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Circularly Polarized Plane Wave Has Spin Angular Momentum

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Abstract: Absorption of a circularly polarized light beam is considered as a laser-material interaction. It is shown that the torque acting on the material is twice as much as the standard electrodynamics predicts.
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The beam [1,2] of power $\mathsf{P}=1$, $\mathbf{E}=e^{iz-it}[\mathbf{x}+i\mathbf{y}+\mathbf{z}(i\partial_x-\partial_y)]E_0(r)$, $\mathbf{B}=-i\mathbf{E}$, is considered. Here $E_0=E_0(0)$ if $r<R_0-\delta/2$, $E_0=0$ if $r>R_0+\delta/2$, $\int_{-\infty}^{+\infty}E_0^2dxdy=\int_0^\infty E_0^2\,2\pi r\,dr=E_0^2(0)\pi R_0^2=1$ (see Fig. 1). This circularly polarized beam is absorbed by a plane $z=0$, and the mechanical stress produced in the plane by the beam is calculated. It is shown that the central part of the beam produces a torque at the central region of the plane due to the spin of the beam, and the skin of the beam produces an additional torque due to the orbital angular momentum of the beam. The total torque acting on the plane equals $\tau_{\text{tot}}=2\mathsf{P}/\omega$. This fact contradicts the standard electrodynamics, which predicts the torque equals $\tau_{\text{tot}}=\mathsf{P}/\omega$, and means the standard electrodynamics, as well as the whole classical field theory, is incomplete. An introducing of a spin tensor corrects the electrodynamics. Here is our calculation.

Components of the momentum density $\mathbf{p}$ in the circularly polarized beam are

$$p^x=<(\mathbf{E}\times\mathbf{B})_x>=\Re(E_y\overline{B}_z-E_z\overline{B}_y)/2=\partial_yE_0^2/2,\quad p^y=-\partial_xE_0^2/2,\quad p^\phi=-\partial_rE_0^2/2. \quad (1)$$

(We set $k=\omega=c=1$). The beam exerts the force density $f^\phi=-\partial_rE_0^2/2$ on the plane and gives rise to a 2-dimensional stress tensor density T^{ik} of the plane, according to $\nabla_kT^{ik}\equiv\partial_kT^{ik}+\Gamma^i_{jk}T^{jk}=f^i$ where ∇_kT^{ik} is the covariant divergence of the density [3] and Γ^i_{jk} are the Christoffel symbols of the used cylindrical coordinates r,ϕ,z:
$\Gamma^r_{\phi\phi}=-r,\ \Gamma^\phi_{\phi r}=\Gamma^\phi_{r\phi}=1/r,\ \Gamma^k_{kr}=\Gamma^k_{rk}=1/r$ (we ignore the light pressure). Because $T^{\phi r}=T^{r\phi}$, we have

$$\partial_rT^{\phi r}+\Gamma^\phi_{jk}T^{jk}\equiv\partial_rT^{\phi r}+\Gamma^\phi_{\phi r}T^{\phi r}+\Gamma^\phi_{r\phi}T^{r\phi}\equiv\partial_rT^{\phi r}+2T^{\phi r}/r=f^\phi=-\partial_rE_0^2/2. \quad (2)$$

This equation has a solution, which is depicted in Fig. 2,

$$T^{\phi r}=C(r)/r^2,\quad \partial_rC/r^2=-\partial_rE_0^2/2,\quad C(r)=-r^2E_0^2/2+\int_0^r rE_0^2dr. \quad (3)$$

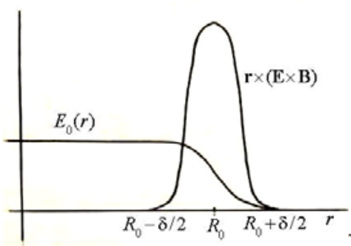

Fig. 1. The profile of the electric field of the beam and the angular momentum density.

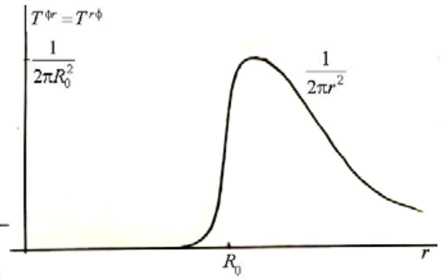

Fig. 2. The component of the stress tensor arisen from the orbital angular momentum.

