theory and a $U(4) \times U(4)$ Yang-Mills multiplet in $4D$, as required, if one wishes to incorporate the $SU(3)$ and $SU(2)$ groups.

As mentioned previously, a complex gravitational theory in $4D$ involves a Hermitian complex metric $g_{(\mu\nu)} + ig_{[\mu\nu]}$ where the symmetric components

$g_{(\mu\nu)}$ belong to the usual metric in ordinary gravity; the anti-symmetric component can be identified with the Kalb-Ramond field $B_{\mu\nu}$ in string theory. The *complex* Maxwell-like field can be assigned to a *dyon*-field with *complex* charges/couplings, i.e; charges with both electric *and* magnetic components.

A breaking of $U(4) \times U(4) \longrightarrow SU(2)_L \times SU(2)_R \times SU(4)$ leads to the Pati-Salam GUT group [8] which contains the Standard Model Group, which in turn, breaks down to the ordinary Maxwell Electro-Magnetic (EM) $U(1)_{EM}$ and color (QCD) group $SU(3)_c$ after the following chain of symmetry breaking patterns

$$SU(2)_L \times SU(2)_R \times SU(4) \rightarrow SU(2)_L \times U(1)_R \times U(1)_{B-L} \times SU(3)_c \rightarrow$$
$$SU(2)_L \times U(1)_Y \times SU(3)_c \rightarrow U(1)_{EM} \times SU(3)_c. \tag{2.43}$$

where $B-L$ denotes the Baryon minus Lepton number charge; $Y=$ hypercharge and the Maxwell EM charge is $Q = I_3 + (Y/2)$ where I_3 is the third component of the $SU(2)_L$ isospin. A recent exposition of the algebraic structures behind the GUT groups $SO(10), SU(5), SU(2) \times SU(2) \times SU(4)$ can be found in [31].

Having explained how one generates the Standard model group and Gravity one must not forget the scalar $\Phi^I, \widetilde{\Phi}^I$ multiplets and the rank two antisymmetric tensor field $A^I_{\mu\nu}$ multiplet. The scalar Φ^I admits the $2^5 = 32$ components $\phi,\ \phi^i,\ \phi^{[ij]}, \phi^{[ijk]}, \phi^{[ijkl]}, \phi^{[ijklm]}$ associated with the $Cl(5,C)$ gauge group. Similar results apply to the $\widetilde{\Phi}^I$ components. The ϕ and $\tilde{\phi}$ fields are gauge-singlets that can be identified with the *dilaton* and *axion* scalar fields in modern Cosmology. The other scalar fields carry gauge charges and some of them can be interpreted as the Higgs scalars that will break the Weyl Conformal symmetry leading to ordinary gravity, and break the $U(4) \times U(4)$ symmetry leading to the Standard Model Group. The rank two antisymmetric tensor field $A^I_{\mu\nu}$ multiplet leads to a *generalized* Yang-Mills theory based on *tensorial* antisymmetric gauge fields of rank *two* [19]. Such antisymmetric fields do appear in the massive spectrum of strings and in the physics of membranes. Therefore, the Clifford gauge field theory in C-spaces presented here yields findings compatible with string/M theory.

Despite the appealing nature of our construction one can improve it. It is more elegant not to have to recur to the algebra $Cl(5,C)$ but instead to stick to the $Cl(4,C)$ algebra associated with the tangent space of a *complexified* $4D$ spacetime (like it occurs in Twistor theory). In this case one has then a $U(4) \times U(4)$ Yang-Mills sector corresponding to $A^A_\mu\ \Gamma_A,\ \widetilde{A}^A_\mu\ \Gamma_A$, respectively, where the Γ_A generators, $A = 1, 2, 3, ..., 16$ belong to the 16-dim $Cl(4,C)$ algebra. The key question is now : How do we incorporate gravity into the picture ? The answer to this question lies in the novel physical interpretation behind the anti-symmetric tensor gauge field of rank two $A^A_{\mu\nu}\ \Gamma_A$. It has been shown in [25] when we constructed the generalized gravitational theories in *curved* C-spaces (Clifford spaces) that covariant derivatives in C-spaces of a poly-vector $A_M(\mathbf{X})$ with respect to the area *bivector* coordinate $x^{\mu\nu}$ involves generalized connections (with more indices) in C-space and which are related to the Torsion $T^\rho_{\mu\nu} =$

