# INTEGER ALGORITHMS TO SOLVE LINEAR EQUATIONS AND SYSTEMS

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**Abtract.** Original integer general solutions, together with examples, are presented to solve linear equations and systems.

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**Introduction.**

The present work includes some of the author's original researches on the integer solutions of equations and linear systems:

1. The notion of "general integer solution" of a linear equation with two unknowns is extended to linear equations with n unknowns and then, to linear systems.
2. The proprieties of the general integer solution are determined (both of a linear and system).
3. Seven original integer algorithms (two for linear equations and five for linear systems) are presented. The algorithms are strictly demonstrated and an example for each of them is given. These algorithms can be easily introduced in a computer.

## INTEGER SOLUTIONS OF LINEAR EQUATIONS

Definitions and properties of the integer solutions (integer solution) of linear equations (1.e):

Let 1.e:

n

(1) $\sum_{i=1} a_i x_i = b$, with all $a_i \neq 0$ and b from Z

Again, let $h \in N$ be, and the functions $f_i : Z^h \to Z$, $i = 1, \overline{1,n}$.

<u>Definition 1</u>

$x_i = x_i^0$, $i = \overline{1,n}$, is the particular integer solution of equation (1), if all $x_i^0 \in Z$ and $\sum_{i=1}^{n} a_i x_i^0 = b$.

<u>Definition 2</u>

$x_i = f_i(k_1, \ldots k_h)$, $i = \overline{1,n}$, is the general integer solution of equation (1) if:

(a) $\sum_{i=1}^{n} a_i f_i\,(k_1, \ldots k_h) = b \quad \forall (k_1, \ldots k_h) \in Z^h$,

(b) Irrespective of $f_i(x_1^0, \ldots x_n^0)$ there is a particular integer solution for (1) $(k_1^0, \ldots k_h^0) \in Z^h$ so that $x_i^0 = f_i\,(k_1^0, \ldots k_h^0)$ for all $i = \overline{1,n}$.

We will further see that the general integer solution can be expressed by linear functions.

We consider for $1 \leq i \leq n$, the functions $f_i = \sum_{j=1}^{h} c_{ij} k_j + d_i$ with all $c_{ij}, d_i \in Z$.

Definition 3

$A = (c_{ij})_{i,j}$ is referred to as the matrix associated to the general solution of equation (1).

Definition 4

The integers $k_1, \ldots k_s$, $1 \leq s \leq h$, are independent if all the corresponding column vectors of matrix A are linearly independent.

Definition 5

An integer solution is s - times undetermined if the maximal number of independent parameters is s.

Theorem 1. The general integer solution of equation (1) is (n-1) - times undetermined.

Proof

We suppose that the particular integer solution is of the form:

$$(2) \quad x_i = \sum_{e=1}^{r} i_{ie}P_e + v_i, \; i = \overline{1,n}, \text{ with all } u_{ie}, v_i \in Z,$$

$P_e$ = are parameters from Z, while a $\leq$ r < n - 1.

0 0

Let $(x_1, \ldots x_n)$ a general integer solution of equation (1) (we are not interested in the case when the equation does not accept integer solution). The solution

$$x_j = a_n l_j + x_j^0 \qquad j = 1,n - 1$$

$$x_n = -\left(\sum_{j=1}^{n-1} a_j k_j - x_n^0\right)$$

is (n - 1) - times undetermined (it can be easily checked that the order of the associated matrix is n - 1). Hence, there are n - 1 undetermined solutions. Let, in the general case, a solution n - 1 times be undetermined:

$$x_i = \sum_{j=1}^{n-1} c_{ij} k_j + d_i \quad i = 1,n, \text{ with all } c_{ij}, d_i \in Z.$$

The case when b = 0

Then $\sum_{i=1}^{n} a_i x_i = 0$. It follows $\sum_{i=1}^{n} a_i x_i =$

$$= \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} c_{ij} k_j + d_i \right) = \sum_{i=1}^{n} a_i \sum_{j=1}^{n-1} c_{ij} k_j + \sum_{i=1}^{n} a_i d_i = 0.$$

n

For $k_{j_0} = 1$ and $k_j = 0$, $j \neq j_0$, it follows $\sum_{i=1} a_i c_{ij_0} = 0$.

Let the homogenous linear system of n equations with n unknowns be:

$$\sum_{i=1}^{n} x_i c_{ij} = 0, \quad j = 1,n - 1;$$

$$\sum_{i=1}^{n} x_i d_i = 0$$

which, obviously accepts the solution $x_i = a_i$, $i = 1,n$ different from the trivial one. Hence the determinant of the system is zero, i.e., the vectors $C_j = (c_{1j}, \ldots C_{nj})^t$, $j = 1,n - 1$, $D = (d_1, \ldots, d_n)^t$ are linearly dependent. But the solution being n - 1 times undetermined it follows that $C_j$, $j = 1,n - 1$ are linearly independent. Then $(C_1, \ldots C_{n-1})$ determines a free submodule Z of the order n - 1 in $Z_n$ of solutions for the given equation.

Let us see what can be obtained from (2). We have:

$$0 = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left( \sum_{e=1}^{r} u_{ie} P_e + v_i \right).$$ As above,

we obtain:

$$\sum_{i=1}^{n} a_i v_i = 0 \quad \text{and} \quad \sum_{e=1}^{r} a_i u_{ie_0} = 0;$$ and similarly, the

vectors $U_h = (u_{1h}, \ldots, u_{nh})$ are linearly independent,

$h = 1,r$. $U_h$, $h = 1,r$ and $V = (v_z, \ldots, v_n)$ are particular integer solutions of the homogenous linear equation.

