On Dark Energy, Weyl Geometry and Brans-Dicke-Jordan Scalar Field

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314, castro@ctps.cau.edu

November 2008, Revised December 2008

Abstract

We review firstly why Weyl's Geometry, within the context of Friedman-Lemaitre-Robertson-Walker cosmological models, can account for *both* the *origins* and the *value* of the observed vacuum energy density (dark energy). The source of dark energy is just the dilaton-like Jordan-Brans-Dicke scalar field that is required to implement Weyl invariance of the most *simple* of all possible actions. A *nonvanishing* value of the vacuum energy density of the order of $10^{-123} M^4_{Planck}$ is derived in agreement with the experimental observations. Next, a Jordan-Brans-Dicke gravity model within the context of ordinary Riemannian geometry, yields also the observed vacuum energy density (cosmological constant) to very high precision. One finds that the temporal flow of the scalar field $\phi(t)$ in ordinary Riemannian geometry, from $t = 0$ to $t = t_o$, has the same numerical effects (as far as the vacuum energy density is concerned) as if there were Weyl scalings from the field configuration $\phi(t)$, to the constant field configuration ϕ_o, in Weyl geometry. Hence, Weyl scalings in Weyl geometry can recapture the flow of time which is consistent with Segal's Conformal Cosmology, in such a fashion that an expanding universe may be visualized as Weyl scalings of a static universe. The main novel result of this work is that one is able to reproduce the observed vacuum energy density to such a degree of precision $10^{-123} M^4_{Planck}$, while still having a Big-Bang singularity at $t = 0$ when the vacuum energy density blows up. This temporal flow of the vacuum energy density, from very high values in the past, to very small values today, is not a numerical coincidence but is the signal of an underlying Weyl geometry (conformal invariance) operating in cosmology, combined with the dynamics of a Brans-Dicke-Jordan scalar field.

Keywords: Dark Energy, Weyl Geometry, Brans-Dicke-Jordan Gravity, Segal Conformal Cosmology.

1 Introduction : Why Weyl Geometry

The problem of dark energy is one of the most challenging problems facing Cosmology today with a vast numerable proposals for its solution, we refer to the recent monograph [1] and references therein. In this introductory section we will review [3] how Weyl's geometry (and its scaling symmetry) is instrumental to solve this dark energy riddle. Before starting we must emphasize that our procedure is quite different than previous proposals [2] to explain dark matter (instead of dark energy) in terms of Brans-Dicke gravity. It is not only necessary to include the Jordan-Brans-Dicke scalar field ϕ but it is essential to have a Weyl geometric extension and generalization of Riemannian geometry (ordinary gravity). It will be shown why the scalar ϕ has a *nontrivial* energy density despite having *trivial* dynamics due entirely to its potential energy density $V(\phi = \phi_o)$ and which is precisely equal to the observed vacuum energy density of the order of $10^{-123} M^4_{Planck}$.

Weyl's geometry main feature is that the norm of vectors under parallel infinitesimal displacement going from x^μ to $x^\mu + dx^\mu$ change as follows $\delta ||V|| \sim ||V|| A_\mu dx^\mu$ where A_μ is the Weyl gauge field of scale calibrations that behaves as a connection under Weyl transformations :

$$A'_\mu = A_\mu - \partial_\mu \Omega(x). \quad g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu}. \tag{1}$$

involving the Weyl scaling parameter $\Omega(x^\mu)$.

The Weyl covariant derivative operator acting on a tensor $\mathbf{T}$ is defined by $D_\mu \mathbf{T} = (\nabla_\mu + \omega(\mathbf{T}) A_\mu) \mathbf{T}$; where $\omega(\mathbf{T})$ is the Weyl weight of the tensor $\mathbf{T}$ and the derivative operator $\nabla_\mu = \partial_\mu + \Gamma_\mu$ involves a connection Γ_μ which is comprised of the ordinary Christoffel symbols plus extra A_μ terms in order for the metric to obey the condition $D_\mu(g_{\nu\rho}) = 0$. The Weyl weight of the metric $g_{\nu\rho}$ is 2. The meaning of $D_\mu(g_{\nu\rho}) = 0$ is that the angle formed by two vectors remains the same under parallel transport despite that their lengths may change. This also occurs in conformal mappings of the complex plane.

The Weyl covariant derivative acting on a scalar ϕ of Weyl weight $\omega(\phi) = -1$ is defined by

$$D_\mu \phi = \partial_\mu \phi + \omega(\phi) A_\mu \phi = \partial_\mu \phi - A_\mu \phi. \tag{2}$$

The Weyl scalar curvature in D dimensions and signature $(+, -, -, -....)$ is

$$\mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} - (D-1)(D-2) A_\mu A^\mu + 2(D-1) \nabla_\mu A^\mu. \tag{3}$$

For a signature of $(-, +, +, +, ...)$ there is a *sign* change in the second and third terms due to a sign change of $\mathcal{R}_{Riemann}$.