$T^a_{\mu\nu}\ V^\rho_a$ and Riemannian curvature $R^\sigma_{\mu\nu\rho} = R^{ab}_{\mu\nu}\ V^\sigma_a\ V_{b\rho}$ tensors of the underlying spacetime (V^a_μ is the tetrad/vielbein field). The generalized curvature scalar in *curved* C-spaces [25] admits an expansion in terms of sums of powers of ordinary curvature and torsion tensors; i.e. it looks like a higher derivative theory. Therefore, the components $A^a_{\mu\nu}\ \Gamma_a$ and $A^{ab}_{\mu\nu}\ \Gamma_{ab}$ of the anti-symmetric tensor gauge field of rank two $A^A_{\mu\nu}\ \Gamma_A$ can be identified with the Torsion and Riemannian curvature two-forms as follows

$$(A^a_{\mu\nu}\ \Gamma_a\ )\ dx^{\mu\nu} \ \longleftrightarrow\ (T^a_{\mu\nu}\ P_a)\ dx^\mu \wedge dx^\nu. \qquad (2.44a)$$

$$(A^{ab}_{\mu\nu}\ \Gamma_{ab}\ )\ dx^{\mu\nu} \ \longleftrightarrow\ (R^{ab}_{\mu\nu}\ \Sigma_{ab})\ dx^\mu \wedge dx^\nu. \qquad (2.44b)$$

where P_a corresponds to the Poincare group translation operator and $\Sigma_{ab} = \frac{1}{4}[\Gamma_a, \Gamma_b] = \frac{1}{2}\Gamma_{ab}$ is the Lorentz generator. This is not surprising since the area-bivector differential $dx^{\mu\nu}$ has a similar structure as $dx^\mu \wedge dx^\nu$. The only subtletly arises in the $P_a \leftrightarrow \Gamma_a$ correspondence because we know $[P_a, P_b] = 0$ but $[\Gamma_a, \Gamma_b] = 2\ \Gamma_{ab}$. A more accurate correspondence would be like the one displayed in (2.27)

$$(A^a_{\mu\nu}\ \Gamma_a + A^{a5}_{\mu\nu}\ \Gamma_a\ \Gamma_5\ )\ dx^{\mu\nu} \ \longleftrightarrow\ (\widetilde{F}^a_{\mu\nu}[P]\ P_a + \widetilde{F}^a_{\mu\nu}[K]\ K_a\ )\ dx^\mu \wedge dx^\nu \quad (2.45)$$

where the torsion two form is defined in terms of the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ and vielbein one forms $V^a = V^a_\mu dx^\mu$ as $\widetilde{F}^a[P] = T^a = dV^a + \omega^a{}_b \wedge V^b$; the curvature two form is defined as $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$. The conformal-boost field strength is $\widetilde{F}^a_{\mu\nu}[K]$.

Therefore, in this more natural fashion by performing the key identifications (2.44, 2.45) relating C-space quantities to the curvature and torsion of ordinary spacetime, we may encode gravity as well, in addition to the $U(4) \times U(4)$ Yang-Mills structure *without* having to use the $Cl(5, C)$ algebra which has an intrinsic $5D$ nature, but instead we retain only the $Cl(4, C)$ algebra that is intrinsic to the *complexified* $4D$ spacetime. A *real* slice must be taken in order to extract the real four-dimensional theory from the four complex dimensional one (8 real dimensions) with complex coordinates z_1, z_2, z_3, z_4. A real slice can be taken for instance by setting $z_3 = \bar{z}_1,\ z_4 = \bar{z}_2$. In our opinion, this is the most important result of this work. How Gravity and $U(4) \times U(4)$ Yang-Mills unification in $4D$ can be obtained from a $Cl(4, C)$ gauge theory in the C-space (Clifford space) comprised of poly-vector valued coordinates $\varphi, x^\mu, x^{\mu\nu}, x^{\mu\nu\rho}$..... and poly-vector valued gauge fields $A_0, A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}, \ldots\ldots.$ $A_0 = \Phi$ is the Clifford scalar. The only caveat with the C-space/spacetime correspondence of eqs-(2.44,2.45) is that it involves imposing constraints among the $A_{\mu\nu}$ and A_μ components of a poly-vector A_M since the field strengths $F_{\mu\nu}$ are defined in terms of A_μ. In doing so, one needs to verify that no inconsistencies arise in C-space. Poly-vector valued coordinates correspond to -1-branes (instantons whose world history is a point); 0-branes (points whose world history is a line described by x^μ); 1-branes (strings whose world history are areas described by the area-coordinates $x^{\mu\nu}$), 2-branes (membranes whose world history are described by volumes $x^{\mu\nu\rho}$

