$4D$ Quantum Gravity via W_∞ Gauge Theories in $2D$, Collective Fields and Matrix Models

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Abstract

It is shown how Quantum Gravity in $D = 3$ can be described by a W_∞ Matrix Model in $D = 1$ that can be solved *exactly* via the Collective Field Theory method. A quantization of $4D$ Gravity can be attained via a $2D$ Quantum W_∞ gauge theory coupled to an infinite-component scalar-multiplet ; i.e. the quantization of Einstein Gravity in $4D$ admits a reformulation in terms of a $2D$ Quantum W_∞ gauge theory coupled to an infinite family of scalar fields. Since higher-spin W_∞ symmetries are very relevant in the study of $2D$ W_∞ Gravity, the Quantum Hall effect, large N QCD, strings, membranes, topological QFT, gravitational instantons, Noncommutative $4D$ Gravity, Modular Matrix Models and the Monster group, it is warranted to explore further the interplay among all these theories.

Keywords: Quantum Gravity, W_∞-gravity, W_∞ Gauge Theories, Higher spins, Holography, Moyal Brackets, Collective Field Theory, Strings, Branes, Matrix Models.

1 Gravity in $D = m + n$ as an m-dim Gauge Theory of diffeomorphisms of an internal n-dim space and Holography

Some time ago Park [1] showed that $4D$ Self Dual Gravity is equivalent to a WZNW model based on the group $SU(\infty)$. Namely, $4D$ Self Dual Gravity is the non-linear sigma model based in $2D$ whose target space is the "group manifold" of area-preserving diffs of another $2D$-dim manifold. Roughly speaking, this means that the effective $D = 4$ manifold, where Self Dual Gravity is defined, is "spliced" into two $2D$-submanifolds: one submanifold is the original $2D$ base manifold where the non-linear sigma model is defined. The other $2D$ submanifold is the target group manifold of area-preserving diffs of a two-dim sphere S^2.

The authors [2] went further and generalized this particular Self Dual Gravity case to the full fledged gravity in $D = 2 + 2 = 4$ dimensions, and in general, to *any* combinations of $m + n$-dimensions. Their main result is that $m + n$-dim Einstein gravity can be identified with an m-dimensional generally invariant gauge theory of $Diffs\ N$, where N is an n-dim manifold. Locally the $m + n$-dim space can be written as $\Sigma = \mathcal{M} \times \mathcal{N}$ and the metric G_{AB} decomposes as:

$$G_{AB} = \begin{pmatrix} g_{\mu\nu}(x,y) \ + \ e^2 g_{ab}(x,y)\ A^a_\mu(x,y)\ A^b_\nu(x,y) & eA^a_\mu(x,y)\ g_{ab}(x,y) \\ eA^a_\mu(x,y)\ g_{ab}(x,y) & g_{ab}(x,y) \end{pmatrix}, \tag{1.1}$$

It must not be confused with the Kaluza-Klein reduction where one imposes an isometry restriction on the γ_{AB} that turns A^a_μ into a gauge connection associated with the gauge group G generated by isometry. Dropping the isometry restrictions allows *all* the fields to depend on *all* the coordinates x, y. Nevertheless $A^a_\mu(x,y)$ can still be identified as a connection associated with the infinite-dim gauge group of $Diffs\ N$. The gauge transformations are now given in terms of Lie-brackets and Lie derivatives:

$$\delta A^a_\mu = -\frac{1}{e} D_\mu \xi^a = -\frac{1}{e}(\partial_\mu \xi^a - e[A_\mu, \xi]^a) = -\frac{1}{e}(\partial_\mu - e\mathcal{L}_{A_\mu})\xi^a,$$

$$A_\mu \equiv A^a_\mu \partial_a,$$

$$\mathcal{L}_{A_\mu}\xi^a \equiv [A_\mu, \xi]^a,$$

$$\delta g_{ab} = -[\xi, g]_{ab} = \xi^c \partial_c g_{ab} + g_{ac}\partial_b \xi^c + g_{cb}\partial_a \xi^c,$$

$$\delta g_{\mu\nu} = -[\xi, g_{\mu\nu}]. \tag{1.2}$$

In particular, if the relevant algebra is the area-preserving diffs of S^2, given by the suitable basis dependent limit $SU(\infty)$ [22], one induces a natural Lie-Poisson structure generated by the gauge fields A_μ. The Lie derivative of f along a vector ξ is the Lie bracket $[\xi, f]$, which coincides in this case with the Poisson bracket $\{\xi, f\}$. This implies that the Lie brackets of two generators of the area-preserving diffs S^2 is given precisely by the generator associated with their respective Poisson brackets (a Lie-Poisson structure):

$$[L_f, L_g] = L_{\{f,g\}}. \tag{1.3}$$

This relation is derived by taking the vectors ξ^a_1, ξ^a_2, along which we compute the Lie derivatives, to be the symplectic gradients of two functions $f(\sigma^1, \sigma^2), g(\sigma^1, \sigma^2)$:

$$\xi_1^a = \Omega^{ab}\partial_b f, \quad \xi_2^a = \Omega^{ab}\partial_b g. \tag{1.4}$$

When nontrivial topologies are involved one must include harmonic forms ω into the definition of ξ^a [6] allowing central terms for the algebras. This relation can be extended to the volume-preserving diffs of N by means of the Nambu-Poisson brackets:

$$\{A_1, A_2, A_3, ...A_n\} = Jacobian = \frac{\partial(A_1, A_2, A_3, ..., A_n)}{\partial(\sigma^1, \sigma^2, ...\sigma^n)} \Rightarrow$$

$$[L_{A_1}, L_{A_2}, ..., L_{A_n}] = L_{\{A_1, A_2, ..., A_n\}}, \tag{1.5}$$

which states that the Nambu-commutator of n-generators of the volume-preserving diffs of $\mathcal{N}$ is given by the generator associated with their corresponding Nambu-Poisson brackets.

Using eq-(1.1) the authors [2] have shown that the curvature scalar $R^{(m+n)}$ in $m+n$-dim decomposes into:

$$R^{(m+n)} \;=\; g^{\mu\nu}R^{(m)}_{\mu\nu} \;+\; \frac{e^2}{4} g_{ab}F^a_{\mu\nu}F^b_{\rho\tau}g^{\mu\rho}g^{\nu\tau} \;+\; g^{ab}R^{(n)}_{ab} \;+$$

$$\frac{1}{4}g^{\mu\nu}g^{ab}g^{cd}D_\mu g_{ab}D_\nu g_{cd} \;+\; \frac{1}{4}g^{ab}g^{\mu\nu}g^{\rho\tau}[\;\partial_a g_{\mu\rho}\partial_b g_{\nu\tau} \;-\; \partial_a g_{\mu\nu}\partial_b g_{\rho\tau}\;] \tag{1.6}$$

plus total derivative terms given by

$$\partial_\mu(\;\sqrt{|det\;g_{\mu\nu}|}\;\sqrt{|det\;g_{ab}|}\;J^\mu\;) \;-\partial_a(\;\sqrt{|det\;g_{\mu\nu}|}\;\sqrt{|det\;g_{ab}|}\;eA^a_\mu J^\mu) \;+$$

$$\partial_a(\;\sqrt{|det\;g_{\mu\nu}|}\;\sqrt{|det\;g_{ab}|}\;\;J^a\;), \tag{1.7}$$

with the currents:

$$J^\mu = g^{\mu\nu}g^{ab}D_\nu g_{ab}, \quad J^a = g^{ab}g^{\mu\nu}\partial_b g_{\mu\nu}, \tag{1.8}$$

$$S \;=\; \frac{1}{16\pi G}\;\int\; d^m x\; d^n y\;\sqrt{|det(g_{\mu\nu})|}\;\sqrt{|det(g_{ab})|}\;R^{(m+n)}(x,y). \tag{1.9}$$