The Jordan-Brans-Dicke action involving the scalar ϕ and $\mathcal{R}_{Weyl}$ is

$$S = - \int d^4x \sqrt{|g|} \; [\; \phi^2 \; \mathcal{R}_{Weyl} \;]. \tag{4}$$

Under Weyl scalings,

$$\mathcal{R}_{Weyl} \rightarrow e^{-2\Omega} \mathcal{R}_{Weyl}; \quad \phi^2 \rightarrow e^{-2\Omega} \phi^2. \tag{5}$$

to compensate for the Weyl scaling (in $4D$) of the measure $\sqrt{|g|} \rightarrow e^{4\Omega} \sqrt{|g|}$ in order to render the action (4) Weyl invariant.

When the Weyl integrability condition is imposed $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \Rightarrow A_\mu = \partial_\mu \Omega$, the Weyl gauge field A_μ does not have dynamical degrees of freedom; it is pure gauge and barring global topological obstructions, one can choose the gauge in eq-(4)

$$A_\mu = 0; \quad \phi_0^2 = \frac{1}{16\pi G_N} = constant. \tag{6}$$

such that the action (4) reduces to the standard Einstein-Hilbert action of Riemannian geometry

$$S = -\frac{1}{16\pi G_N} \int d^4x \sqrt{|g|} \; [\mathcal{R}_{Riemann}(g)]. \tag{7}$$

The Weyl integrability condition $F_{\mu\nu} = 0$ means physically that if we parallel transport a vector under a closed loop, as we come back to the starting point, the *norm* of the vector has not changed; i.e, the rate at which a clock ticks does not change after being transported along a closed loop back to the initial point; and if we transport a clock from A to B along different paths, the clocks will tick at the same rate upon arrival at the same point B. This will ensure, for example, that the observed spectral lines of identical atoms will not change when the atoms arrive at the laboratory after taking different paths (histories) from their coincident starting point. If $F_{\mu\nu} \neq 0$ Weyl geometry may be responsible for the alleged variations of the physical constants in recent Cosmological observations. A study of the Pioneer anomaly based on Weyl geometry was made by [4]. The literature is quite extensive on this topic.

Our starting action is

$$S = S_{Weyl}(g_{\mu\nu}, A_\mu) \; + S(\phi). \tag{8}$$

with

$$S_{Weyl}(g_{\mu\nu}, A_\mu) \;\; = \;\; - \int d^4x \; \sqrt{|g|} \;\; \phi^2 \;\; [\; \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) \;]. \tag{9}$$

where we define $\phi^2 = (1/16\pi G)$. The Newtonian coupling G is spacetime dependent in general and has a Weyl weight equal to 2. The term $S(\phi)$ involving the Jordan-Brans-Dicke scalar ϕ is

$$S_\phi = \int d^4x \; \sqrt{|g|} \; [\; \frac{1}{2} g^{\mu\nu} \; (D_\mu \phi)(D_\nu \phi) \; - \; V(\phi) \;]. \tag{10}$$

where $D_\mu \phi = \partial_\mu \phi - A_\mu \phi$. The FRW metric is

$$ds^2 = dt^2 - a^2(t) \; (\frac{dr^2}{1 - k(r/R_0)^2} + r^2 (d\Omega)^2). \tag{11a}$$

where $k = 0$ for a 3-dim spatially flat region; $k = \pm 1$ for regions of positive and negative constant spatial curvature, respectively. The de Sitter metric belongs to a special class of FRW metrics and it admits different forms depending on the coordinates chosen. The Friedman-Einstein-Weyl equations in the gauge $A_\mu = (0,0,0,0)$, in units of $c=1$ are

$$G_{\mu\nu} \; = \; [\; T_{\mu\nu} \; + \; T^{BDJ}_{\mu\nu} \;] \quad \phi^2 = \frac{1}{16\pi G}. \quad T_{\mu\nu} \; = \; - \frac{2}{\sqrt{|g|}} \; \frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \qquad (11b)$$

where the $effective$ stress energy tensor associated with the BDJ scalar field $\Phi \equiv \phi^2$ (when the ω parameter is $\omega = \frac{1}{4}$) is given by

$$T^{BDJ}_{\mu\nu}(\phi) \; = \; - \frac{1}{4\phi^4} \; [\; (D_\mu \phi^2) \; (D_\nu \phi^2) \; - \frac{1}{2} \; g_{\mu\nu} \; (D^\rho \phi^2) \; (D_\rho \phi^2) \;] \; -$$