), etc.... In this way, a unified description of p-branes, for different values of p, was attained in [25]

Other approaches, for instance, to Grand Unification with Gravity based on spinors, C-spaces and Clifford algebras have been proposed by [30], [28] and [29], respectively. A proposal to unify the Gravity and Standard Model gauge groups by using algebraic spinors of the standard four-dimensional Clifford algebra, in left-right symmetric fashion was presented by [30].

The Gravity-Yang-Mills-Maxwell-Matter GUT model by [29] relies on the $Cl(8)$ algebra in $8D$ leading to the observed three fermion families and their masses, force strengths coupling constants, mixing angles, ... In the model by [28] the 16-dim C-space metric G_{MN} (corresponding to $4D$ Clifford algebra) has enough components in principle to accommodate ordinary gravity and the $SU(3) \times SU(2) \times U(1)$ gauge degrees of freedom in the decomposition of the C-space metric $G_{\mu\nu} = g_{\mu\nu} + A^i_\mu \ A^j_\nu \ g_{ij}$.

In the standard Kaluza-Klein compactification procedure from higher to lower dimensions, the isometry group of the physical internal space carries the corresponding gauge degrees of freedom of the (Maxwell) Yang-Mills theory in lower dimensions. The Killing symmetry vectors associated with the group of isometries of the internal manifold are the generators of the corresponding Lie algebra. For example, the group of isometries associated to an $8D$ internal space given by $CP^2 \times S^3 \times S^1$ is large enough to accommodate $SU(3) \times SU(2) \times U(1)$, because $CP^2 = SU(3)/U(2)$, $S^3 \sim SU(2)$, $S^1 \sim U(1)$. However, the 12 generators $\mathbf{1}, \gamma_5, \gamma_\mu\gamma_5, \gamma_{\mu\nu}$ orthogonal to the generator γ_μ in the 16-dim C-space associated with the $Cl(3,1)$ algebra in $4D$, clearly *cannot* generate the group $SU(3) \times SU(2) \times U(1)$. Therefore, the Extended Gravitational Theory in the C-space [25] associated with the $Cl(3,1)$ algebra in $4D$ does *not* contain enough physical degrees of freedom to generate a Grand Unified Theory (GUT) of ordinary gravity with the other forces in Nature.

Another geometric approaches to unification (see [28] and the many references therein) have been based in gauging the transformations in C-space which leave invariant the norm-squared of a polyvector $\mathbf{X}$ given by the scalar part of the Clifford geometric product of $< \widetilde{\mathbf{X}}^* \ \mathbf{X} >_s$ where the tilde operation represents a reversal in the order of the gamma factors and $*$ denotes a complex conjugation of the components of $\mathbf{X}$. These transformations (poly-rotations or spin gauge transformations) are of the form $\mathbf{X'} = \mathbf{R} \ \mathbf{X} \ \mathbf{L}$ such that $\mathbf{R}\widetilde{\mathbf{R}}^* = \mathbf{1}$ and $\mathbf{L}\widetilde{\mathbf{L}}^* = \mathbf{1}$; i.e. the combined right/left actions can be assigned to the direct product group $U(4) \times U(4)$ which is large enough to accommodate the Standard model group but it *leaves* out Conformal Gravity. A Weyl unitary trick yields $U(2,2) \times U(4)$ which includes the Conformal Gravity-Maxwell-like theory but the remaining group $U(4)$ is not large enough to accommodate $SU(3) \times SU(2)$; i.e $U(4)$ contains *separately* $SU(3)$ and $SU(2)$, but it does *not* contain them *simultaneously*. For this reason, one needs to recur to the dual gauge field $\widetilde{A}^I_\sigma$ of the antisymmetric rank 3 tensor gauge field $A^I_{\mu\nu\rho}$ in C-space in order to incorporate all the degrees of freedom involving Gravity and the Standard Model.