Subcase (a1)

U, $h = 1,r$ are linearly dependent. It gives $\{U_1, \ldots, U_r\}$ = the free submodule of order r in $z^n$ of solutions of the equation. Hence, there are solutions from $\{V_1, \ldots, V_{n-1}\}$ which are not from $\{U_1, \ldots, U_r\}$, this contradicts the fact that (2) is the general integer solution.

Subcase (a2)

$U_h$, $h = 1,r$, V are linearly independent. Then, $\{U_1, \ldots, U_r\} + V$ is a linear variety of the dimension $< n - 1 = \dim \{V_1, \ldots, V_{n-1}\}$ and the conclusion can be similarly drawn. The case when $b \neq 0$

So, $\sum^{n} a_i x_i = b$. Then $\sum^{n} a_i \left( \sum^{n-1} c_{ij} k_j + d_i \right) =$

i=1 i=1 j=1

$$= \sum_{j=1}^{n-1} \left( \sum_{i=1}^{n} a_i c_{ij} \right) k_j + \sum_{i=1}^{n} a_i d_i = b, \ \forall (k_1, \ldots, k_{n-1} \in$$

$\in Z^{n-1}$. As in the previous case we get $\sum_{i=1}^{n} a_i d_i = b$ and

$\sum_{i=1}^{n} a_i c_{ij} = 0 \ \forall_j = \ 1,n - 1$. The vectors $C_j =$

$= (c_{ij}, \ldots, c_{nj})^t$, $j = 1,n - 1$, are linearly independent because the solution is n - 1 times undetermined.

Conversely, if $C_1, \ldots C_{n-1}$, D (where $D = (d_1, \ldots, d_n)^t$) were linearly dependent, it would mean that $D = \sum_{j=1}^{n-1} s_j C_j$, with all $s_j$ scalar; it would also mean that $b = \sum_{i=1}^{n} a_i d_i =$

$$= \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} s_j c_{ij} \right) = \sum_{j=1}^{n-1} s_j \left( \sum_{i=1}^{n} a_i c_{ij} \right) = 0.$$ Impossible

(3) then $\{C_1, \ldots, C_{n-1}\} + D$ is a linear variety.

Let us see what we can obtain from (2). We have:

$$b = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \left( \sum_{e=1}^{r} u_{ie} P_e + v_i \right) = \sum_{e=1}^{r} \left( \sum_{i=1}^{n} a_i u_{ie} \right) P_e + \sum_{i=1}^{n} a_i v_i$$

and, similarly: $\sum_{i=1}^{n} a_i v_i = b$ and $\sum_{i=1}^{n} a_i u_{ie} = 0$, $\forall$ e = 1,r,

respectively. The vectors $U_e = (u_{1e}, \ldots, u_{ne})^t$, e = 1,r are linearly independent because the solution is r - times undetermined.

A procedure like that applied in (3) gives that $U_1, \ldots, U_r$, V are linearly independent, where $V = (v_1, \ldots, v_n)^t$. Then $\{J_1, \ldots U_r\} + V$ = a linear variety = free submodule of order $r < n - 1$. That is, we can find vectors from $C_1, \ldots, C_{n-1} + D$ which are not from $\{U_1, \ldots, U_r\} + V$, contradicting the "general" characteristic of the integer number solution. Hence, the general integer solution is n - 1 times undetermined.

Theorem 2. The general integer solution of the homogenous linear equation $\sum_{i=1}^{n} a_i x_i = 0$ (all $a_i \in Z \setminus \{0\}$) can be written under the form:

(4) $x_i = \sum_{j=1}^{n-1} c_{ij} k_j$, i = 1,n (with $d_1 = \ldots = d_n = 0$).

Definition 6. This is called the standard form of the general integer solution of a homogenous linear equation.

Proof

We consider the general integer solution under the form: $x_i = \sum_{j=1}^{n-1} c_{ij}P_j + d_i$, $i = 1,n$ with not all $d_i = 0$. We show that it can be written under the form (4). The homogenous equation admits the trivial solution $x_i = 0$, $i = 1,n$. There is $(p_1^0, \ldots, p_{n-1}^0) \in Z^{n-1}$ so that $\sum_{j=1}^{n-1} c_{ij}p_j^0 + d_i = 0$ $\forall i = 1,n$. Substituting: $P_j = k_j + p_j^0$, $j = 1,n-1$ in the form from the beginning of the demonstration we will obtain form (4). We have to mention that the substitution does not diminish the degree of generality as $P_j \in Z \Leftrightarrow k_j \in Z$ because $j \in 1,n-1$.

Theorem 3. The general integer solution of a nonhomogeneous linear equation of its associated homogenous linear equation + any particular integer solution of the nonhomogeneous linear equation.