$$\frac{1}{\phi^2} \; [\; (D_\mu \; D_\nu \; \phi^2) \; - \; g_{\mu\nu} \; (D_\rho \; D^\rho \; \phi^2) \;] \; + \; g_{\mu\nu} \; \frac{V(\phi^2)}{2\phi^2}. \qquad (11c).$$

the second terms in eq-(11c) stem from the variation $\phi^2 \; (\delta R_{\mu\nu} / \delta g_{\rho\sigma})$ which is *no* longer given by a *total derivative* because ϕ^2 is no longer a constant. Eq-(11c) are the *corrections* to our derivation of the vacuum energy density [3] where we erroneously omitted these second terms in eq-(11c). The equations of motion read

$$3 (\frac{(da/dt)}{a})^2 \; + \; (\frac{3k}{a^2 R_0^2}) \; = \; 8\pi G(t) \; \rho_\phi. \qquad (12)$$

and

$$-2 \; (\frac{(d^2 a/dt^2)}{a}) - (\frac{(da/dt)}{a})^2 \; - \; (\frac{k}{a^2 R_0^2}) = \; 8\pi G(t) \; p_\phi. \qquad (13)$$

where the density ρ_ϕ and pressure terms p_ϕ must include now the *extra* contributions to the $effective$ stress energy tensor $T^{BDJ}_{\mu\nu}$ given by the second terms of eq-(11c). From eqs-(12-13) one can infer the important relation :

$$- \; (\frac{(d^2 a/dt^2)}{a}) \; = \; \frac{4\pi G(t)}{3} \; (\rho + 3p). \qquad (14)$$

The Jordan-Brans-Dicke scalar ϕ must obey the generalized Klein-Gordon equations of motion

$$(\; D_\mu D^\mu \; + \; 2\mathcal{R}_{Weyl} \;) \; \phi \; + \; (\frac{dV}{d\phi}) = 0 \qquad (15)$$

notice that because the Weyl covariant derivative obeys the condition $D_\mu (g_{\nu\rho}) = 0 \; \Rightarrow D_\mu (\sqrt{|g|}) = 0$ there are *no* terms of the form $(D_\mu \sqrt{|g|})(D^\mu \phi)$ in the generalized Klein-Gordon equation like it would occur in ordinary Riemannian geometry $(\partial_\mu \sqrt{|g|})(\partial^\mu \phi) \neq 0$. In addition, we have the crucial *constraint* equation obtained from the variation of the action w.r.t to the A^μ field :

$$\frac{\delta S}{\delta A^{\mu}} \ = \ 0 \ \Rightarrow 6 \ (A_{\mu}\phi^2 \ + \ \partial_{\mu}(\phi^2)) \ + \ \frac{1}{2} \ (A_{\mu}\phi^2 \ - \ \partial_{\mu}(\phi)^2 \ ) \ = \ 0. \qquad (16)$$

The last constraint equation in the gauge $A_{\mu} = 0$, forces $\partial_{\mu}\phi = 0 \ \Rightarrow \phi = \phi_o = constant$. Consequently $G \sim \phi^{-2}$ is also constrained to a constant G_N and one may set $16\pi \ G_N \ \phi_o^2 = 1$, where G_N is the observed Newtonian constant today.

Furthermore, in the gauge $A_{\mu} = 0$, due to the constraint eq-(16), one can infer that $D_{\mu}\phi = 0, \ \Rightarrow D^{\mu}D_{\mu}\phi = 0$ because $D_t \ \phi(t) = \partial_t \ \phi - A_t \ \phi = \partial_t \ \phi = 0$, and $D_i\phi(t) = -A_i\phi(t) = 0$. These results will be used in the generalized Klein-Gordon equation.

Therefore, the stress energy tensor $T^{\mu}_{\mu} = diag \ (\rho, -p, -p, -p)$ corresponding to the constant scalar field configuration $\phi(t) = \phi_o$, in the $A_{\mu} = 0$ gauge, becomes :

$$\rho_{\phi} = \frac{1}{2}(\partial_t \ \phi - A_t \ \phi)^2 \ + \ V(\phi) \ + \ extra \ terms \ = \ V(\phi);$$
$$p_{\phi} = \frac{1}{2}(\partial_t \ \phi - A_t \ \phi)^2 \ - \ V(\phi) \ + \ extra \ terms \ = \ - V(\phi). \qquad (17)$$

the extra terms in (17) stemming from the second terms in eq-(11c) also vanish because $D_{\mu}\phi^2 = 0$ and $D_{\rho}D^{\rho}\phi^2 = 0$ when $\phi = \phi_o = constant$ and $A_{\mu} = 0$. Therefore, from (17) one arrives at