If one wishes to incorporate string theory into the picture, one needs to start with the geometrical C-space (Clifford space) corresponding to the 5 complex dimensional spacetime (10 real dimensions) and associated to the complex Clifford algebra $Cl(5,C)$. In this case the $Cl(5,C)$ symmetry is the one associated with the tangent space to the 5-complex dim-spacetime. This is *another* arena where the extended gravitational theory of the C-space belonging to the $Cl(5,C)$ algebra has enough of degrees of freedom to retrieve the physics of the Standard Model and Gravity in four real dimensions. 10 real dimensions is the dimensions of the anomaly-free superstring theory. If one wishes to incorporate F theory the natural setting would be a 6 complex dim space (12 real dimensional) corresponding the $Cl(6,C)$ algebra isomorphic to the 8×8 matrix algebra over the complex numbers.

To finalize, Complex, Quaternionic and Octonionic Gravity in connection to GUT have been analyzed further in [32], [33]. Star Product deformations of the Clifford Gauge Field Theory discussed in this work, furnishing Noncommutative versions of the action, etc.... are straightforward generalizations of the work by [26]. The wedge star product of two Clifford-valued one-forms is defined as

$$\mathbf{A} \wedge_* \mathbf{A} \;=\; \left((\mathcal{A}^A_\mu \;*\; \mathcal{A}^B_\nu)\,\Gamma_A\,\Gamma_B \right)\; dx^\mu \wedge dx^\nu \;=$$

$$\frac{1}{2}\;\left((\mathcal{A}^A_\mu *_s \mathcal{A}^B_\nu)\;[\Gamma_A,\;\Gamma_B] \;+\; (\mathcal{A}^A_\mu *_a \mathcal{A}^B_\nu)\;\{\Gamma_A,\;\Gamma_B\}\;\right)\; dx^\mu \;\wedge\; dx^\nu. \quad (2.44)$$

In the case when the coordinates don't commute $[x^\mu, x^\nu] = \theta^{\mu\nu}$ (constants), the cosine (symmetric) star product is defined by [26]

$$f *_s g \;\equiv\; \frac{1}{2}(f*g + g*f) \;=\; f\,g + \left(\frac{i}{2}\right)^2 \theta^{\mu\nu}\,\theta^{\kappa\lambda}\,(\partial_\mu\,\partial_\kappa f)\,(\partial_\nu\,\partial_\lambda g) + O(\theta^4). \quad (2.45)$$

and the sine (anti-symmetric Moyal bracket) star product is

$$f *_a g \;\equiv\; \frac{1}{2}(f*g - g*f) \;=\; \left(\frac{i}{2}\right)\,\theta^{\mu\nu}\;(\partial_\mu f)\;(\partial_\nu g) \;+$$

$$\left(\frac{i}{2}\right)^3\;\theta^{\mu\nu}\;\theta^{\kappa\lambda}\;\theta^{\alpha\beta}\;(\partial_\mu\;\partial_\kappa\;\partial_\alpha f)\;(\partial_\nu\;\partial_\lambda\;\partial_\beta g)\;+\;O(\theta^5). \quad (2.46)$$

Notice that both commutators *and* anticommutators of the gammas appear in the star deformed products in (2.44). For example, in the $U(1)$ *Abelian* gauge field theory case, the deformed field strength in C-spaces is

$$\mathbf{F} \;=\; D*A \;=\; d*A \;+\; A*A \;=\; (\Gamma^M\partial_M)*(\Gamma^N A_N) \;+\; (\Gamma^M A_M)*(\Gamma^N A_N). \quad (2.47)$$