Therefore, Einstein gravity in $m+n$-dim describes an m-dim generally invariant field theory under the gauge transformations or Diffs $\mathcal{N}$. Notice how A^a_μ couples to the graviton $g_{\mu\nu}$, meaning that the graviton is charged /gauged in this theory and also to the g_{ab} fields. The "metric" g_{ab} on N can be identified as a non-linear sigma field whose self interaction potential term is given by: $g^{ab}R^{(n)}_{ab}$. The currents J^μ, J^a are functions of $g_{\mu\nu}, A_\mu, g_{ab}$. Their contribution to the action is essential when there are boundaries involved; i.e. like in the AdS/CFT correspondence.

When the internal manifold $\mathcal{N}$ is a homogeneous compact space one can perform a harmonic expansion of the fields w.r.t the internal y coordinates, and after integrating w.r.t these y coordinates, one will generate an infinite-component field theory on the m-dimensional space. A reduction of the Diffs $\mathcal{N}$, via the inner automorphims of a subgroup G of the Diffs $\mathcal{N}$, yields the usual Einstein-Yang-Mills theory interacting with a nonlinear sigma field. But in general, the theory described in (1.9) is by far *richer* than the latter theory. A crucial fact of the decomposition (2.6, 2.7) is that *each* single term in (1.6, 21.7) is by itself independently invariant under Diffs $\mathcal{N}$. The second term of (1.6), for example,

$$\frac{1}{16\pi G}\ \sqrt{|det(g_{\mu\nu})|}\ \sqrt{|det(g_{ab})|}\ \frac{e^2}{4} g_{ab} F^a_{\mu\nu} F^b_{\rho\tau} g^{\mu\rho} g^{\nu\tau}, \tag{1.10}$$

is precisely the one that is related to the large N limit of $SU(N)$ YM [11].

The decomposition of the higher-dim Einstein-Hilbert action shown in eq-(1.6, 1.7) required to use a non-holonomic basis of derivatives $\partial_\mu - eA^a_\mu \partial_a$ and ∂_a that allows a diagonal decomposition of the metric and simplifies the computation of all the geometrical quantities. In this sense, the lower m-dimensional spacetime gauged "Ricci scalar" term $g^{\mu\nu}(x,y) R^{(m)}_{\mu\nu}(x,y)$ and the internal space "Ricci scalar" term $g^{ab}(x,y) R^{(n)}_{ab}(x,y)$ are obtained. In the special case when $g_{\mu\nu}(x)$ depends solely on x and $g_{ab}(y)$ depends on y then the spacetime gauged "Ricci scalar" coincides with the ordinary Ricci scalar $g^{\mu\nu}(x) R^{(m)}_{\mu\nu}(x)$ and the internal space "Ricci scalar" becomes the true Ricci scalar of the internal space. However, the gauge field $A_\mu(x,y)$ still retains its full dependence on both variables x, y.

We have shown [4] that in this particular case the $D = m + n$ dimensional gravitational action restricted to $AdS_m \times S^n$ backgrounds admits a *holographic* reduction to a lower $d = m$-dimensional Yang-Mills-like gauge theory of diffs of S^n, interacting with a charged/gauged nonlinear sigma model plus boundary terms, by a simple tuning of the radius of S^n and the size of the throat of the AdS_m space. Namely, in the case of $AdS_5 \times S^5$, the holographic reduction occurs if, and only if, the size of the AdS_5 throat *coincides* precisely with the radius of S^5 ensuring a *cancellation* of the scalar curvatures $g^{\mu\nu} R^{(m)}_{\mu\nu}$ and $g^{ab} R^{(n)}_{ab}$ in eq-(1.6) [4]:

$$\begin{aligned} R^{(10)} \ &= \ \frac{e^2}{4}\ g_{ab}(y)\ F^a_{\mu\nu}(x,y)\ F^b_{\rho\tau}(x,y)\ g^{\mu\rho}(x)\ g^{\nu\tau}(x)\ + \\ & \frac{1}{4} g^{\mu\nu}(x)\ g^{ab}(y)\ g^{cd}(y)\ (D_\mu g_{ab})\ (D_\nu g_{cd}) \end{aligned} \tag{1.11}$$

plus total derivative terms (boundary terms)

$$D_\mu\ g_{ab}\ =\ \partial_\mu g_{ab}\ +\ [A_\mu, g_{ab}].$$

where the Lie-bracket is

$$\begin{aligned} [\ A_\mu\ ,\ g_{ab}\ ]\ =\ (\partial_a\ A^c_\mu(x^\mu, y^a))\ g_{bc}(x^\mu, y^a)\ +\ (\partial_b\ A^c_\mu(x^\mu, y^a))\ g_{ac}(x^\mu, y^a)\ + \\ A^c_\mu(x^\mu, y^a)\ \partial_c\ g_{ab}(x^\mu, y^a). \end{aligned} \tag{1.12}$$

and the Yang-Mills like field strength is

$$F^a_{\mu\nu} \ = \ \partial_\mu A^a_\nu \ - \partial_\nu A^a_\mu \ - \ [A_\mu, A_\nu]^a \ = $$
$$\partial_\mu A^a_\nu \ - \partial_\nu A^a_\mu \ - \ A^c_\mu \partial_c A^a_\nu \ + A^c_\nu \partial_c A^a_\mu. \tag{1.13}$$

Eq-(1.11) is nothing but the *holographic* reduction of the $D = 10$-dim pure gravitational action to a 5-dim Yang-Mills-like action (of diffeomorphisms of the internal S^5 space) interacting with a charged nonlinear sigma model (involving the g_{ab} field) plus boundary terms. The previous argument can also be generalized to gravitational actions restricted to de Sitter spaces, like $dS_m \times H^n$ backgrounds as well, where H^n is an internal hyperbolic noncompact space of constant negative curvarture, and dS_m is a de Sitter space of positive constant scalar curvature. The decomposition (1.11) provided a very straightfoward explanation of why AdS spaces played a crucial importance in the Maldacena AdS/CFT duality conjecture, because the algebra of area-preserving diffs of the sphere is isomorphic to the large N (basis dependent) limit of $SU(N)$, as shown by Hoppe long ago [22]; i.e. why higher-dim gravity admits a holographic reduction to a lower-dim $SU(\infty)$ YM theory. It is unfortunate that the important work of [2], [3] that already contained the seeds of the holographic principle was largely ignored by the physics community.

