Universal scaling near the onset of chaos and the spectrum of lepton magnetic moments

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Abstract

Starting from the nonlinear dynamics of Renormalization Group (RG) equations, we show that the spectrum of lepton magnetic moments follow a Feigenbaum-like scaling pattern. Based on this approach, we find that the predicted moment of the τ - lepton falls in line with current experimental data.

Key words: Renormalization Group; Feigenbaum scaling; Standard Model

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1. Introduction

The Standard Model for particle physics (SM) is a solid theoretical framework that has impressively passed a large number of high-precision tests. For example, measurements of the anomalous magnetic of the electron (a_e) and muon (a_μ) have reached the astonishing relative precision of 0.7 parts-per-billion and 0.5 parts-per-million, respectively [11]. This unsurpassed level of experimental precision has revealed a minute yet non-vanishing discrepancy between the experimental and SM values of a_μ [9]. It is generally assumed that this difference could be tracked down to an experimental error, an incorrect evaluation of hadronic loop-diagrams or a hint for "new physics" (NP), beyond the energy range of SM. While a_e is relatively insensitive to the weak and strong sectors of SM, a_μ contains contributions from all SM sectors. As a result, precise tests of a_μ are thought to provide an excellent opportunity to reveal or constrain the signature of NP, including, for example, supersymmetry and related field models [10-12, 16]

Drawing on previous studies where nonlinear dynamics and complexity assume the leading role [13-15], we explore in this work a scenario in which lepton magnetic moments are organized in a hierarchical pattern. The rationale for this pattern stems from the universal scaling behavior of RG equations near the onset of chaos [13-15]. This conjecture enables us to predict the magnetic moment of the τ - lepton and to further compare it with its most updated experimental values.

The paper is structured as follows: section 2 contains a brief overview on the theory of anomalous magnetic moment of leptons. Section 3 introduces general aspects related to the chaotic dynamics of the renormalization group flow (RG) and the emergence of Feigenbaum scaling. The hierarchical pattern of lepton moments and the predicted value of the τ - lepton moment form the topic of section 4. Concluding remarks are presented in the last section.

2. Lepton magnetic moment: a brief overview

For a charged lepton $l = \{e, \mu, \tau\}$, the Dirac equation predicts the following magnetic moment [11-12]

$$\overrightarrow{\mu_l} = g_l \frac{e\hbar}{2m_l} \frac{\vec{S}}{\hbar} \tag{1}$$

in which $S = \frac{1}{2}$ represents the lepton spin, m_l the lepton mass and g_l its gyromagnetic ratio. The cumulative contribution of quantum fluctuations leads to a small deviation from the Dirac value $g_l = 2$ that is parameterized by the *anomalous magnetic moment*:

$$a_l \equiv \frac{1}{2}(g_l - 2) \tag{2}$$

This quantity can be accurately measured in experiments and, through the perturbative framework of SM, precisely predicted. The difference between experiment and theory ($\Delta a_l = \left| a_l^{OBS} - a_l^{SM} \right|$) provides a stringent test of SM at its quantum loop level. The overall SM contribution to a_l may be split into three independent components

$$a_l^{SM} = a_l^{QED} + a_l^{EW} + a_l^{Had} \tag{3}$$

denoting the QED, electroweak and hadronic sectors, respectively. The contribution a_l^{SM} is generally expected to scale linearly with $(m_l/\Lambda_{NP})^2$, where Λ_{NP} stands for the energy scale that corresponds to the onset of NP [10-12]. It is thus apparent from these remarks that a_μ, a_τ are a much more sensitive signal for NP effects on or above the Λ_{NP} scale, since $(m_\mu/m_e)^2 = 4.275\times10^4$ and $(m_\tau/m_e)^2 = 1.209\times10^7$.

The theory of the lepton anomalous magnetic moment is an active topic of research in particle physics. It includes multi-loop calculations, precision low-energy hadron physics, isospin violations and scattering of light by light [1-8, 10-12]. Current precision of the experimental value for a_μ has significantly improved in the past several years thanks to experiments conducted at Brookhaven National Laboratory [12, 16]. By contrast, measurement of a_τ is a significantly more difficult task due to the short lifetime of the

τ -lepton. With the current testing equipment and technology, one can only determine the most probable range associated with a_τ [11, 12].