Proof

Let $x_i = \sum_{j=1}^{n-1} c_{ij}k_j$, $i = 1,n$ be the general integer solution of the associated homogenous linear equation and, again, let $x_i = v_i$, $i = 1,n$ be a particular integer solution of the nonhomogeneous linear equation. Then, $x_i = \sum_{j=1}^{n-1} c_{ij}k_j + v_i$, $i = 1,n$ is the general integer solution of the nonhomogeneous linear equation. Actually, $\sum_{i=1}^{n} a_i x_i =$

$$= \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} c_{ij}k_j + v_i \right) = \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{n-1} c_{ij}k_j \right) + \sum_{i=1}^{n} a_i v_i = b;$$

if $x_i = x_i^0$, $i = 1,n$ is a particular integer solution of the nonhomogeneous linear equation, then $x_i = x_i - v_i$, $i = 1,n$ is a particular integer solution of the homogenous linear equation; hence, there is $(k_1^0, \ldots, k_{n-1}^0) \in Z^{n-1}$ so that $\sum_{j=1}^{n-1} c_{ij}k_j^0 = x_i^0 - v_i$, $\forall_i = 1,n$, i.e., $\sum_{j=1}^{n-1} c_{ij}k_j^0 + v_i = x_i^0$, $\forall_i = 1,n$, which was to be proven.

<u>Theorem 4</u>. If $x_i = \sum_{j=1}^{n-1} c_{ij}k_j$, $i = 1,n$, is the general integer solution of a homogenous linear equation of a homogenous linear equation $(c_{ij}, \ldots, c_{nj}) \sim 1$, $\forall_j = 1,n-1$.

The demonstration is made by reductio ad absurdum. If $\exists\ j_0,\ 1 \leq j_0 \leq n - 1$, so that $(c_{1j_0}, \dots, c_{nj_0}) \sim d_{j_0} \neq \pm 1$, then $c_{ij_0} = c'_{ij_0} d_{j_0}$ with $(c'_{1j_0}, \dots, c'_{nj_0}) \sim 1$, $\forall i = 1,n$.

But $x_i = c'_{ij_0}$, $i = 1,n$ represents a particular integer solution as

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i c'_{ij_0} = 1/d_{j_0} \bullet \sum_{i=1}^{n} a_i c_{ij_0} = 0$$

(because $x_i = c_{ij_0}$, $i = 1,n$ is a particular integer solution from the general integer solution by introducing $k_{j_0} = 1$ and $k_j = 0,\ j \neq j_0$). But the particular integer solution $x_i = c'_{ij_0}$, $i = 1,n$ cannot be obtained, by introducing whole number parameters (as it should), from the general integer solution, as, from the linear system of n equations and n - 1 unknowns, which is compatible, we obtain:

$$x_i = \sum_{\substack{j=1 \\ j \neq j_0}}^{n} c_{ij} k_j + c'_{ij_0} d_{j_0} k_{j_0} = c'_{ij_0},\ i = 1,n$$

Leaving aside the last equation--which is a linear combination of the other n - 1 equations--a Kramerian system is obtained. It follows:

$$k_{j_0} = \frac{\begin{vmatrix} c_{11} \ldots c'_{1j_0} \ldots c_{1,n-1} \\ \vdots \\ c_{n-1,1} \ldots c'_{n-1,j_0} \ldots c_{n-1,n-1} \end{vmatrix}}{\begin{vmatrix} c_{11} \ldots c'_{1j_0} d_{j_0} \ldots c_{1,n-1} \\ \vdots \\ c_{n-1,1} . c'_{n-1,j_0} d_{j_0} . c_{n-1,n-1} \end{vmatrix}} = \frac{1}{d_{j_0}} \in Z.$$

The assumption is false and of the demonstration.

<u>Theorem 5</u>. Considering the equation (1) with $(a_1, \ldots, a_n) \sim 1$, $b = 0$ and the general integer solution $x_i = \sum_{j=1}^{n-1} c_{ij}k_j$, $i = \overline{1,n}$ then $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \sim$

$\sim (c_{i1}, \ldots, c_{in-1})$, $\forall i \in \overline{1,n}$. The demonstration is made by double divisibility. Let $i_0$, $1 \leq i_0 \leq n$ be arbitrary but

fixed. $x_i = \sum_{j=1}^{n-1} c_{i_0 j}k_j$. Consider the equation $\sum_{i \neq 1_0} a_i x_i =$

$= -a_{i_0} x_{i_0}$. We have shown that $x_i = c_{ij}$, $i = \overline{1,n}$ is a particular integer solution irrespective of j, $a \leq j \leq$

$\leq$ n - 1. The equation $\sum_{i \neq i_0} a_i x_i = -a_{i_0} c_{i_0 j}$, obviously,

admits the integer solution $x_i = c_{ij}$, $i \neq i_0$. Then $(a_1, \ldots, a_{i_0 -1}, a_{i_0 +1}, \ldots, a_n)$ divides $-a_{i_0} c_{i_0 j}$, but, as we have assumed that $(a_1, \ldots, a_n) \sim 1$, it follows that $(a_1, \ldots, a_{i_0 -1}, a_{i_0 +1}, \ldots, a_n)$ $c_{i_0 j}$, irrespective of j. Hence $(a_1, \ldots, a_{i_0 -1}, a_{i_0 -1}, \ldots, a_n)$ $(c_{i_0 1}, \ldots, c_{i_0 n-1})$,

$V_i \in 1,n$, and the divisibility in one sense was proven.