Introducing the light-cone coordinates u, v such that

$$u \ = \ \frac{1}{\sqrt{2}} \ (x^0 + x^1), \quad v \ = \ \frac{1}{\sqrt{2}} \ (x^0 - x^1). \tag{1.14}$$

and define

$$A^a_u \ = \ A^a_+ \ = \ \frac{1}{\sqrt{2}} \ (A^a_0 + A^a_1), \quad A^a_v \ = \ A^a_- \ = \ \frac{1}{\sqrt{2}} \ (A^a_0 - A^a_1). \tag{1.15}$$

the Polyakov ansatz is [24]

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 2h_{++} \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} -2h_{++} & -1 \\ -1 & 0 \end{pmatrix}, \quad det \ g_{\mu\nu} = -1. \tag{1.16}$$

$$g_{ab} \ = \ e^\sigma \ \rho_{ab}; \quad det \ \rho_{ab} \ = \ 1. \tag{1.17}$$

The covariant derivative of a tensor *density* ρ_{ab} with weight 1 is

$$D_\mu \ \rho_{ab} \ = \ \partial_\mu \ \rho_{ab} \ - [\ A_\mu \ , \ \rho \ ]_{ab} \ + (\partial_c A^c_\mu) \ \rho_{ab} \ = $$

$$\partial_\mu \ \rho_{ab} \ - A^c_\mu \partial_c \ \rho_{ab} \ - (\partial_a A^c_\mu) \ \rho_{cb} \ - (\partial_b A^c_\mu) \ \rho_{ac} \ + (\partial_c A^c_\mu) \ \rho_{ab}. \tag{1.18}$$

the covariant derivative on the scalar density $\Omega = e^\sigma$ of weight -1 is

$$D_\mu \Omega \ = \ \partial_\mu \Omega \ - A^a_\mu \partial_a \Omega \ - (\partial_a A^a_\mu) \Omega \ \Rightarrow$$

$$D_\mu \sigma \ = \ \partial_\mu \sigma \ - A^a_\mu \partial_a \sigma \ - (\partial_a A^a_\mu). \tag{1.19}$$

after factoring the e^σ terms. Notice the extra term $w(\rho_{ab})(\partial_c A^c_\mu)\rho_{ab}$ in the definition of the covariant derivative acting on a tensor density ρ_{ab} whose weight is $w(\rho_{ab}) = 1$. Similarly there is an extra term $-(\partial_a A^a_\mu)\Omega$ in the covariant derivative of the scalar density Ω of weight -1. The Yang-Mills like field strength is

$$F^a_{+-} = \partial_+ A^a_- - \partial_- A^a_+ - [A_+, A_-]^a =$$
$$\partial_+ A^a_- - \partial_- A^a_+ - A^c_+ \partial_c A^a_- + A^c_- \partial_c A^a_+. \quad (1.20)$$

The gauged-Ricci scalar becomes [3]

$$\sqrt{det\ g_{ab}}\, g^{\mu\nu}\, R_{\mu\nu} \rightarrow 2h_{++}\, e^\sigma\, [\, D^2_- \sigma + \frac{1}{2}(D_-\sigma)^2 + \frac{1}{4}\, \rho^{ab}\, \rho^{cd}\, (D_-\rho_{ac})\, (D_-\rho_{bd})\,]. \quad (1.21)$$

The Polyakov ansatz (1.16) leads to

$$det\ g_{\mu\nu} = -1 \Rightarrow g^{\mu\nu}\, \partial_a\, g_{\mu\nu} = 2(-det\ g_{\mu\nu})^{-1/2}\, \partial_a(-det\ g_{\mu\nu})^{1/2} = 0. \quad (1.22)$$

and one can verify that

$$g^{ab}\, g^{\mu\nu}\, g^{\alpha\beta}\, (\partial_a g_{\mu\alpha})\, (\partial_b g_{\nu\beta}) = 0. \quad (1.23)$$

vanishes identically.

To sum up, after a laborious calculation Yoon [3] arrived finally at the expression for the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\, e^{2\sigma}\, \rho_{ab}\, F^a_{+-}\, F^b_{+-} + e^\sigma\, D_+\sigma\, D_-\sigma - \frac{1}{2}\, e^\sigma\, \rho^{ab}\, \rho^{cd}\, (D_+\rho_{ac})\, (D_-\rho_{bd}) +$$

$$e^\sigma\, \mathcal{R}_2 + 2h_{++}\, e^\sigma\, [\, D^2_-\sigma + \frac{1}{2}(D_-\sigma)^2 + \frac{1}{4}\, \rho^{ab}\, \rho^{cd}\, (D_-\rho_{ac})\, (D_-\rho_{bd})\,] \quad (1.24)$$

plus surface terms. At each point x^μ of the $2D$ base space $\mathcal{M}$, the quantity $\mathcal{R}_2 = g^{ab}R_{ab}$ can be interpreted as the "scalar curvature" of the internal space or fiber $\mathcal{N}_2$ at x^μ. Since $g_{ab}(x^\mu, y^a)$ depends on both the base space and internal space coordinates, the integral $\int d^2y\ e^\sigma \mathcal{R}_2$ is no longer given in terms of the Euler class topological invariant associated with the 2-dim surface $\mathcal{N}_2$. The scalar curvature $g^{ab}R_{ab}$ is interpreted now as the potential $V(g_{ab})$ for the self-interacting non-linear sigma field g_{ab}.

The gauged-Ricci scalar $g^{\mu\nu}R_{\mu\nu}$ of the $2D$ base spacetime $\mathcal{M}$ leads to the those terms multiplying the scalar h_{++} in (1.24) such that h_{++} acts as a Lagrange multiplier enforcing the constraint

$$D^2_-\sigma + \frac{1}{2}(D_-\sigma)^2 + \frac{1}{4}\, \rho^{ab}\, \rho^{cd}\, (D_-\rho_{ac})\, (D_-\rho_{bd}) = 0. \quad (1.25)$$

The area-preserving diffs algebra is generated by vector fields ξ^a tangent to the surface $\mathcal{N}_2$ and which are divergence-free $\partial_a\xi^a = 0$. The condition $\partial_a A^a_\pm = 0$ is invariant under area-preserving diffs, thus by imposing the divergence free condition $\partial_a A^a_\pm = 0$ one will have invariance under area-preserving diffs and such that the covariant derivatives acting on the tensor density ρ_{ab} and scalar σ in eqs-(1.18, 1.19) are now given by

$$D_\pm \sigma \ = \ \partial_\pm \sigma \ - A^a_\pm \partial_a \sigma. \tag{1.26}$$

$$D_\pm \rho_{ab} \ = \ \partial_\pm \rho_{ab} \ - \ [\ A_\pm, \ \rho\ ]_{ab}. \tag{1.27}$$

Under infinitesimal variations, the fields transform

$$\delta\sigma \ = \ - [\xi \ , \sigma] \ = \ - \xi^a \partial_a \sigma, \quad \partial_a \xi^a = 0. \tag{1.28}$$

$$\delta\rho_{ab} \ = \ - [\xi \ , \ \rho]_{ab} \ = \ - \xi^c \partial_c \rho_{ab} \ - (\partial_a \xi^c)\rho_{cb} \ - (\partial_b \xi^c)\rho_{ac}. \tag{1.29}$$

$$\delta A^a_+ \ = \ - D_+ \xi^a \ = \ - \partial_+ \xi^a \ + \ [A_+ \ , \ \xi]^a. \tag{1.30}$$

$$\delta A^a_- \ = \ - \partial_- \xi^a \ . \tag{1.31}$$

since δA^a_- is given by a total derivative one can choose the light-cone gauge $A^a_- = 0$ leaving $A^a_+ \neq 0$.