3. Chaotic dynamics of the RG flow

By definition, the magnetic moment of leptons depends on ratio (e/m). As a result, departures from $g_l = 2$ are linked to quantum processes that distort the relative distribution of charge and mass. Since both charge and mass are scale-dependent, the gyromagnetic ratio dependents also on the observation scale. From the standpoint of Renormalization Group (RG) theory, they represent observables that "run" with the measurement scale μ according to the equation

$$\mu \frac{da_l}{d\mu} \equiv \frac{da_l}{dt} = \beta_a(a_l) = d_1 + d_2 a_l + d_3 a_l^2 + ... \tag{4}$$

in which d_i $(i = 1, 2, 3...)$ denote expansion coefficients and $t \sim \log(\mu/\mu_0)$ is the sliding scale defined relative to an arbitrary reference μ_0. Analysis of the RG flow for couplings and masses in the presence of a generic control parameter reveals the onset of a scaling pattern near the chaotic attractor of the flow [13-15]. This is consistent with the universal scenario for transition to chaos in unimodal maps ([13-15] and Appendix). Hence, with regard to (4), we posit that $\Delta a = \left| a^{OBS} - a^{SM} \right| = a^{NP}$ obeys a scaling relationship described by the geometric progression

$$\Delta a_l \propto a_0 \overline{\delta}^{\,-2^{n(l)}} \tag{5}$$

where $\overline{\delta}$ is a constant (identical to the Feigenbaum constant $\delta = 4.669...$ only when the RG flow (4) is strictly quadratic) and $n(l) > 1$ is a natural index associated with each lepton flavor. For two consecutive terms in the lepton family we have (see (A9))

$$\boxed{\frac{\Delta a_l}{\Delta a_{l+1}} = \mathrm{K}\overline{\delta}^{\,-(2^l)}} \tag{6}$$

Here, constant K parameterizes our "ignorance" with regard to the physics on or above the threshold scale M. It is introduced to enable a closer numerical match between the left hand side of (6) and the scaling factor $\overline{\delta}^{\,-(2^l)}$.

4. Hierarchical structure of magnetic moments

Table 1 summarizes the wealth of current knowledge on lepton anomalous moments, along with their respective references. a^{OBS} denotes the 'observed' data, whereas a^{SM} represents the Standard Model values computed according to (3).

Tab.1

Lepton flavor	a^{OBS}	a^{SM}	$\Delta a = a^{OBS} - a^{SM}$
e	$1159652188.3\times10^{-12}$ [1]	0.0011596521859 [2]	5.05×10^{-12}
μ	116592080×10^{-11} [3]	116591858×10^{-11} [4]	22×10^{-10}
τ	$-0.052 < a_\tau^{OBS} < 0.013$ [5]	$117721(5)\times10^{-8}$ [6]	?
	$\langle a_\tau^{OBS} \rangle = -0.018(17)$ [5], [7]	$117721(5)\times10^{-8}$ [6]	?
	$-0.007 < a_\tau^{OBS} < 0.005$ [8]	$117721(5)\times10^{-8}$ [6]	?

The set of values for Δa_e and Δa_μ are known to a great degree of precision [1-8, 11-12]. Hence applying (6) to the first two components of the charged lepton triplet yields

$$\frac{\Delta a_e}{\Delta a_\mu} = \mathrm{K}\overline{\delta}^{\,-2^{l(e,\mu)}} \Rightarrow \begin{cases} \mathrm{K} \approx 1/2 \\ l(e,\mu) = 2 \end{cases} \tag{7}$$

It follows from (6) and (7) that $l(\mu,\tau)=l(e,\mu)+1=3$ and the estimate of the τ -lepton magnetic moment is given by

$$\boxed{\Delta a_{\tau}=\frac{\Delta a_{\mu}}{\mathrm{K}\bar{\delta}^{-8}}\Rightarrow a_{\tau}^{OBS}=2.179917\times10^{-3}} \tag{8}$$

A quick glance at Tab. 1 shows that this prediction complies with the range of experimental data recently reported in [5-8]. Narrowing the range Δa_{τ} through new rounds of high-precision tests on the τ -lepton will either confirm or refute the validity of (8).

5. Concluding remarks

The starting point of this brief report was the universal scaling behavior of RG equations near the onset of chaos. On this basis it was postulated that the magnetic moments of leptons are organized in a hierarchical pattern. The predicted moment of the τ - lepton was shown to be consistent with recent measurements. Future testing and data analysis are required to validate, improve or falsify this prediction.