$$\rho + \ 3p \ = \ - \ 2V(\phi) \ = -2V(\phi). \qquad (18)$$

This completes the proof why the above ρ and p terms, in the gauge $A_{\mu} = 0$, become $\rho(\phi) = V(\phi) = -p(\phi)$ such that $\rho + 3p = -2V(\phi)$ (that will be used in the Einstein-Friedman-Weyl equations (13b)). This is the *key* reason why Weyl's geometry and symmetry is essential to explain the origins of a $non-vanishing$ vacuum energy (dark energy). The latter relation $\rho(\phi) = V(\phi) = -p(\phi)$ is the *key* to derive the vacuum energy density in terms of $V(\phi = \phi_o)$, because such relation resembles the dark energy relation $p_{DE} = -\rho_{DE}$. Had one not had the constraint condition $D_t \ \phi(t) = (\partial_t - A_t)\phi = \partial_t \ \phi = 0$, and $D_i\phi(t) = -A_i\phi(t) = 0$, in the gauge $A_{\mu} = 0$, enforcing $\phi = \phi_o$, one would not have been able to deduce the crucial condition $\rho(\phi = \phi_o) \ = \ - p(\phi = \phi_o) \ = \ V(\phi = \phi_o)$ that will furnish the observed vacuum energy density today.

We will find now solutions of the Einstein-Friedman-Weyl equations in the gauge $A_{\mu} = (0, 0, 0, 0)$ after having explained why A_{μ} can (and must) be gauged to zero. The most relevant case corresponding to de Sitter space :

$$a(t) = e^{H_o t}; \ \ A_{\mu} = (0, 0, 0, 0); \ \ k = 0; \ \mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} = -12 \ H_0^2. \qquad (19)$$

where we will show that the potential is

$$V(\phi) = 12 H_0^2 \phi^2 \ + V_o. \qquad (20)$$

one learns in this case that $V(\phi = \phi_o) \neq 0$ since this non-vanishing value is precisely the one that shall furnish the observed vacuum energy density today (as we will see below) . We shall begin by solving the Einstein-Friedman-Weyl equations eq-(12-13) in the gauge $A_\mu = (0,0,0,0)$ for a spatially flat universe $k = 0$ and $a(t) = e^{H_0 t}$, corresponding to de Sitter metric :

$$ds^2 = dt^2 \ - \ e^{2H_o t} \ \ (dr^2 + r^2 (d\Omega)^2). \tag{21}$$

the Riemannian scalar curvature when $k = 0$ is

$$\mathcal{R}_{Riemann} = - \ 6 \ [\ (\frac{(d^2 a/dt^2)}{a}) + (\frac{(da/dt)}{a})^2 \ ] \ = -12 \ H_0^2 \tag{22}$$

(the negative sign is due to the chosen signature $+,-,-,-$).

To scalar Weyl curvature $\mathcal{R}_{Weyl}$ in the gauge $A_\mu = (0,0,0,0)$ is the same as the Riemannian one $\mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} = -12 \ H_0^2$. Inserting the condition $D_\mu \phi = D_t \phi(t) = (\partial_t \phi - A_t \phi) = \partial_t \ \phi = 0$, in the gauge $A_\mu = 0$, the generalized Klein-Gordon equation (3.20) will be satisfied if, and only if, the potential density $V(\phi)$ is chosen to satisfy

$$(\ 12 \ H_0^2) \ \phi \ = \ \frac{1}{2} \ (\frac{dV}{d\phi}) \ \Rightarrow \ V(\phi) \ = \ 12 \ H_0^2 \ \phi^2 \ + \ V_o \tag{23}$$

One must firstly differentiate w.r.t the scalar ϕ , and only afterwards, one may set $\phi = \phi_o$. $V(\phi)$ has a Weyl weight equal to -4 under Weyl scalings in order to ensure that the full action is Weyl invariant. H_0^2 and ϕ_o^2 have both a Weyl weight of -2, despite being constants, because as one performs a Weyl scaling of these quantities (a change of a scales) they will acquire then a spacetime dependence. H_0^2 is a masslike parameter, one may interpret H_0^2 (up to numerical factors) as the "mass" squared of the Jordan-Brans-Dicke scalar. We will see soon why the integration constant V_o plays the role of the "cosmological constant".

The potential density is $V = 12 H_o^2 \ \phi^2 + V_o$ where the integration constant V_o will be determined next. Some important remarks are in order prior to determining V_o. The potential density $V(\phi)$ has a Weyl weight of -4 under Weyl scalings to compensate for the Weyl weight of the measure of integration $\sqrt{|g|} \rightarrow e^{4\Omega} \ \sqrt{|g|}$ in the action. This implies that the Weyl weight of the term $H_o^2 \ \phi^2$ is $-2-2 = -4$, as well as the weight of V_o. This means that constants like H_o and ϕ_o behave like parameter-like scalars of weight -1 under Weyl scalings. There is no contradiction in assigning nontrivial Weyl weights to parameters like H_o, ϕ_o, V_o in Weyl geometry. It is the *dimensionless* ratio of parameters that is Weyl invariant.