2 $w_\infty, w_{1+\infty}$ Gauge Field Theory in $2D$ from $4D$ Gravity

2.1 $w_\infty, w_{1+\infty}$ as Area-preserving Diffs Algebras

Zamolodchikov [5] was the first to pioneer the theory of higher conformal spin algebras W_N, $N = 2, 3, 4, ..., N$ in $2D$ that are the higher conformal spin extensions of the Virasoro algebra that arise in various physical systems as $2D$ quantum gravity, the quantum Hall effect, the membrane, the large N QCD, gravitational instantons, topological QFT, etc.... see [7] for an extensive review and references. The $w_{1+\infty}$ algebra is isomorphic to the area-preserving diffs algebra of the cylinder $S^1 \times R^1$:

$$[v^i_m \ , \ v^j_n] \ = \ [(j+1)m \ - \ (i+1)n] \ v^{i+j}_{m+n}. \tag{2.1}$$

where the index $i, j = -1, 0, 1, 2, ...$ is related to the $su(1,1)$ conformal spin $s = 1, 2, 3, ...$ and m, n label their respective Fourier modes. The spin $s = 1$ correspond to an extra spin 1 current. The w_∞ algebra is the area-preserving diffs algebra of the two-dim plane and is comprised of higher spin generators whose conformal spin range is $s = 2, 3, 4, ...$ and it is a subalgebra of $w_{1+\infty}$; whereas $su(\infty)$ is the area-preserving diffs algebra of a sphere S^2. A realization of the higher conformal spin generators of $w_{1+\infty}$ is

$$v^l_m \;=\; -\,i\; e^{im\theta}\; y^l\; [-im\; y\; \partial_y \;+ (l+1)\; \partial_\theta]. \tag{2.2}$$

A complete set of functions (not orthogonal) on the cylinder $S^1 \times R^1$ is

$$u^l_m \;=\; -\,ie^{im\theta}\; y^{l+1}; \quad -\infty \;\le\; m \;\le\; \infty, \quad l \;\ge\; -1. \tag{2.3}$$

where the conformal $su(1,1)$-spin s in $D=2$ is given by $s = l+2 \ge 1$.

The $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian density was constructed by [13]

$$\mathcal{L} \;=\; \sum_{\vec{i},\vec{j}} \; (\Phi^6(x))^{-\vec{i}-\vec{j}}\; \mathcal{F}^{\vec{i}}_{+-}(x)\; \mathcal{F}^{\vec{j}}_{+-}(x) \;+$$

$$\sum_{\vec{k}} \; (\mathcal{D}_+\Phi^{-\vec{k}}(x))\; (\mathcal{D}_-\Phi^{\vec{k}}(x)) \;+\; V(\Phi^{\vec{k}}(x)). \tag{2.6}$$

The gauge field $A^{\vec{k}}_\mu$ is Hemitian (w.r.t a well defined scalar product) $(A^{\vec{k}}_\mu)^* = A^{-\vec{k}}_\mu \;=\; A_{\mu,\vec{k}}$ and belongs to the adjoint representation $V_{\alpha,\beta}$ constructed by Feigin-Fuks-Kaplansky (FFK) [14] with $\alpha \;=\; 1, \beta \;=\; 0$. $\Phi^{\vec{k}}$ is an infinite-component complex scalar multiplet belonging to the infinite-dim vector representation $V_{\alpha,\beta}$ with $\alpha = -1/2, \beta = 0$. In order to write invariant actions based on a scalar product the weights must obey $\alpha^* + \alpha + 1 = 0$ and $\beta^* - \beta = 0$ where α^*, β^* are the weights of the *dual* representation $V^*_{\alpha,\beta} = V_{-1-\alpha,-\beta}$. For further details we refer to [13]. The gauge invariant Lagrangian based on the Virasoro w_2 algebra involving only the conformal spin 2 current (stress energy tensor) was constructed by [12] and can be obtained from the w_∞ Lagrangian by a simple truncation.

The field strength in the adjoint representation of FFK is

$$\mathcal{F}^{\vec{k}}_{+-} \;=\; \partial_+\mathcal{A}^{\vec{k}}_- \;-\; \partial_-\mathcal{A}^{\vec{k}}_+ \;-ie\; [\; \mathcal{A}_+,\; \mathcal{A}_-\;]^{\vec{k}}. \tag{2.7}$$

The commutator of the gauge fields in the adjoint representation is [13]

$$[\; \mathcal{A}_+,\; \mathcal{A}_-\;]^{\vec{k}} \;=\; [\; m_1\; (k_2+2) \;-\; (m_2+1)\; k_1\;]\; \mathcal{A}^{\vec{m}}_+\; \mathcal{A}^{\vec{k}-\vec{m}}_-. \tag{2.8}$$

where $\vec{k}$ denotes a two-dim lattice index

$$\vec{k} \;=\; (k_1,k_2),\; \vec{m} \;=\; (m_1,m_2), \tag{2.9a}$$

and their values are constrained by

$$k_2 \ge -1;\;\; m_2 \ge -1;\;\; -\infty \;\le\; k_1 \;\le\; \infty;\; -\infty \;\le\; m_1 \;\le\; \infty. \tag{2.9b}$$

since the conformal $su(1,1)$-spin s associated with the $2D$ higher conformal spin generators $v^{k_2}_{k_1}$ of the $w_{1+\infty}$ algebra is given by $s \;=\; k_2 + 2 \;\ge\; 1$ such that $s \;=\; 1,2,3,......$ Whereas, the index k_1 labels the infinite Fourier modes associated with each one of the conformal spin-s generators. The covariant derivative is

$$(\mathcal{D}_\pm \Phi^{\vec{k}}) \;=\; \partial_\pm \Phi^{\vec{k}} \,+\, ie\,[\,(m_2{+}1)(\frac{m_1}{2}{-}k_1) \,-\, (\frac{m_2}{2}{-}k_2)m_1\,]\,\mathcal{A}^{\vec{m}}_\pm\,\Phi^{\vec{k}-\vec{m}}. \quad (2.10)$$