<u>Inverse Divisibility</u>

Let us suppose the contrary and say that $\exists\ i_1 \in$

$\in 1,n$ for which $(a_1, \ldots, a_{i_1 -1}, a_{i_1 +1}, \ldots, a_n) \sim d_{i_1 1} \neq d_{i_1 2} \sim$ $\sim (c_{i_1 1}, \ldots, c_{i_1 n-1})$; we have considered $d_{i_1 1}$ and $d_{i_1 2}$ without restricting the generality. $d_{i_1 1}$ $d_{i_1 2}$ according to the first part of the demonstration. Hence, $\exists\ d \in Z$ so that $d_{i_1 2} = d \bullet d_{i_1 1}$, $d \neq 1$.

$$x_i = \sum_{j=1}^{n-1} c_{i_1 j} k_j = d \bullet d_{i_1 1} \sum_{j=1}^{n-1} c'_{i_1 j} k_j; \quad \sum_{i=1}^{n} a_i x_i = 0 \Rightarrow \sum_{i \neq i_1} a_i x_i =$$

$$= -a_{i_1} x_{i_1} \sum_{i \neq i_1} a_i x_i = -a_{i_1} d \bullet d_{i_1 1} \sum_{j=1}^{n-1} c'_{i_1 j} k_j, \text{ where}$$

$(c_{i_1 1}, \dots, c_{i_1 n-1}) \sim 1$

The nonhomogeneous linear equation $\sum_{i \neq i_1} a_i x_i = -a_{i_1} d_{i_1}$ admits integer solution because $a_{i_1} d_{i_1}$ is divisible by $(a_1, \dots, a_{i_1 -1}, a_{i_1 +1}, \dots, a_n)$. Let $x_i = x_i^0$, $i \neq i_1$ be its particular integer solution. It follows that the equation $\sum_{i=1}^{n} a_i x_i = 0$ admits the particular solution $x_i = x_i^0$, $i \neq i_1$, $x_{i_1} = d_{i_1}$, which is written as (5). We show that (5) cannot be obtained from the general solution by integer number parameters:

$$\sum_{j=1}^{n-1} c_{ij} k_j = x_i^0, \quad i \neq i_1;$$

$$d \bullet d_{i_1 1} \sum_{j=1}^{n-1} c_{ij} k_j = d_{i_1 1} \qquad (6)$$

But equation (6) does not admit integer solution because $d \bullet d_{i_1 1} \mid d_{i_1 1}$ thus, contradicting, thus, the "general" characteristic of the integer solution.

As a conclusion we can write:

Theorem 6. Let the homogenous linear equation be:

$\sum_{i=1}^{n} a_i x_i = 0$ with all $a_i \in Z \setminus \{0\}$, and $(a_1, ..., a_n) \sim 1$.

Let $x_i = \sum_{j=1}^{h} c_{ij} k_j$, $i = \overline{1,n}$ be, with all $c_{ij} \in Z$, all $k_j$ whole integer parameters and $h \in N$, a general integer solution of the equation. Then,

$1^0$ the solution is n - 1 times undetermined;

$2^0$ $\forall j = \overline{1,n-1}$ we have $(c_{1j}, ..., c_{nj}) \sim 1$;

$3^0$ $\forall i = \overline{1,n}$ we have $(c_{i1}, ..., c_{in-1}) \sim (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$. The proof results from Theorems 1, 4 and 5.

Note 1. The only equation of the form (1) which is n - times undetermined is the trivial equation $0 \bullet x_1 + ... + 0 \bullet x_n = 0$.

Note 2. The converse of theorem 6 is not true.

Counterexample:

$$\begin{aligned} x_1 &= -k_1 + k_2 \\ x_2 &= 5k_1 + 3k_2 \\ x_3 &= 7k_1 - k_2,\ k_1, k_2 \in Z \end{aligned} \quad (7)$$

is not the general integer solution of the equation

$$-13x_1 + 3x_2 - 4x_3 = 0 \qquad (8)$$

although the solution (7) verifies the points $1^0$, $2^0$ and $3^0$ of theorem 6. (1, 7, 2) is the particular integer solution of (8) but cannot be obtained by introducing integer number parameters in (7) because from

$$-k_1 + k_2 = 1$$
$$5k_1 + 3k_2 = 7$$
$$7k_1 - k_2 = 2$$

follows that $k = 1/2 \in Z$ and $k = 3/2 \in Z$ (unique roots).

Reference:

[1] Smarandache, Florentin, "Whole number solutions of linear equations and systems", M. Sc. Thesis, 1979, University of Craiova (under the supervision of Assoc. Prof. Dr. Alexandru Dincä).

# AN INTEGER NUMBER ALGORITHM
# TO SOLVE LINEAR EQUATIONS

An algorithm is given which ascertains whether a linear equation admits integer number solutions or not; if it does, the general integer solution is determined.

Input

A linear equation $a_1x_1 + ... + a_nx_n = b$ with $a_i\ b \in Z$,

$x_i$ being integer number unknowns, $i = \overline{1,n}$ and not all $a_i = 0$.

Output

Decision on the integer solution of this equation; if the equation has solutions in Z, its general solution is obtained.

Method

Step 1. Calculate $d = (a_1, ..., a_n)$.

Step 2. If d/b then "the equation has integer solution"; go on to Step 3. If d/b then "the equation does not admit integer solution"; stop.

Step 3. Consider h: = 1. If $d \neq 1$ divides the equation by d; consider $a_i$: = $a_i/d$, i = 1,n, b: = b/d.