The reason why constants like H_o, ϕ_o admit non-trivial weights is the following. A constant, like mass m in ordinary flat space is defined by $\partial_\mu (m) = 0$. A scalar "constant" like m of weight -1 in Weyl geometry is defined by $D_\mu (m) = (\partial_\mu - A_\mu)(m) = 0$, from which one can infer that $A_\mu \sim \partial_\mu \ log(m)$ and that leads to the conclusion that "constants" are compatible with Weyl's geometry if, and onli if, the Weyl gauge field A_μ is pure gauge, a total derivative. When m is set to a constant m_o independent of the coordinates this is tantamount of choosing the trivial gauge $A_\mu = 0$ condition. Under a Weyl gauge transformation, the constant m_o transforms into $m'_o = e^{-\Omega(x)}\ m_o$ and $A'_\mu = A_\mu - \partial_\mu \Omega$ and which is again compatible with the condition that $D_\mu(m'_o) = (\partial_\mu - A'_\mu)(m'_o) = 0 \Rightarrow A'_\mu \sim \partial_\mu\ log(m'_o) \neq 0$ because now m'_o has acquired an x^μ dependence through the scaling factor $m'_o = e^{-\Omega(x)}\ m_o$.

Despite that the potential density contribution to the action $\int \sqrt{|g|} V(\phi)$ does not break conformal symmetry when one sets $\phi = \phi_o$ in $V(\phi = \phi_o)$, because the parameters H_o, ϕ_o, V_o still scale properly under Weyl scalings, it *is* the gravitational term in the action (8) $\int \sqrt{|g|}\phi^2\ \mathcal{R}_{Weyl}$, that will *break* the Weyl scaling symmetry when it becomes $\int \sqrt{|g|}\phi_o^2\ \mathcal{R}_{Riemann}$, because the $\mathcal{R}_{Riemann}$ scalar curvature does *not* transform homogeneously under Weyl scalings. This is one of the most salient features of our findings because one is inclined to look for *quartic* potentials $V = \lambda\ \phi^4$ [15] which also scale properly under Weyl scalings, instead of recurring to *quadratic* potentials like we have found here within the framework of Weyl's geometry $V = 12 H_o^2 \phi^2 + V_o$. This fact, of quadratic versus quartic potentials is the key to obtaining the observed vacuum energy density.

An important remark is in order. Even if we included other forms of matter in the Einstein-Fredmann-Weyl equations, in the very large t regime, their contributions will be *washed* away due to their scaling behaviour. We know that ordinary matter ($p = 0$); dark matter ($p_{DM} = w \rho_{DM}$ with $-1 < w < 0$) and radiation terms ($p_{rad} = \frac{1}{3}\rho_{rad}$) are all *washed* away due to their scaling behaviour :

$$\rho_{matter} \sim R(t)^{-3}. \quad \rho_{radiation} \sim R(t)^{-4}. \quad \rho_{DM} \sim R(t)^{-3(1+w)}. \tag{24}$$

where $R(t) = a(t) R_0$. The dark energy density remains *constant* with scale since $w = -1$ and the scaling exponent is zero, $\rho_{DE} \sim R^0 = costant$. For this reason it is the only contributing factor at very large times.

Now we are ready to show that eqs-(12-13) are indeed satisfied when $a(t) = e^{H_0 t};\ k = 0;\ A_\mu = 0;\ \phi = \phi_o \neq 0$. Eq-(13b), due to the conditions $\rho + 3p = -2V(\phi)$ and $\phi(t) = \phi_o$ (resulting from the constraint eq-(16) in the $A_\mu = 0$ gauge) gives :

$$- (\frac{(d^2 a/dt^2)}{a}) \ = \ - H_0^2 = \ \frac{4\pi G_N}{3}\ (\rho + 3p) =$$

$$- \ (\frac{8\pi\ G_N\ V(\phi = \phi_o)}{3}) = - \ (\frac{8\pi\ G_N\ 12\ H_0^2\ \phi_o^2}{3}) \ - \ \frac{8\pi G_N\ V_o}{3}. \tag{25}$$

Eq-(12) (with $k = 0$) is just the same as eq-(13b) but with an overall *change* of sign because $\rho(\phi = \phi_o) = V(\phi = \phi_o)$. Using the definition $16\pi\ G_N\ \phi_o^2 = 1$ in (25) one gets

$$- H_0^2 \ = - \ (\frac{8\pi\ G_N\ 12\ H_0^2\ \phi_o^2}{3}) \ - \ \frac{8\pi\ G_N\ V_o}{3} \ = -2\ H_0^2 \ - \ \frac{8\pi\ G_N\ V_o}{3} \ \Rightarrow$$

$$- \frac{8\pi\ G_N\ V_o}{3} \ = \ H_0^2 \ \Rightarrow \ - 8\pi\ G_N\ V_o = 3\ H_0^2 \tag{26}$$

Therefore, we may identify the term $-\ V_o$ with the vacuum energy density so the quantity $3H_0^2 = -8\pi\ G_N\ V_o = \Lambda$ is nothing but the cosmological constant.