The integration of the Yang-Mills-like terms of eq-(1.24) w.r.t the *internal* coordinates of the two-dim surface $\mathcal{N}_2$ furnishes the correspondence with the terms of the $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian [13] associated with the two-dim base spacetime $\mathcal{M}_2$. Integrating over a cylinder $S^1 \times R^1$ whose base S^1 has unit radius yields

$$\int dy\, d\theta\, e^{2\sigma(x;y,\theta)}\,[\,F^y_{+-}\,F^y_{+-} + F^\theta_{+-}\,F^\theta_{+-}\,] \;=\; \sum_{\vec{i},\vec{j}}\,(\Phi^6(x))^{-\vec{i}-\vec{j}}\,\mathcal{F}^{\vec{i}}_{+-}(x)\,\mathcal{F}^{\vec{j}}_{+-}(x). \quad (2.11)$$

where one has set $\rho_{ab} = \delta_{ab}$. The scalar kinetic terms correspondence based on eq-(1.24) is

$$\int dy\; d\theta\;\; e^{\sigma(x;y,\theta)}\; D_+\sigma\; D_-\sigma \;\;=\;\; \sum_{\vec{k}}\; (\mathcal{D}_+\Phi^{-\vec{k}}(x))\;(\mathcal{D}_-\Phi^{\vec{k}}(x)). \qquad (2.12)$$

And integrating

$$\int dy\; d\theta\; e^\sigma\; g^{ab}\; R_{ab} \;\;=\;\; V(\Phi^{\vec{k}}). \qquad (2.13)$$

yields the self-interacting potential $V(\Phi^{\vec{k}})$.

The resulting integrals in eqs-(2.12, 2.13) yield the functional relations among the infinite component fields $\sigma_{lm}(x^\mu)$, $A^a_{\pm,lm}(x^\mu)$ in the decomposition

$$\sigma \;=\; \sigma(x^\mu, y, \theta) \;=\; \sum_{lm}\; \sigma_{lm}(x^\mu)\; e^{im\theta}\; y^{l+1}. \qquad (2.14a)$$

$$A^a_\pm \;=\; A^a_\pm(x^\mu, y, \theta) \;=\; \sum_{lm}\; A^a_{\pm,lm}(x^\mu)\; e^{im\theta}\; y^{l+1}. \qquad (2.14b)$$

with $s = l + 2 \geq 1$, $\;l = -1, 0, 1, 2, 3, ..., -\infty \leq m \leq \infty$, and the fields

$$\Phi^{\vec{k}}(x^+, x^-),\; \mathcal{A}^{\vec{k}}_\pm(x^+, x^-), \quad \vec{k} = (k_1, k_2). \qquad (2.15)$$

associated with the FFK representations of the $w_{1+\infty}$ algebra and which appear in the $2D$ Lagrangian density of the $w_{1+\infty}$-gauge field theory [13]. Therefore, the 1+1-dim Lagrangian density of the $w_{1+\infty}$ gauge theory is inherently present in the $1+1$-dim description of the algebraically special class of space-times in 4-dim that contain a twist-free null vector field [3].

2.2 Moyal Star Product Deformations of $su(\infty), w_{1+\infty}, w_\infty$ Gauge Theories

The authors [9] have shown that upon quantization of field theories exhibiting symmetries provided by classical algebras $w_{1+\infty},\ w_\infty$ these symmetries get *deformed* to the quantum algebras $W_{1+\infty},\ W_\infty$ whose commutation relations are

$$[V^i_m\ ,\ V^j_n]\ =\ \sum_l\ g^{ij}_{2l}(m,n)\ V^{I+j-2l}_{m+n}\ +\ c_i(m)\ \delta^{ij}\ \delta_{m+n,0}. \tag{2.16}$$

where the structure constants $g^{ij}_{2l}(m,n)$ are complicated expressions given in terms of the generalized Saalschutzian hyper-geometric functions, binomial coefficients, ... and the c_i are the central charges associated with all of the higher spin sectors [9] [25]. The deformation of the classical algebras $w_{1+\infty},\ w_\infty$ can be obtained from a Moyal-Fedosov-Kontesevich star product deformation program as shown by [8] and in this fashion one may generate the structure constants $g^{ij}_{2l}(m,n)$ that were originally obtained by a tour de force method [9]. In addition, there are also *modifications* in the central charges where the central charge term present only in the Virasoro sector [8] is extended to *all* of the higher conformal spin sectors of the quantum $W_\infty, W_{1+\infty}$ algebras. The origin of the modifications of the central charge terms is due to universal gauge anomalies of the algebras [25].

The ordinary Moyal star-product of two functions in phase space $f(x,p), g(x,p)$ is :

$$(f*g)(x,p) = \sum_s \frac{\hbar^s}{s!}\sum_{t=0}^{s}(-1)^t C(s,t)\ (\partial_x^{s-t}\partial_p^t f(x,p))\ (\partial_x^t\partial_p^{s-t} g(x,p)) \tag{2.17}$$

where $C(s,t)$ is the binomial coefficient $s!/t!(s-t)!$. In the $\hbar \rightarrow 0$ limit the star product $f*g$ reduces to the ordinary pointwise product fg of functions. The Moyal product of two functions of the $2n$-dim phase space coordinates (q_i, p_i) with $i = 1,2...n$ is:

$$(f*g)(x,p) = \sum_i^n \sum_s \frac{\hbar^s}{s!}\sum_{t=0}^{s}(-1)^t C(s,t)\ (\partial_{x_i}^{s-t}\partial_{p_i}^t f(x,p))\ (\partial_{x_i}^t\partial_{p_i}^{s-t} g(x,p)) \quad (2.18$$

The noncommutative, associative Moyal bracket is defined:

$$\{f,g\}_{MB} = \frac{1}{i\hbar}(f*g - g*f). \tag{2.19}$$

In the $\hbar \rightarrow 0$ limit the star product $f*g$ reduces to the ordinary pointwise product fg of functions and the Moyal bracket reduces to the Poisson one. Thus, the Moyal deformations of the Yang-Mills-like terms are

$$\int dy\ d\theta\ \ e_*^{2\sigma}\ *\ [\ F^y_{+-}\ *\ F^y_{+-}\ +\ F^\theta_{+-}\ *\ F^\theta_{+-}\ ]. \tag{2.20}$$

$$\mathcal{F}_{+-}\ =\ \partial_+\mathcal{A}_-\ -\ \partial_-\mathcal{A}_+\ -ie\ \{\ \mathcal{A}_+,\ \mathcal{A}_-\ \}_{MB}. \tag{2.21}$$

$$D_\pm\sigma\ =\ \partial_\pm\sigma\ +\ \{\ A_\pm\ ,\ \sigma\ \}_{MB}. \tag{2.22}$$

$$D_\pm\ e_*^\sigma\ =\ \partial_\pm\ e_*^\sigma\ +\ \{\ A_\pm\ ,\ e_*^\sigma\ \}_{MB}. \tag{2.23}$$