Step 4. Calculate $a = \min_{a_s \neq 0} a_s$ and determine an i so that $a_i = a$.

Step 5. If $a \neq 1$, go on to Step 7.

Step 6. If a = 1, then:

(A) $x_i = -(a_1x_1 + ... + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} +$

$+ ... a_nx_n - b) \bullet a_i$

(B) Substitute the value of $x_i$ in the values of the other determined unknowns.

(C) Substitute integer number parameters for all the variables of the unknown values in the right term: $k_1$, $k_2$, ..., $k_{n-2}$ and $k_{n-1}$, respectively.

(D) Write down the general solution thus determined; stop.

Step 7. Write down all $a_j$, $j \neq i$ and b under the form:

$a_j = a_iq_j + r_j$

$b = a_iq + r$, where $q_j = \left[\frac{a_j}{a_i}\right]$, $q = \left[\frac{b}{a_i}\right]$

Step 8. Write $x_i = - q_1x_1 - ... - q_{i-1}x_{i-1} - q_{i+1}x_{i+1} -$ $- ... - q_nx_n + q - t_h$. Substitute the value of $x_i$ in the values of the other determined unknowns.

Step 9. $a_1 := 4_1$ and $a_i := -a_i$

$b := r$

Consider $x_i := t_h$

$h := h + 1$

$a_i - 1 := r$

$a_{i+1} := r_{i+1}$

.

.

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$a_n := r_n$ and go back to Step 4.

<u>Lemma 1</u>. The previous algorithm is finite.

<u>Proof</u>

Let the initial linear equation be $a_1x_1 + \ldots + a_nx_n = b$ with not all $a_i = 0$; it is considered that $\min_{a_s \neq 0} a_s = a_1 \neq 1$ (if not, it is renumbered). Following the algorithm, once we pass from this initial equation to a new equation: $a'_1t_1 + a'_2x_2 + \ldots + a'_nx_n = b'$, with $a'_1 < a_i$ for $i = 2,n$, $b' < b$ and $a'_1 = -a_1$. It follows that $\min_{a'_s \neq 0} a'_s < \min_{a_s \neq 1} a_s$. We continue similarly and after a finite number of steps we get, at Step 4, at $a := 1$ (as, every time, at this step the actual a is strictly smaller than the previous a, according to the former note) and in this case the algorithm terminates.

<u>Lemma 2</u>. Let the linear equation be: (25) $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ with $\min_{a_s \neq 0} a_s = a_1$ and the

equation: (26) $- a_1t_1 + r_2x_2 + ... + r_nx_n = r$ with $t_1 =$

$= - x_1 - q_2x_2 - ... - q_nx_n + q$ where $r_i = a_i - a_iq_i$, $i =$

$= \overline{2,n}$, $r = b - a_1q$ while $q_i = \left[\frac{a_i}{a}\right]$, $r = \left[\frac{b}{a_1}\right]$. Then

$x_1 = x_1^0, x_2 = x_2^0, ..., x_n = x_n^0$ is particular solution of equation (25) if and only if $t_1 = t_1^0 = - x_1 - q_2x_2^0 - ... - q_nx_n^0 + q$, $x_2, ..., x_n = x_n^0$ is a particular solution of equation (26).

<u>Proof</u>

$x_1 = x_1^0, x_2 = x_2^0, ..., x_n = x_n^0$ is a particular solution of equation (25) $\Leftrightarrow a_1x_1^0 + a_2x_2^0 + ... + a_nx_n^0 = b \Leftrightarrow a_1x_1^0 +$ $+ (r_2 + a_1q_2)\, x_2^0 + ... + (r_n + a_1q_n)\, x_n^0 = a_1q + r \Leftrightarrow r_2x_2^0 +$ $+ ... + r_nx_n^0 - a_1 (-x_1^0 -q_2x_2^0 - ... - q_nx_n^0 + q) = r \Leftrightarrow - a_1t_1^0 +$ $+ r_2x_2^0 + ... + r_nx_n^0 = r \Leftrightarrow t_1 = t_1^0, x_2 = x_2^0, ..., x_n = x_n^0$ is a particular solution of equation (26).

<u>Lemma 3</u>. $x_i = c_{i1}k_1 + ... + c_{in-1}k_{n-1} + d_i$, $i = \overline{1,n}$ is the general solution of equation (25) if and only if:

(28)

$t_1 = - (c_{11} + q_2c_{21} + ... + q_nc_{n1})k_1 - ... - (c_{1n-1} +$

$+ q_2c_{2n-1} + ... + q_nc_{nn-1})K_n - (d_1 + q_2d_2 + ... + q_nd_n) +$

$+ q,\ x_j = c_{cj1}k_1 + \dots + c_{jn-1}k_{n-1} + d_j,\ j = \overline{2,n}$

is a general solution for equation (26).