Hence one has from the last term of eq-(26) :

$$-V_o = \rho_{vacuum} = \frac{3H_0^2}{8\pi\ G_N}. \quad (27)$$

and finally, when we set $H_0^2 = (1/R_0^2) = (1/R_{Hubble}^2)$ and $G_N = L_{Planck}^2$ in the last term of eq-(26), as announced, the vacuum density ρ_{vacuum} observed today is *precisely* given by :

$$-V_o = \rho_{vacuum} \ = \ \frac{3H_0^2}{8\pi\ G_N} \ = \ \frac{3}{8\pi}\ (L_{Planck})^{-2}\ (R_{Hubble})^{-2} \ =$$

$$\frac{3}{8\pi}\ (\frac{1}{L_{Planck}})^4\ (\frac{L_{Planck}}{R_{Hubble}})^2\ \sim 10^{-123}\ (M_{Planck})^4. \quad (28)$$

Concluding this analysis of the Einstein-Friedman-Weyl eqs-(12-13) : By invoking the principle of Weyl scaling symmetry in the context of Weyl's geometry; when $k=0$ (spatially flat Universe), $a(t) = e^{H_0 t}$ (de Sitter inflationary phase) ; $H_o =$ Hubble constant today; $\phi(t) = \phi_o = constant$, such $16\pi G_N\ \phi_o^2 = 1$, one finds that

$$V(\phi = \phi_o) = 12\ H_0^2\ \phi_o^2\ +\ V_o =\ 2\rho_{vacuum}\ -\ \rho_{vacuum}\ =\ \rho_{vacuum} =$$

$$6H_0^2\phi_o^2 = \frac{3H_0^2}{8\pi\ G_N} \sim 10^{-123}\ M_{Planck}^4. \quad (29)$$

is precisely the observed vacuum energy density (28) . Therefore, the observed vacuum energy density is intrinsically and inexorably linked to the potential density $V(\phi = \phi_o)$ corresponding to the Jordan-Brans-Dicke scalar ϕ required to build Weyl invariant actions and evaluated at the special point $\phi_o^2 = (1/16\pi G_N)$.

Another way of obtaining the *same* value for the vacuum energy density is by rewriting the generalized Klein-Gordon equation (15) in the form

$$-D^\mu D_\mu \Phi \ = \ \frac{1}{2\omega + 3}\ [\ \Phi\ \frac{\partial V(\Phi)}{\partial \Phi}\ -\ 2V(\Phi)\ ]; \quad \Phi \equiv \phi^2; \quad \omega = \frac{1}{4}. \quad (30)$$

which can be derived directly by taking the trace of the Einstein-Weyl equations (11b, 11c) (when $T_{\mu\nu}^{matter} = 0$)

$$-\mathcal{R}_{Weyl} \ = \ \frac{1}{4\Phi^2}\ (D^\mu \Phi)\ (D_\mu \Phi)\ +\ \frac{3}{\Phi}\ D^\mu D_\mu(\Phi)\ +\ \frac{2}{\Phi} V(\Phi); \quad \omega = \frac{1}{4}. \quad (31)$$

and by substituting the Weyl scalar curvature in the generalized Klein-Gordon equation in terms of the expression given by (31). When $D^\mu D_\mu \Phi = 0$, eq-(30) leads to

$$\Phi \frac{\partial V(\Phi)}{\partial \Phi} - 2V(\Phi) = 0 \Rightarrow V(\Phi) = \mathcal{V}_o \left(\frac{\Phi}{\Phi_o}\right)^2 = \mathcal{V}_o \left(\frac{\phi}{\phi_o}\right)^4. \qquad (32)$$