Due to the fact that for higher derivatives

$$\partial^n_{y^a}\ e^{\sigma(x^\mu,y^a)}\ \neq\ e^{\sigma(x^\mu,y^a)}\ \partial^n_{y^a}\ \sigma(x^\mu,y^a)\ \Rightarrow$$

$$\{\ A_\pm\ ,\ e^\sigma\ \}_{MB}\ \neq\ e^\sigma\ \{\ A_\pm\ ,\ \sigma\ \}_{MB}. \tag{2.24}$$

and

$$\{\ A_\pm\ ,\ e_*^\sigma\ \}_{MB}\ \neq\ e_*^\sigma\ \{\ A_\pm\ ,\ \sigma\ \}_{MB}. \tag{2.25}$$

the correct Moyal deformations of the scalar kinetic terms are

$$\int dy\ d\theta\ \ e_*^\sigma\ *\ [\ e_*^{-\sigma}\ *\ e_*^{-\sigma}\ *\ D_+\ e_*^\sigma\ *\ D_-\ e_*^\sigma\ ]. \tag{2.26}$$

where the star-deformed exponential function is defined by

$$e_*^\sigma\ =\ 1\ +\ \sigma\ +\ \frac{1}{2!}\sigma*\sigma\ +\frac{1}{3!}\sigma*\sigma*\sigma\ +\ .... \tag{2.27}$$

The star-deformed potential $V_*(\varphi)$ is defined as the star-deformed Taylor expansion of the original potential $V(\varphi)$

$$V_*(\varphi(x,y))\equiv\sum_n g_n\varphi*\varphi*\varphi.....*\varphi. \tag{2.28}$$

where the couplings g_n are obtained by taking the n-th derivatives of $V(\varphi)$ w.r.t φ and evaluated at $\varphi=0$

$$g_n\equiv\frac{1}{n!}\frac{\partial^n V(\varphi)}{\partial\varphi^n}(\varphi=0). \tag{2.29}$$

The Moyal deformed-action S_* is highly nontrivial. The leading terms $\hbar^0$ coincide with the undeformed action based on the Poisson bracket algebra of area-preserving diffs of the two-dim internal $\mathcal{N}_2$ surface. In the case that the internal two-dim space has the topology of a sphere, this Poisson bracket algebra is isomorphic to the basis-dependent limit of the $N\to\infty$ limit of $SU(N)$ [22]. For arguments refuting the ismorphism behind the large N limits of $su(N)$ algebras and the area-preserving diffs of a sphere S^2 see [23] . The Moyal-deformations of the area-preserving-diffs S^2 symmetry transformations that leave invariant the Moyal-deformed gravitationally induced action-density $\mathcal{L}_*(x)$ are given by

$$\delta \mathcal{A}_\mu(x,y) = -\frac{1}{e}[\partial_\mu \xi(x,y) - e\{\mathcal{A}_\mu(x,y), \xi(x,y)\}_{MB}]. \tag{2.30a}$$

one may set $e = 1$ for convenience.

$$\delta \mathcal{F}_{\mu\nu}(x,y) = -\{\xi(x,y), \mathcal{F}_{\mu\nu}(x,y)\}_{MB}. \tag{2.30b}$$

$$\delta \varphi(x,y) = -\{\xi(x,y), \varphi(x,y)\}_{MB}. \tag{2.30c}$$

$$\delta D_\mu \varphi = -\{\xi(x,y), D_\mu \varphi\}_{MB}. \tag{2.30d}$$

$$\delta V_*(\varphi) = -\{\xi, V_*(\varphi)\}_{MB}. \tag{2.30e}$$

and the variation of $\mathcal{L}_*(x)$ is given by a sum of *total derivatives* that vanishes after integration by parts since the internal sphere has no boundaries

$$\delta L_*(x,y) = -\{\xi, L_*(x,y)\}_{MB} \ \Rightarrow \ \delta \mathcal{L}_*(x) = \int d^2y \ \delta L_*(x,y) \ =$$

$$-\int d^2y \ \{\xi, L_*(x,y)\}_{MB} \ = \ \int \ (sum\ of\ total\ derivatives) \ = \ 0. \tag{2.31}$$

To show this requires the use of the Liebnitz property of the Moyal Brackets

$$\{\xi \ , \ F_{\mu\nu} \ F^{\mu\nu}\}_{MB} \ = \ \{\xi \ , \ F_{\mu\nu}\}_{MB} \ F^{\mu\nu} \ + \ F^{\mu\nu} \ \{\xi \ , \ F_{\mu\nu}\}_{MB}. \tag{2.32}$$

and

$$\int d^2y \, F_{\mu\nu} * F^{\mu\nu} \ = \ \int d^2y \, (F_{\mu\nu} \, F^{\mu\nu} + total\ derivatives) \ = \ \int d^2y \, F_{\mu\nu} \, F^{\mu\nu} \ \Rightarrow$$

$$\delta \int d^2y \ F_{\mu\nu} \ * \ F^{\mu\nu} \ = \ \delta \int d^2y \ F_{\mu\nu} \ F^{\mu\nu} \ =$$

$$\int d^2y \ \{\xi \ , \ F_{\mu\nu}\}_{MB} \ F^{\mu\nu} \ + \ F_{\mu\nu} \ \{\xi \ , \ F^{\mu\nu}\}_{MB} \ =$$

$$\int d^2y \ \{\xi \ , \ F_{\mu\nu} \ F^{\mu\nu}\}_{MB} \ = \ \int \ sum\ of\ total\ derivatives \ = \ 0. \tag{2.33}$$

if there are no boundaries or if the fields vanish fast enough at infinity. Similar results follow for the kinetic terms. For further details see [16], [17]. In general, the generators of $w_\infty, w_{1+\infty}$ admit a parametrization in terms of an infinity family of functions f as

$$L_f \ = \ \omega^{ab} \ \partial_b f \ \partial_a, \quad \omega^{ab} = symplectic\ structure. \tag{2.34}$$

where the Lie-Poisson structure is deformed into a Lie-Moyal one upon quantization

$$[L_f \ , \ L_g] \ = \ L_{\{f,g\}} \ \rightarrow \ [L_f \ , \ L_g]_* \ = \ L_{\{f,g\}_*} \ . \tag{2.36}$$

For instance, When the topology of the internal two-dim surface is that of a cylinder $S^1 \times R^1$ one may expand the function f and generators L_f as

$$f(y,\theta) \ = \ f_{lm} \, e^{im\theta} \, y^{l+1}; \ \ L_f \ = \ f_{lm} \, v^{lm} \ = \ f_{lm} \ e^{im\theta} \, [m \, y^{l+1} \, \partial_y + i(l+1) \, y^l \, \partial_\theta]. \tag{2.37}$$

from which one may read the commutation relations of the (deformed)currents v^l_m, V^l_m from the Lie-Poisson and Lie-Moyal algebraic structures upon deformation quantization. Similar results follow for the sphere and the two-dim plane by choosing the appropriate basis of functions. The algebras admit central charges or not depending on the genus of the two-dim surfaces [6].

3 $4D$ Quantum Gravity via $2D$ Quantum W_∞ Gauge Theories, Collective Fields and Matrix Models

In this section we shall show how Quantum Gravity in $D = 3$ can be described by a W_∞ Matrix Model in $D = 1$ that can be solved *exactly*. $4D$ Quantum Gravity is more complicated, nevertheless its quantization program can be attained from the perspective of a $2D$ Quantum W_∞ gauge theory coupled to an infinite-component scalar-multiplet whose action is described by eq-(2.6); i.e. Quantization of Einstein Gravity in $4D$ admits a reformulation in terms of a $2D$ Quantum W_∞ gauge theory coupled to an infinite family of scalars.