Proof

$t_1 = \overset{0}{t_1} = - \overset{0}{x_1} - q_2\overset{0}{x_2} - \dots - q_n\overset{0}{x_n} + q,\ x_2 = \overset{0}{x_2}, \dots,$
$\dots, x_n = \overset{0}{x_n}$ is a particular solution of the equation (25) $\Leftrightarrow x_1 = \overset{0}{x_1},\ x_2 = \overset{0}{x_2}, \dots, x_n = \overset{0}{x_n}$ is a particular solution of equation (26) $\Leftrightarrow \exists\ k_1 = \overset{0}{k_1} \in Z, \dots, k_n = \overset{0}{k_n} \in Z$ so that

$x_i = c_{i1}\overset{0}{k_1} + \dots + c_{in-1}\overset{0}{k_{n-1}} + d_i = \overset{0}{x_i},\ i = \overline{1,n} \Leftrightarrow \exists\ k_1 =$
$= \overset{0}{k_1} \exists Z, \dots, k_n\overset{0}{k_n} \in Z$ so that $x_1 = c_{i1}\overset{0}{k_1} + \dots + c_{in-1}k_{n-1} +$

$+ d_i = \overset{0}{x_i},\ i = \overline{2,n}$ and $t_1 = - (c_{11} + q_2c_{21} + \dots + q_nc_{n1})\overset{0}{k_1} -$
$- \dots - (c_{1n-1} + q_2c_{2n-1} + \dots + q_nc_{nn-1})\overset{0}{k_{n-1}} - (d_1 + q_2d_2 +$
$+ \dots + q_nd_n) + q = - \overset{0}{x_1} - q_2\overset{0}{x_2} - \dots - q_n\overset{0}{x_n} + q = \overset{0}{t_1}.$

Lemma 4. The linear equation (29) $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ with $a_1 = 1$ has the general solution: (30)

$$x_1 = - (a_2k_2 + \dots + a_nk_n - b)a_1,$$

$$x_i = k_i \in Z,$$

$$i = \overline{2,n}.$$

Proof

Let $x_1 = \overset{0}{x_1},\ x_2 = \overset{0}{x_2}, \dots, x_n = \overset{0}{x_n}$ be a particular

0 0

solution of the equation (29). $\exists$ $k_2 = x_2$, ..., $k_n = k_n$ so that $x_1 = -(a_2 x_2^0 + \ldots + a_n x_n^0 - b)a_1 = x_1^0$, $x_2 = x_2^0$, ..., ..., $x_n = x_n^0$.

Lemma 5. Let the linear equation be $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$, with $\min_{a_s \neq 0} a_s = a_1$ and $a_i = a_1 q_i$, $i = \overline{2,n}$.

Then, the general solution of the equation is:

$$x_1 = -(q_2k_2 + \ldots + q_nk_n - q),$$

$$x_i = k_i \in Z,$$

$$i = \overline{2,n}.$$

Proof

Dividing the equation by $a_1$ the conditions of Lemma 4 are met.

Theorem of Correctness. The preceding algorithm correctly calculates the general solution of the linear equation $a_1x_1 + \ldots + a_nx_n = b$ with not all $a_i = 0$.

Proof

The algorithm is finite according to Lemma 1. The correctness of steps 1, 2, and 3 is obvious. At step 4 there is always $\min_{a_s \neq 0} a_s$ as not all $a_i = 0$. The correctness of substep 6(A) results from Lemmas 4 and 5, respectively. This algorithm represents a procedure of obtaining the general solution of the initial equation by means of the general solutions of the linear equation obtained after the algorithm was followed several times (according to Lemmas 2 and 3); from Lemma 3 it follows that to obtain the general solution of an initial linear equation is equivalent to calculate the general solution of an equation at step 6(A), equations whose general solution is given in algorithm (according to Lemmas 4 and 5). The theorem of correctness has been fully proven.

Note. At step 4 of the algorithm we consider $a := \min_{a_s \neq 0} a_s$ so that the number of iterations be as small as possible. The algorithm works if we consider $a := a_i \neq \max_{s=\overline{1,n}} a_s$ but it takes longer. The algorithm can be introduced in the computer.

## Application

Calculate the integer solution of the equation:

$$6x_1 - 12x_2 - 8x_3 + 22x_4 = 14$$

## Solution

The former algorhythmus is applied.

1. (6, - 12, - 8, 22) = 2
2. 2 14 so that the solution of the equation is in Z
3. h: = 1

   2 $\neq$ 1; dividing the equation by 2 we get:

   $3x_1 = 6x_2 - 4x_3 + 11x_4 = 7$
4. a: = min { 3 , -6 , -4 , 11 } = 3, i = 1
5. a $\neq$ 1
7. -6 = 3.(-2) + 0

   -4 = 3.(-2) + 2

   11 = 3.3 + 2

   7 = 3.2 + 1
8. $x_1 = 2x_2 + 2x_3 - 3x_4 + 2 - t_1$
9. $a_2$: = 0 $a_1$: = -3

   $a_3$: = 2 b: = 1

   $a_4$: = 2 $x_1$: = $t_1$

   h: = 2
4. We have a new equation:

   $-3t_1 + 0 \bullet x_2 + 2x_3 + 2x_4 = 1,$

   a = min { -3 , 2 , 2 } = 2, and

   i = 3
5. a $\neq$ 1
7. $-3 = 2 \bullet (-2) + 1$

$0 = 2 \bullet 0 + 0$

$2 = 2 \bullet 1 + 0$

$1 = 2 \bullet 0 + 0$

8. $x_3 = 2t_1 - 0 \bullet x_2 - x_4 + 0 - t_2$. Substituting the value of $x_3$ in the value determined for $x_1$ we get:

$x_1 = 2x_2 - 5x_4 + 3t_1 - 2t_2 + 2$

9. $a_1: = 1$ $a_3: = -2$

$a_2: = 0$ $b: = 1$

$a_4: = 0$ $x_3: = t_2$

$h: = 3$

4. We have obtained the equation:

$1 \bullet t_2 + 0x_2 - 2t_2 + 0 \bullet x_4 = 1$,

$a = 1$, and

$i = 1$

6. (A) $t_1 = -(0 \bullet x_2 - 2t_2 + 0 \bullet x_4 - 1) \bullet 1 = 2t_2 + 1$

(B) Substituting the value of $t_1$ in the values of $x_1$ and $x_3$ previously determined, we get:

$x_1 = 2x_2 - 5x_4 + 4t_2 + 5$ and

$x_3 = -x_4 + 3t_2 + 2$

(C) $x_2: = k_1$, $x_4: = k_2$, $t_2 = k_3$, $k_1, k_2, k_3 \in Z$

(D) The general solution of the initial equation is:

$x_1 = 2k_1 - 5k_2 + 4k_3 + 5$

$x_2 = k_1$

$x_3 = -k_2 + 3k_3 + 2$

$x_4 = k_2$

$k_1, k_2, k_3$ are parameters $\in Z$

Reference

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# ANOTHER INTEGER NUMBER ALGORITHM TO SOLVE LINEAR EQUATIONS (USING CONGRUENCY)

In the present part a new integer number algorithm for linear equations is presented. This is more "rapid" than W. Sierpinski's presented in [1] in the sense that it reaches the general solution after a smaller number of iterations. Its correctness will be strictly demonstrated.

## INTEGER NUMBER ALGORITHM TO SOLVE LINEAR EQUATIONS

Let us consider the equation (1);
(the case $a_i$, b are in Q,

i = 1,n, is reduced to the case (1) by bringing to the same denominator and eliminating the denominators).
Let $d = (a_1, ..., a_n)$. If d b, then the equation does not admit integer solutions, while if d b, the equation admits integer solutions (according to a well-known theorem from the theory of numbers).

If the equation accepts solutions and $d \neq 1$, we divide the equation by d. Then, we can agree that d = 1 (we do not make any restriction if we consider the maximal co-divisor positive).

(a) Also, if all $a_i = 0$, the equation is trivial; it admits the general integer solution $x_i = k_i \in Z$, $i = \overline{1,n}$ when $b = 0$ (the only case when the general integer solution is n - times undetermined) and does not have solutions when $b \neq 0$.

(b) If $\exists$ i, $1 \leq i \leq n$, so that $a_i = \pm 1$, then the general integer solution is:

$$x_i = -a_i \left( \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j k_j - b \right) \text{ and } x_s = k_s \in Z,\ s \in \{1, ..., n\} \setminus \{i\}$$

The proof of this assertion was give in [4]. All these cases being trivial, we will leave them aside. The following algorithm can be written:

<u>Input</u>

A linear equation: (2)

$$\sum_{i=1}^{n} a_i x_i = b,\ a_i,\ b\ Z,\ a_i \neq \pm 1,\ i = \overline{1,n}$$

with not all $a_i = 0$ and $(a_1, ..., a_n) = 1$.

Output

The general solution of the equation.

Method

1. $h := 1$, $p := 1$

2. Calculate $\min_{1 \le i,j \le n} \{ r \; , \; r \equiv a_i \pmod{a_j},$

$r \neq 0, \; r < a_j \}$

and determine r and the paid (i,j) for which this minimum can be obtained (when there are more possibilities we have to choose one of them).

3. If $r \neq 1$ go on to step 4.
If $r = 1$, then

$$x_i := r(-a_j t_h - \sum_{\substack{s=1 \\ s \notin \{i,j\}}}^{n} a_s x_s + b)$$

$$x_j := r(a\ t + \frac{a_i - r}{a_j} \bullet \sum_{\substack{s=1 \\ s \notin \{i,j\}}}^{n} a_s x_s + \frac{r - a_i}{a_j} b)$$

(A) Substitute the values thus determined of these unknowns in all the relations (p), p = 1, 2 ... (if possible).

(B) From the last relation (p) obtained in the algorhythmus substitute in all relations:

(p - 1), (p - 2), ..., (1).

(C) Every relation, starting in order from

(p - 1) should be applied the same

procedure as in (B); then (p - 2), ..., ..., (3), respectively.

(D) Write the values of the unknowns $x_i$,

$i = 1,n$ from the initial equation writing the corresponding integer number parameters from the right term of these unknowns with $k_1$, ..., $k_{n-1}$), STOP.

4. Write the relation (p):

$$x_j = t_h - \frac{a_i - r}{a_j} x_i;$$

5. Consider $x_j : = t_h$ $h : = h + 1$

$a_i : = r$ $p : = p + 1$

The other coefficients and variables remaining unchanged and go back to step 2.

## The Correctness of the Algorithm

Let us consider linear equation (2). Under these conditions, the following proprieties exist:

Lemma 1. The set $M = \{ r , r \equiv a_I \pmod{a_j}, 0 < r < a_j \}$ has a minimum.

### Proof

Obviously $M \subset N^*$ and M is finite because the equation has a finite number of coefficients: n and considering all the possible combinations of these, by twos, there is the maximum $AR_n^2$ (arranged with repetition) = n elements.