After inserting the value $\mathcal{R}_{Weyl} = -12\ H_o^2$ obtained from eq-(19) in the gauge $A_\mu = 0$; gauging ϕ to a constant $\phi = \phi_o \Rightarrow D_\mu(\Phi_o) = 0$, and substituting the expression for the potential $V(\Phi) = V(\phi^2)$ found in eq-(32) into eq-(31), when $\phi = \phi_o$, gives

$$12\ H_o^2\ \phi_o^2 = 2\ \mathcal{V}_o \left(\frac{\phi_o}{\phi_o}\right)^4 = 2\ \mathcal{V}_o \Rightarrow$$

$$\mathcal{V}_o = 6\ H_o^2\ \phi_o^2 = \frac{3H_0^2}{8\pi\ G_N} \sim 10^{-123}\ M_{Planck}^4 \qquad (33)$$

which is again the observed value of the vacuum energy density. Having determined the value of the constant $\mathcal{V}_o = 6H_o^2\phi_o^2$ appearing in the potential found in (32) it yields the most general expression for other values of ϕ

$$V(\Phi) = \mathcal{V}_o \left(\frac{\Phi}{\Phi_o}\right)^2 = \mathcal{V}_o \left(\frac{\phi}{\phi_o}\right)^4 = 6\ H_o^2\ \phi_o^2 \left(\frac{\phi}{\phi_o}\right)^4 \qquad (34)$$

thus, it is the *particular* value $V(\phi = \phi_o) = 6\ H_o^2\ \phi_o^2$ that leads to the observed vacuum energy density *today*.

Concluding, one has been able to reproduce the observed vacuum energy density (cosmological constant) to very high precision such that the temporal flow of the scalar field $\phi(t)$ from $t = 0$ to $t = t_o$, in Riemannian geometry, has the same numerical effects (as far as the vacuum energy density is concerned) as Weyl scalings from the field configuration $\phi(t)$ to the constant field configuration ϕ_o. We believe this temporal flow of the vacuum energy density, from very high values in the past, to very small values today t_o, is not a numerical coincidence but is the signal of an underlying Weyl geometry (conformal invariance) operating in cosmology and combined with the dynamics of a Brans-Dicke-Jordan scalar field. Therefore, Weyl scalings in Weyl geometry can recapture the flow of time consistent with Segal's Conformal Cosmology, see [16], [4] and references therein, in such a fashion that an expanding universe may be visualized as Weyl scalings of a static universe. Scalings as time's arrow has been investigated by others [11] within a different context than Weyl's geometry and Brans-Dicke-Jordan gravity. These ideas deserves further investigation.

The most general Lagrangian involving dynamics for A_μ is

$$\mathcal{L} = -\phi^2 \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (D_\mu\phi)(D_\nu\phi) - V(\phi) + L_{matter} + ... \qquad (35)$$

The L_{matter} must involve the full fledged Weyl gauge covariant derivatives acting on scalar and spinor fields . It is a well known fact to the experts that the electron neutrino mass $m_\nu \sim 10^{-3}\ eV$ is of the same order as $(m_\nu)^4 \sim$

$10^{-123}\ M^4_{Planck}$ and that the SUSY breaking scale in many models is given by a geometric mean relation : $m^2_{SUSY} = m_\nu\ M_{Planck} \sim (5\ TeV)^2$.

This completes our review and corrections to the derivation of the vacuum energy density [3] and a new derivation based on eqs-(30-33). The main new lesson found here is *why* the quadratic potential $V(\phi) = 12 H_o^2\ \phi^2 + V_o$ does not break the Weyl scaling symmetry after fixing the gauge $\phi = \phi_o$, leading to the observed vacuum energy density $V(\phi_o) = 6 H_o^2 \phi_o^2$, but it is the Brans-Dicke-Jordan gravitational terms $\phi^2\ \mathcal{R}_{Weyl}$ that break the Weyl invariance when they become $\phi_o^2\ \mathcal{R}_{Riemann} = \frac{1}{16\pi G_N}\ \mathcal{R}_{Riemann}$ and leading to the standard Einstein-Hilbert action. The reason being that the scalar Riemann curvature does *not* transform properly under Weyl transformations. The cosmological constant in the gauge-fixed action (8), when $A_\mu = 0 \Rightarrow \phi = \phi_o$ and $D_\mu \phi = 0$, is defined by $8\pi G_N\ \rho_{vacuum}$:

$$\Lambda \ = \ (8\pi G_N)\ (6 H_o^2\ \phi_o^2) \ = \ (8\pi G_N)\ (6 H_o^2)\ (\frac{1}{16\pi G_N}) \ = \ \frac{3}{R_H^2}; \quad H_o = \frac{1}{R_{Hubble}}.$$

as observed, an ever (accelerating) expanding de Sitter Universe with a very small (bot not zero) cosmological constant of the order of $10^{-123} M^2_{Planck}$.