It has been known for some time [20] that the bosonization program of non-relativistic fermions in one space dimension can be used to describe the low energy excitations of a Fermi gas in terms of a Fermi fluid of various shapes with the *same area* as the ground state configuration if one insists in fermion number conservation. The Fermi fluid exists in the 2-dim phase space of the single fermion and changes in the state of the Fermi theory correspond to *area−preserving* shape changes of the Fermi fluid.

The Das-Jevicki-Sakita [19] collective field theory approximation studies the fluctuations of the phase-space density and in the semi classical limit describes the low energy excitations of the Fermi fluid near the Fermi surface when one restricts the shapes of the Fermi fluid to have a quadratic profile for the Fermi energy $\mu_F = \frac{1}{2}(p^2 - q^2)$ related to an *inverted* one-dim harmonic oscillator potential.

A direct proof of bosonization of non-interacting non-relativistic fermions in one space dimension was derived by Wadia et al [20] by using W_∞ coherent states in the fermion path-integral. The bosonized action was derived earlier by the method of coadjoint orbits associated with the W_∞ algebra. The classical limit of the bosonized theory and the precise nature of the truncation of the full theory that leads to the Das-Jevicki-Sakita collective field theory was also described by [20].

The use of W_∞ coherent states in the fermionic path-integral was made possible by the observation [18] that the bosonized problem is analogous to that of a spin in a magnetic field. This system has a W_∞ spectrum generating algebra that follows from the existence of the w_∞ symmetry of the harmonic oscillator. It is natural to rewrite the collective field theory in $0+1$ dimensions as a $1+1$ relativistic field theory [21] so the collective field theory is a theory of a massless boson that reproduces the fluctuations in the density.

The quantum algebra W_∞ may be realized [21] as the algebra of modes of the Fermion bilinears $: \partial^k \Psi(z) \partial^l \Psi(z) :$. A bosonization relates the fermion-bilinears to the bosonic currents $\frac{1}{s} : e^{-\phi(z)} \partial^s e^{\phi(z)} :$ and similarly to the left movers by replacing $z \rightarrow \bar{z}$. The key point was that although the collective field theory is *not* a free theory it has a spectrum generating algebra given by charges

$$Q_{lm} \ = \ \int dx \int_{p_-}^{p^+} \ dp \ (p+x)^{l+m+1} \ (p-x)^{l-m+1}. \tag{3.1}$$

that satisfy a w_∞ algebra isomorphic to the Poisson-bracket algebra of the charges $\{ Q_{lm} \ , \ Q_{l'm'} \ \}_{PB}$.

After this historical this preamble one may notice that the action (1.24) obtained from the decomposition of Einstein gravity in $D = 1+2$ (instead of $D = 2+2$) is much *simpler* since there are *no* Yang-Mills-like terms and a gauged-Ricci scalar curvature term in a one-dimensional base space $\mathcal{M}_1$, and when $\rho_{ab} = \delta_{ab}$ the Einstein-Hilbert action in $D = 1+2$ action reduces to

$$S \ = \ \int \ dt \ \mathcal{L} \ = \ \int \ dt \ \int \ dy \ d\theta \ \ e^{\sigma} \ [\ D_+ \sigma \ D_- \sigma \ + \ \mathcal{R}_2 \ ] \ =$$

$$\int \ dt \ \int \ dy \ d\theta \ \ e^{\sigma} \ [\ D_+ \sigma \ D_- \sigma \ + \ V(\sigma) \ ]. \tag{3.2}$$

where $\mathcal{R}_2 \ = \ V(\sigma)$. The Moyal star product deformation is

$$\int \ dt \ \mathcal{L}_* \ = \ \int \ dt \ \int \ dy \ d\theta \ \ e^{\sigma}_* * [\ e^{-\sigma}_* * e^{-\sigma}_* * D_+ \ e^{\sigma}_* * D_- \ e^{\sigma}_* + V_*(\sigma) \ ]. \tag{3.3}$$

and has the same functional form (up to scaling factors in the integration measure) as the W_∞ and w_∞ Matrix Model Lagrangians in $D = 1$ studied by [18]

$$\mathcal{L} \ = \ trace \ [\ \frac{1}{2} \ (\ \partial_t \ \mathbf{M}(t) \ + \ \{ \ \mathbf{A}_t \ (t) \ , \ \mathbf{M}(t) \ \} \ )^2 \ - V(\mathbf{M}) \ ] \ =$$

$$\int \ d^2 z \ \frac{1}{2} \ [\ \partial_t \ M(t,z,\bar{z}) \ + \ \{ \ A_t \ (t,z,\bar{z}) \ , \ M(t,z,\bar{z}) \ \}_{MB} \ ]^2_* \ - V_*(M). \tag{3.4a}$$

where the infinite-dimensional trace operation is replaced by an integration $trace \rightarrow \int d^2z$ and

$$(\partial_t M + \{ A_t, , M \}_{MB})^2_* =$$

$$(\partial_t M + \{ A_t, , M \}_{MB}) * (\partial_t M + \{ A_t , M \}_{MB})$$

the one-dim w_∞ Matrix Model is based on the Lagrangian

$$\mathcal{L} = \int d^2y \frac{1}{2} [\partial_t M(t,y^1,y^2) + \{ A_t (t,y^1,y^2) , M(t,y^1,y^2) \}_{PB}]^2 - V(M). \tag{3.4b}$$

the internal coordinates y^1, y^2 of the two-dim surface $\mathcal{N}_2$ are represented by the complex coordinates $z = \frac{1}{\sqrt{2}l}(y^1 + iy^2), \bar{z} = \frac{1}{\sqrt{2}l}(y^1 - iy^2)$ associated with the coherent-states representation and l is length scale parameter . The Moyal brackets of two functions $\xi_1(z,\bar{z}), \xi_2(z,\bar{z})$ in units of $\hbar = c = 1$ is

$$\{ \xi_1(z,\bar{z}) , \xi_2(z,\bar{z}) \}_{MB} =$$

$$i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [\partial_z^n(\xi_1(z,\bar{z})) \partial_{\bar{z}}^n(\xi_2(z,\bar{z})) - \partial_{\bar{z}}^n(\xi_1(z,\bar{z})) \partial_z^n(\xi_2(z,\bar{z}))]. \tag{3.5}$$