6. Appendix: *RG flow equations in the presence of perturbations and the transition to dynamical chaos.*

For convenience, we re-iterate here the line of arguments developed in [14]. The parameters of the Standard Model $\boldsymbol{\sigma}=(\sigma_i)$; $i=1,2,...,n$ evolve according to the free-flow equations

$$\mu\frac{d\sigma_i}{d\mu}=\frac{d\sigma_i}{dt}=\beta_i(\sigma_i) \tag{A1}$$

In what follows, we consider that a generic set of perturbations acts as control parameter that drives the dynamics described by (A1). In the presence of perturbations $\lambda_i = \lambda_i(\sigma_i, t)$ [A1] may be written as

$$\frac{d\sigma_i}{dt} = \beta_i[\sigma_i, \lambda_i(\sigma_i, t)] \tag{A2}$$

For the sake of concision and simplicity, we limit the analysis to the simplest case of a single stationary perturbation having constant amplitude

$$\lambda_i(\sigma_i, t) = \lambda \tag{A3}$$

Under these circumstances, (A2) assumes the form of a generic autonomous system of ordinary differential equations (ODE)

$$\frac{d\sigma_i}{dt} = \beta_i(\sigma_i, \lambda) \tag{A4}$$

Based on the line of arguments developed in [17], we proceed with the following assumptions:

a1) (A4) is a smooth family of nonlinear autonomous systems of ODE in three-dimensional phase space M that is dependent on the single control parameter λ $(\sigma \in M \subset \mathrm{R}^3,\ \lambda \in \mathrm{I} \subset \mathrm{R})$.

a2) (A4) are analytic functions of λ.

a3) the limit cycle $\sigma_0(t, \lambda)$ of period $T(\lambda)$ represents a solution of (A4) for all $\lambda \in \mathrm{I}$.

a4) the limit cycle $\sigma_0(t, \lambda)$ is stable for $\lambda < 0$ and it becomes unstable at $\lambda = 0$ after a period-doubling bifurcation created as a result of crossing the imaginary axis by one of the Floquet exponents.

According to the theorem 4.4 of [17], the first stage of the transition to chaos driven by the continuous variation of $\lambda > 0$ represents a Feigenbaum cascade of period-doubling bifurcations for $\sigma_0(t,\lambda)$. Numerous examples of this scenario [17] show that the sequence of critical values λ_n, $n \in \mathrm{N}$, leading to the onset of super-stable orbits, satisfies the geometric progression

$$\lambda_n - \lambda_\infty \approx \mathrm{K}\,\overline{\delta}^{-n} \tag{A5}$$

Based on a2), we expand $\sigma_0(t,\lambda)$ around the critical value of $\lambda = \lambda_\infty$ that leads to fully developed chaos

$$\sigma_0(t,\lambda_n) = \sigma_0(t,\lambda_\infty) + (\lambda_n - \lambda_\infty)\left.\frac{\partial\sigma_0(t,\lambda)}{\partial\lambda_n}\right|_{\lambda_\infty} + \frac{(\lambda_n - \lambda_\infty)^2}{2}\left.\frac{\partial^2\sigma_0(t,\lambda)}{\partial\lambda_n^2}\right|_{\lambda_\infty} + \ldots \tag{A6}$$

This yields

$$\sigma_0(t,\lambda_n) = \sigma_0(t,\lambda_\infty) + (\mathrm{K}\overline{\delta}^{-n})\left.\frac{\partial\sigma_0(t,\lambda)}{\partial\lambda_n}\right|_{\lambda_\infty} + \frac{(\mathrm{K}\overline{\delta}^{-n})^2}{2}\left.\frac{\partial^2\sigma_0(t,\lambda)}{\partial\lambda_n^2}\right|_{\lambda_\infty} + \ldots \tag{A7}$$

For $n = 2^l$, where the generic index is $l \geq 1$, the ratio of two consecutive terms in the series then takes the form

$$\frac{\Delta\sigma_{0,n}}{\Delta\sigma_{0,n+1}} = \frac{\sigma_0(t,\lambda_n) - \sigma_0(t,\lambda_\infty)}{\sigma_0(t,\lambda_{n+1}) - \sigma_0(t,\lambda_\infty)} = \frac{\sum_k c_k (\mathrm{K}\overline{\delta}^{-n})^k}{\sum_k c_k [\mathrm{K}\overline{\delta}^{-(n+1)}]^k} \tag{A8}$$

Under the assumption $c_1 \neq 0$ and $\overline{\delta}^{-n} \propto O(\varepsilon)$ corresponding to $l \gg 1$, we obtain

$$\frac{\Delta\sigma_{0,2^{l+1}}}{\Delta\sigma_{0,2^l}} \approx \overline{\delta}^{-(2^l)} \tag{A9}$$

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