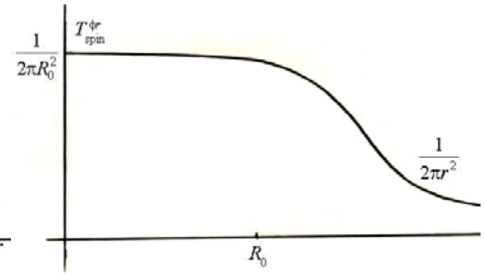

Fig. 3. The component of the stress tensor arisen from the spin angular momentum

It is easy to verify that $T^{rr}=T^{\phi\phi}=0$. The solution (3) means that $T^{\phi r}=0$ if $r<R_0-\delta/2$ and $T^{\phi r}=1/(2\pi r^2)$ if $r>R_0+\delta/2$, i.e. there is no mechanical stress in the central region of the target plane, while the outside part of the plane ($r>R_0+\delta/2$) experiences the torque

$$\tau = rT^{\phi r}2\pi r = 1 \qquad (4)$$

This is because the electric and magnetic fields of the beam have a nonzero *z*-component only within the skin region of δ-thickness of the beam. Having *z*-component within this region implies the possibility of a nonzero *z*-component of angular momentum within this region. Since the fields is identically zero outside the skin and constant inside the skin region, the skin region is the only one in which the *z*-component of angular momentum does not vanish [2,4]. The result (4) is in accordance with the common opinion that $\tau = \mathsf{P}/\omega$ (we set $\omega=1$).

However, the central region of the plane evidently catches spin angular momentum and experiences a torque. This torque arises from the fact that the dielectric constant ε is a tensor. Consequently, the electric intensity $\mathbf{E}$ is not parallel to the electric polarization $\mathbf{P}$ in the medium of the plane [5,6]. But this torque is not connected with a moment of the momentum (1) and with the torque (4), and requires a concept of the electrodynamics' spin tensor [7-11], $\Upsilon^{\lambda\mu\nu}=(A^{[\lambda}\partial^{|\nu|}A^{\mu]}+\Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]})/2$, where A^λ and Π^λ are magnetic and electric vector potentials: $2\partial_{[\mu}A_{\nu]}=F_{\mu\nu}$, $2\partial_{[\mu}\Pi_{\nu]}=-e_{\mu\nu\alpha\beta}F^{\alpha\beta}$ and $F^{\alpha\beta}=-F^{\beta\alpha}$, $F_{\mu\nu}=F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ ($\mu,\nu,...=0,1,2,3$) is the field strength tensor. The sense of a spin tensor $\Upsilon^{\lambda\mu\nu}$ is as follows. The component Υ^{ij0} is a volume density of spin. This means that $dS^{ij}=\Upsilon^{ij0}dV$ is the spin of electromagnetic field inside the spatial element dV. The component Υ^{ijk} is a flux density of spin flowing in the direction of the x^k axis. For example, $dS_z/dt=dS^{xy}/dt=d\tau^{xy}=\Upsilon^{xyz}da_z$ is the *z*-component of spin flux passing through the surface element da_z per unit time, i.e. the torque acting on the element.

We take account of the spin torque here. For our beam, the calculation [11] gives $\Upsilon^{r\phi z}=E_0^2$.

As is known, the local conservation law $\nabla_k T^{ik}=0$ is accompanied by the angular momentum conservation law (see, e.g., [12] p. 64) $2T^{[ij]}=\nabla_k\Upsilon^{ijk}$. In our case $2\underset{3}{T}{}^{[r\phi]}=\partial_z\Upsilon^{r\phi z}$ where $\underset{3}{T}{}^{[r\phi]}$ is an antisymmetric part of the 3-dimensional stress tensor density in material of the absorbing plane. By the integration $\int_0^\infty 2\underset{3}{T}{}^{[r\phi]}dz=\int_0^\infty\partial_z\Upsilon^{r\phi z}dz$, we arrive at an a*symmetric* 2-dimensional stress tensor density, T^{ij}_{spin}, which characterizes the medium absorbing spin angular momentum flux and satisfies the two equations, instead of (2),

$$2T_{\text{spin}}^{[r\phi]} = -\Upsilon^{r\phi z} = -E_0^2\,, \quad \partial_r T_{\text{spin}}^{\phi r} + 2T_{\text{spin}}^{(\phi r)}/r = 0\,. \qquad (5)$$

These equations have a solution (see Fig. 3): $T_{\text{spin}}^{\phi r} = -T_{\text{spin}}^{r\phi} = 1/(2\pi R_0^2)$ if $r < R_0 - \delta/2$, $T_{\text{spin}}^{\phi r} = T_{\text{spin}}^{r\phi} = 1/(2\pi r^2)$ if $r > R_0 + \delta/2$. It means that stress tensor is antisymmetric in the central region.

The component of the total stress tensor density is $T_{\text{tot}}^{\phi r} = T^{\phi r} + T_{\text{spin}}^{\phi r} = 1/(\pi r^2)$ if $r > R_0 + \delta/2$. Thus, the outside part of the plane experiences the torque

$$\tau_{\text{tot}} = rT_{\text{tot}}^{\phi r}\,2\pi r = 2 \qquad (6)$$

The result (6) is in accordance with the formula $\tau = 2\mathsf{P}/\omega$ [7-11] and contradicts the common formula $\tau = \mathsf{P}/\omega$. This means also that a circularly polarized plane wave has spin angular momentum described by the spin tensor [13]

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