To finalize, there are many differences among our approach and that of [12]. (i) The Cheng-Weyl approach [12] to account for dark energy and matter (including phantom) does *not* use the Weyl scalar curvature with a variable Newtonian coupling $16\pi\ G = \phi^{-2}$ for the gravitational part of the action, but the ordinary Riemannian scalar curvature with the standard Newtonian gravitational constant . (ii) There was no use of Weyl covariant derivatives in the matter terms. The Weyl covariant derivative is *only* used in the kinetic $(D_\mu \phi)^2$ terms for the Jordan-Brans-Dicke scalar ϕ . And, (iii) the authors [12] introduced a triplet of Cheng-Weyl gauge fields $A^1_\mu, A^2_\mu, A^3_\mu$ whereas here we have only one field A_μ. The role of conformal transformations in accelerated cosmologies has bee studied by [6]. Weyl invariance has been used in [14] to construct Weyl-Conformally Invariant Light-Like p-Brane Theories with numerous applications in Astrophysics, Cosmology, Particle Physics Model Building, String theory,.....

Concerning Weyl geometry and matter creation in the universe see the work of [10]. A thorough study of the unification of geometric and random structures in Physics within the framework of Riemann-Cartan-Weyl spacetimes has been performed by [13]. Conformal Transformations and Accelerated Cosmologies have been studied by [6]. The vacuum energy problem from the Finsler geometry perspective has been analyzed by [8]; modified $f(R)$ gravity actions as another approach to the dark energy problem can be found in [7] and references therein. Energy conditions in $f(R)$ gravity and Brans-Dicke Theories have been studied recently by [9].

Acknowledgements

We thank Frank (Tony) Smith for discussions and M. Bowers for assistance.

References

[1] S. Weinberg, *Cosmology* (Oxford University Press, 2008).

[2] H. Kim, " Can the Brans-Dicke gravity possibly with Λ be a theory of Dark matter ? "astro-ph/0604055. M. Arik and M. Calik, " Can Brans-Dicke scalar field account for dark energy and dark matter ? gr-qc/0505035. Chao-Jun Feng, "Ricci Dark Energy in Brans-Dicke Theory" arXiv : 0806.0673.

[3] C. Castro, How Weyl Geometry solves the Riddle of Dark Energy in *Quantization Astrophysics, Brownian Motion and Supersymmetry* pp. 88- 96 (eds F. Smarandache and V. Christianato, Math. Tiger, Chennai, India 2007). Foundations of Physics **vol 37**, no. 3 (2007) 366. Mod. Phys. Lett **A17** (2002) 2095-2103 Mod. Phys. Lett A 21 , no. 35 (2006) 2685-2701

[4] E. Scholz, " On the Geometry of Cosmological Model Building" gr-qc/0511113

[5] H. Wei and R-G Cai, " Cheng-Weyl Vector Field and Cosmological Application" astro-phy/0607064.

[6] J. Crooks and P. Frampton, " Conformal Transformations and Accelerated Cosmologies " astro-ph/0601051. M.Cardoni, " Conformal Symmetry of Gravity and the Cosmological Constant Problem" hep-th/0606274.

[7] S. Capozzielo, V. Cardone and A. Troisi, " Jour. of Cosmology and Astroparticle Physics **08** (2006) 001. S. Capozziello, S. Nojiri and S. Odintsov, Phys. Letts **B 634** (2006) 93.

[8] H. Brandt, J. Modern Optics 51, 2753-2759 (2004).

[9] K. Atazadeh, A. Khaleghi, H. Sepangi and Y. Tavakoli, "Energy conditions in $f(R)$ gravity and Brans-Dicke Theories" arXiv : 0811.4269.

[10] M. Israelit, *Found. Phys* **29**, 1303 (1999). *Found. Phys* **32**, 295 (2002). *Found. Phys* **32**, 945 (2002). *The Weyl-Dirac theory and our universe* (Nova Science Publishers 1999).

[11] M. Kafatos, S. Roy and R. Amoroso, ″Scaling in cosmology and the arrow of time" in *Studies on the Structure of Time*, (edited by Buccheri et al; Kluwer Academic Publishers. New York , 2000). J. Glanz, *Science* **282**, 2156 (1998).

[12] H. Wei and R-G Cai, ″Cheng-Weyl vector field and cosmological application" [arXiv.org : astro-phy/0607064].

[13] D. Rapoport, *Found. Phys* **35**, 1205 (2005). *Found. Phys* **35**, 1383 (2005).

[14] E. Guendelman and A. Kaganovich, *Phys. Rev* **D 53**, 7020 (1996). *Phys. Rev* **D 60**, 065004 (1999). *Int. J. Mod. Phys* **A 17**, 417 (2002). E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, ″Weyl-conformally invariant light-like p-brane Theories ” [arXiv.org: hep-th/0409078].

[15] C. Wetterich, ”Dilatation symmetry in higher dimensions and the vanishing of the cosmological constant” arXiv : 0806.0741.

[16] F.D. Smith Jr, ”The Physics of E_8 and $Cl(16) = Cl(8) \otimes Cl(8)$ ” www.tony5m17h.net/E8physicsbook.pdf (Carterville, Georgia, June 2008, 367 pages).