The star product deformations of the gauge field strengths in the case of $U(2,2)$ were given by [26] and the expressions are very cumbersome. In four dimensions, the star product deformed action studied by [26] reads

$$I \;=\; i\int_M Tr\;\left(\Gamma_5\;\widetilde{F}\;*\;\widetilde{F}\right)\;=\;i\int_M d^4x\;\epsilon^{\mu\nu\rho\sigma}\;Tr\left(\Gamma_5\;\widetilde{F}_{\mu\nu}\;*\;\widetilde{F}_{\rho\sigma}\right)\;=$$

$$i \int_M d^4x \; \epsilon^{\mu\nu\rho\sigma} \; \left(2 \; \widetilde{F}^1_{\mu\nu} \; *_s \; \widetilde{F}^5_{\rho\sigma} \; + \; \epsilon_{abcd} \; \widetilde{F}^{ab}_{\mu\nu} \; *_s \; \widetilde{F}^{cd}_{\rho\sigma} \right). \tag{2.48}$$

The generalization of the action (2.48) to C-spaces is the subject of future investigations.

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References

[1] C. Castro, *IJGMMP* **4**, No. 8 (2007) 1239; "The Exceptional E_8 Geometry of Clifford (16) Superspace and Conformal Gravity Yang-Mills Grand Unification" to appear in the IJGMMP (May 2009); "An Exceptional E_8 Gauge Theory of Gravity in $D = 8$, Clifford Spaces and Grand Unification", to appear in the IJGMMP (Sept 2009).

[2] M. Cederwall and J. Palmkvist, "The octic E_8 invariant", hep-th/0702024.

[3] N. Batakis, Class and Quantum Gravity **3** (1986) L 99.

[4] I. Bars and M. Gunaydin, *Phys. Rev. Lett* **45** (1980) 859; N. Baaklini, *Phys. Lett* **91 B** (1980) 376; S. Konshtein and E. Fradkin, *Pis'ma Zh. Eksp. Teor. Fiz* **42** (1980) 575; M. Koca, *Phys. Lett* **107 B** (1981) 73; R. Slansky, *Phys. Reports* **79** (1981) 1.

[5] S. Adler, "Further thoughts on Supersymmetric E_8 as family and grand unification theory", hep-ph/0401212.

[6] K. Itoh, T. Kugo and H. Kunimoto, Progress of Theoretical Physics **75**, no. 2 (1986) 386.

[7] S. Barr, *Physical Review* **D 37** (1988) 204.

[8] R. Mohapatra, *Unification and Supersymmetry, the Frontiers of Quark-Lepton Physics* (Springer Verlag, 1986).

[9] J. Hewett and T. Rizzo, *Phys. Reports* **183** (1989) 193.

[10] I. R. Porteous, *Clifford algebras and Classical Groups* (Cambridge Univ. Press, 1995).

[11] C. H Tze and F. Gursey, *On the role of Divison, Jordan and Related Algebras in Particle Physics* (World Scientific, Singapore 1996); S. Okubo, *Introduction to Octonion and other Nonassociative Algebras in Physics* (Cambridge Univ. Press, 2005); R. Schafer, *An introduction to Nonassociative Algebras* (Academic Press, New York 1966); G. Dixon, *Division Algebras, Octonions, Quaternions, Complex Numbers, and the Algebraic*

Design of Physics (Kluwer, Dordrecht, 1994); G. Dixon, *J. Math. Phys* **45** no 10 (2004) 3678; P. Ramond, Exceptional Groups and Physics, hep-th/0301050; J. Baez, *Bull. Amer. Math. Soc* **39** no. 2 (2002) 145.

[12] P. Jordan, J von Neumann and E. Wigner, *Ann. Math* **35** (1934) 2964; K. MacCrimmon, *A Taste of Jordan Algebras* (Springer Verlag, New York 2003); H. Freudenthal, *Nederl. Akad. Wetensch. Proc. Ser* **57 A** (1954) 218; J. Tits, *Nederl. Akad. Wetensch. Proc. Ser* **65 A** (1962) 530; T. Springer, *Nederl. Akad. Wetensch. Proc. Ser* **65 A** (1962) 259.

[13] E. Cremmer, B. Julia and J. Scherk, *Phys. Letts* **B 76** (1978) 409;

[14] S. de Buyl, M. Henneaux and L. Paulot, Extended E_8 invariance of 11-dim Supergravity, hep-th/0512292; P. West, *Class. Quan. Grav* **18** (2001) 4443; V. A. Belinksy, I. M Khalatnikov and E. M. Lifshitz, *Adv. Phys.* **19** (1970) 525.