The canonical quantization leads to the Hamiltonians expressed in terms of momentum variables

$$H = \int d^2z \frac{1}{2} (P(z,\bar{z})) * (P(z,\bar{z})) + V_*(M) =$$

$$\int d^2z \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} [\partial_z^n(P(z,\bar{z})) \partial_{\bar{z}}^n(P(z,\bar{z})) + V_*(M). \tag{3.6a}$$

$$H = \int d^2y \frac{1}{2} (P(y^1,y^2)) (P(y^1,y^2)) + V(M). \tag{3.6b}$$

The W_∞, w_∞ gauge invariance of the actions leads to the following constraints on the state vector $|\Psi>$

$$\int d^2z \{ \xi , M \}_{MB} P(z,\bar{z}) |\Psi> = 0. \tag{3.7a}$$

$$\int d^2y \{ \xi , M \}_{PB} P(y^1,y^2) |\Psi> = 0. \tag{3.7b}$$

Kavalov and Sakita solved the problem by using the techniques based on the collective field method [19] that requires a change of variables from $P(z,\bar{z}), M(z,\bar{z})$ to $\pi(x), \phi(x)$. The procedure is quite elaborate. The end result yields the following Hamiltonians for the collective field associated with the W_∞ algebra

$$H = \int dx [\frac{1}{2}(\partial_x \pi(x))^2 \phi(x) + \frac{\pi}{6} \phi(x)^3 + V(x) \phi(x)] - \lambda(\int dx \phi(x) - N). \tag{3.8a}$$

and

$$H = \int dx \, [\, \frac{1}{2}(\partial_x \pi(x))^2 \phi(x) + \frac{\kappa}{8} \frac{(\partial_x \phi(x))^2}{\phi(x)} + V(x) \, \phi(x) \,] - \lambda(\int dx \, \phi(x) - L^2). \tag{3.8b}$$

associated with the w_∞ algebra. N is the number of fermions, L^2 is the area of the fluid, κ is a numerical parameter and λ a Lagrange multiplier enforcing the constraints. After a suitable scaling transformations, in the $N \to \infty$ limit the excitation spectrum found by Kavalov and Sakita [18] turned out to be

$$H = \frac{1}{2} \sum_{n=0}^{\infty} (p_n^2 + \omega_n^2 \, q_n^2), \quad [q_n, p_n] = i\delta_{mn}, \; \hbar = c = 1. \tag{3.9}$$

for the W_∞ one-dim Matrix model case the frequencies are

$$\omega_n = \frac{n\pi}{T}, \quad T = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E_o - U(x))}}, \quad \frac{1}{\pi} \int_{x_1}^{x_2} dx \, \sqrt{2(E_o - U(x))} = 1. \tag{3.10a}$$

and for the w_∞ one-dim Matrix model the frequencies are obtained from the energy levels of the solutions of the Schroedinger equation

$$\omega_n = E_n - E_o, \quad [-\frac{1}{2}\partial_x^2 + V(x) \,] \, \psi_n(x) = E_n \, \psi_n(x). \tag{3.10b}$$

where

$$U(x) = \sum_n N^{\frac{n}{2}-1} g_n x^n = \sum_n a_n \, x^n. \tag{3.11a}$$

and

$$V(x) = \sum_n \kappa^{n-2} \tilde{g}_n x^n = \sum_n (\kappa l)^{n-2} g_n x^n = \sum_n b_n \, x^n. \tag{3.11b}$$

respectively. As mentioned above, the key point was that although the collective field theory is *not* a free theory it has a w_∞ spectrum generating algebra associated with the harmonic oscillator

As stated earlier, quantization of Einstein Gravity in $4D$ admits a reformulation in terms of a $2D$ Quantum W_∞ gauge theory coupled to an infinite family of scalars. The starting point is the classical $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian density constructed by [13]

$$\mathcal{L} = \sum_{\vec{i},\vec{j}} (\Phi^6(x))^{-\vec{i}-\vec{j}} \, \mathcal{F}^{\vec{i}}_{+-}(x) \, \mathcal{F}^{\vec{j}}_{+-}(x) \; + \\ \sum_{\vec{k}} (\mathcal{D}_+ \Phi^{-\vec{k}}(x)) \, (\mathcal{D}_- \Phi^{\vec{k}}(x)) + V(\Phi^{\vec{k}}(x)). \tag{3.12}$$

A quantization of (2.6, 3.12) will *deform* the classical $w_\infty, w_{1+\infty}$ symmetry algebras of the classical Lagrangian to the quantum $W_\infty, W_{1+\infty}$ symmetry of the quantum theory (a BRST quantization procedure) and such that the latter quantum algebras will be the spectrum generating algebras. Since there are an infinite number of higher conformal spin generators the highest weight representations will generate an infinite number of states at each level. Kac and Radul [29] solved this problem by constructing quasi-finite highest weight representations that were used by [30] to develop the full fledged representation theory of the quantum $W_{1+\infty}$ algebra. Free field realizations, (Super) Matrix generalizations, the structure of subalgebras such as the W_∞ algebra, determinant formulae and character formulae can be found in [30].

The Bars-Witten stringy black hole in $D = 2$ has a nonlinear $\hat{W}_\infty(k = \frac{9}{4})$ for hidden symmetry [8] that can be used as its spectrum generating algebra; a W_∞ symmetry of the Nambu-Goto string in $4D$ was also found in [8] based on a $SU(2)/U(1)$ coset model. Closely related to black-holes in $3D$, Witten has shown [32] that the energy spectrum of three-dimensional gravity with negative cosmological constant associated with the BTZ black-hole can be determined exactly. Witten has argued that the dual Conformal Field Theory (CFT) is very likely to be the Monster theory of Frenkel, Lepowsky, and Meurman. The partition function was found to be a polynomial in the modular invariant Klein function $j(q)$. Manschot has shown more recently that the partition function can be obtained as a modular sum over geometries [33].

Not so long ago, the authors [34] inspired by a formal resemblance of certain q-expansions of modular forms and the master field formalism of matrix models in terms of Cuntz operators, constructed a Hermitian one-matrix model that was coined the "Modular Matrix Model" which naturally encode the Klein elliptic $j(q)$-invariant and the irreducible representations of the Fischer-Griess Monster group resulting from the Moonshine conjecture. These results relating Modular Matrix Models, quantum gravity and the Monster, in particular the role of W_∞ algebras, warrant a furher investigation. For an extensive review of $2D$ Gravity, Matrix Models and String theory see [21].

Isomonodromic quantization of dimensionally reduced Gravity can be found in [28]. The relationship between W_∞ gravity (geometry) and the Fedosov deformation quantization of the $4D$ Self-Dual Gravity [15] associated with the complexified co-tangent space of a two-dim Riemann surface was studied by [17]. String and p-branes actions can be obtained by a Moyal deformation quantization of (Generalized) Yang-Mills as shown in [11]. A natural Fedosov type quantization of generalized Lagrange models and gravity theories with metrics lifted on tangent bundle, or extended to higher dimensions, has been attained by Vacaru [31]. The constructions are possible due to a synthesis of the nonlinear connection formalism developed in Finsler and Lagrange geometries and deformation quantization methods.

Higher spin field theories in $D > 2$ have been extensively studied over the years by Vasiliev [10] and Calixto [26] has constructed Generalized W_∞-type Higher Spin Algebras in Higher dimensions $D > 2$ where non-linear realization methods [27] could be used to build higher spin extensions of Gravity theories. The interplay among quantum membranes, the continuous Toda theories and non-critical W_∞ (super) strings has been analyzed by [35] to show why non-critical W_∞ (super) strings in $D = 27(11)$ dimensions are devoid of BRST anomalies. Such $D = 27(11)$ dimensions coincide with the alleged critical dimensions of the quantum (super) membrane, respectively. To finalize, we must say that Noncommutative $4D$ Gravity based on deformed diffs and Poincare algebras developed by [36] deserves further investigation within the context of $2D$ W_∞ gauge theory.

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