[15] H. De Bie and F. Sommen, " A Clifford analysis to Superspace" arXiv : 0707.2859; " Hermite and Gegenbauer polynomials in superspace using Clifford analysis" arXiv : 0707. 2863.

[16] C. Castro, "Conformal Gravity, Maxwell and Yang-Mills Unification in $4D$ from a Clifford Gauge Field Theory" submitted to Phys. Letts **B** (April 2009). C. Castro, Mod. Phys Letts **A 17**, no. 32 (2002) 2095.

[17] S. W. MacDowell and F. Mansouri: Phys. Rev. Lett **38** (1977) 739; F. Mansouri: Phys. Rev **D 16** (1977) 2456.

[18] A. Chamseddine and P. West, Nuc. Phys. **B 129** (1977) 39.

[19] C.Castro, Annals of Physics **321**, no.4 (2006) 813.

[20] R. Gilmore, *Lie Groups, Lie Algebras and some of their Applications* (John Wiley and Sons Inc, New York, 1974).

[21] F. Hehl, J. McCrea, E. Mielke and Y. Ne'eman, Phys. Reports **258** (1995) 1. C.Y Lee, Class. Quan Grav **9** (1992) 2001. C. Y. Lee and Y. Ne'eman, Phys. Letts **B 242** (1990) 59.

[22] C. Castro, Phys Letts **B 668** (2008) 442.

[23] S. Low: Jour. Phys **A** Math. Gen **35**, 5711 (2002). J. Math. Phys. **38**, 2197 (1997). J. Govaerts, P. Jarvis, S. Morgan and S. Low, " Worldline Quantization of a Reciprocally Invariant System" J. Phys. **A 40** (2007) 12095; P. Jarvis and S. Morgan, "Constraint Quantization of a worldline system invariant under reciprocal Relativity, II " arXiv.org : 0806.4794. P. Jarvis and S. Morgan, "Born reciprocity and the granularity of spacetime" [arXiv : math-ph/0508041].

[24] A. Einstein, Ann. Math **46**, 578 (1945). A. Einstein and E. Strauss, Ann. Math **47**, 731 (1946). J. Moffat, J. Math. Phys **36**, no. 10 (1995) 5897. J. Moffat and D. Boal, Phys. Rev **D 11**, 1375 (1975).

[25] C. Castro, M. Pavsic, *Progress in Physics* **1** (2005) 31; *Phys. Letts B* **559** (2003) 74; *Int. J. Theor. Phys* **42** (2003) 1693;

[26] A. Chamseddine, "An invariant action for Noncommutative Gravity in four dimensions" hep-th/0202137. Comm. Math. Phys **218**, 283 (2001). " Gravity in Complex Hermitian Spacetime" arXiv : hep-th/0610099.

[27] M.Pavsic, *The Landscape of Theoretical Physics: A Global View, From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle* (Kluwer Academic Publishers, Dordrecht-Boston-London, 2001).

[28] M. Pavsic, Int. J. Mod. Phys **A 21** (2006) 5905; Found. Phys. **37** (2007) 1197; J.Phys. **A 41** (2008) 332001.

[29] Frank (Tony) Smith, *The Physics of* E_8 *and* $Cl(16) = Cl(8) \otimes Cl(8)$ www.tony5m17h.net/E8physicsbook.pdf (Carterville, Georgia, June 2008, 367 pages). *Int. J. Theor. Phys* **24** , 155 (1985); *Int. J. Theor. Phys* **25**, 355 (1985). From Sets to Quarks, hep-ph/9708379; The $D_4 - D_5 - E_6 - E_7 - E_8$ Model, [CERN CDS EXT-2003-087] .

[30] F. Nesti, "Standard Model and Gravity from Spinors" arXiv : 0706.3304 (hep-th). F. Nesti and R. Percacci, J. Phys. **A 41** (2008) 075405.

[31] J. Baez and J. Huerta, "The Algebra of Grand Unified Theories" arXiv : 0904.1556 (hep-th).

[32] C. Castro, J. Math. Phys, **48**, no. 7 (2007) 073517.

[33] S. Marques and C. Oliveira, J. Math. Phys **26** (1985) 3131. Phys. Rev **D 36** (1987) 1716.