Let us show, by reductio ad absurdum, that $M \neq \emptyset$,

$M = \emptyset \Leftrightarrow a_i = 0 \pmod{a_j}$ V i, $j \in 1,n$. Hence $a_j =$

$= 0 \pmod{a_i}$, $V_i$, $j \in 1,n$. Or this is possible only

when $a_i = a_j$, $V_i$, $j \in 1,n$, which is equivalent to

$(a_1, \ldots, a_n) = a_i$, $V_i \in 1,n$. But $(a_1, \ldots, a_n) = 1$ according to a restriction from the assumption. It

follows that $a_i = 1,n$, $Vi \in 1,n$ a fact which contradicts the other restrictions of the assumption.

$M \neq 0$ and finite, it follows that M has a minimum.

Lemma 2. If $r = \min_{1 \leq i,j \leq n} M$ then $r < a_i$, $\forall i \in 1,n$.

Proof

We assume, conversely, that $\exists i_0$, $1 \leq i_0 \leq n$, so that $r \geq a_{i_0}$. Then $r \geq \min_{1 \leq j \leq n} \{ a_j \} = a_{j_0} \neq 1$, $1 \leq j_0 \leq n$. Let $a_{p_0}$, $1 \leq p_0 \leq n$ so that $a_{p_0} > a_{j_0}$ and $a_{p_0}$ is not divided by $a_{j_0}$. There is such a coefficient in the equation as $a_{j_0}$ is the minimum and not all the coefficients are equal among themselves (conversely, it would mean that $(a_1, ..., a_n) = a_1 = \pm 1$, which is against the hypothesis) and, again, of the coefficients whose module is greater than $a_{ij_0}$ not all can be divided by $a_j$ (conversely, it would similarly result that $(a_1, ..., a_n) = a_{j_0} \neq \pm 1$). We write $[a_{p_0}/a_{j_0}] = q_0 \in Z$ (the whole number part), and $r = a_{p_0} - q_0 a_{j_0} \in Z$. We have $a_{p_0} \equiv r_0 \pmod{a_{j_0}}$ and $0 < r_0 < a_{j_0} < a_{i_0} \leq r$. Thus, we have found a $r_0$ with $r_0 < r$ which contradicts the definition of minimum given to $r$. Contrary to the assumption. Thus, $r < a_i$, $\forall i \in 1,n$.

Lemma 3. If $r = \min M = 1$, for the pair of indices (i,j), then:

$$x_i = r\,(-a_j t_h - \sum_{\substack{s=1 \\ s \notin \{i,j\}}}^{n} a_s k_s + b)$$

$$x_j = r\,(a_i t_h + \frac{a_i - r}{a_j} \sum_{\substack{s=1 \\ s \notin \{i,j\}}}^{n} a_s k_s + \frac{r - a_i}{a_j}\, b)$$

$$x_s = k_s \in Z,\ s \in \{1, \ldots, n\} \setminus \{i,j\}$$

is the general integer solution of equation (2).

<u>Proof</u>

Let $x_e = x_e^0$, $e = 1,n$ be a particular integer solution of equation (2). Then $\exists\ k_s = x_s^0 \in Z$, $s \in \{1, \ldots, n\} \setminus$ $\setminus \{i,j\}$ and $t_h = x_j^0 + \frac{a_i - r}{a_j} x_i^0 \in Z$ (because $a_i - r = \mathcal{M} a_j$), so that:

$$x_i = r - a_j(x_j^0 + \frac{a_i - r}{a_j} x_i) - \sum_{\substack{s=1 \\ s \notin \{i,j\}}}^{n} a_s x_s^0 + b = x_i^0$$

$$x_j = r - a_j(x_j^0 + \frac{a_i - r}{a_j} x_i^0) + \frac{a_i - r}{a_j} \sum_{s=1}^{n} a_s x_s^0 +$$

$s \notin \{i,j\}$

$$+ \frac{r-a_i}{a_j} b = \overset{0}{x}_j$$

and $x_s = k_s = \overset{0}{x}_s$, $s \in \{1, ..., n\} \setminus \{i,j\}$.

Lemma 4. Let $r \neq 1$ and (i,j) be the pair of indices for which this minimum can be obtained. Again, let the system of linear equation be:

$$a_j t_h + r x_i + \sum_{\substack{s=1 \\ s \neq \{i,j\}}}^{n} a_s x_s = b$$

(3)

$$t_h = x_j + \frac{a_i - r}{a_j} x_i$$

Then, $x_e = \overset{0}{x}_e$, $e = 1,n$ is a particular integer solution for (2) if and only if $x_e = \overset{0}{x}_e$, $e \in \{1, ..., n\} \setminus \{j\}$ and $t_h = \overset{0}{t}_h = \overset{0}{x}_j + \frac{a_i - r}{a_j} x_i$ is the particular integer solution of (3).

Proof

(2) $x_e = x_e$, $e = 1,n$ is a particular integer solution for

$$\Leftrightarrow \sum_{\substack{e=1}}^{n} a_e x_e^0 = b \Leftrightarrow \sum_{\substack{s=1 \\ s\neq\{i,j\}}}^{n} a_s x_s^0 + a_j\left(x_j^0 + \frac{a_i-r}{a_j} x_i^0\right) +$$

$$+ r x_i^0 = b$$

$$\Leftrightarrow a_j t_h^0 + r x_i^0 + \sum_{\substack{s=1 \\ s\neq\{i,j\}}}^{n} a_s x_s^0 = b \text{ and } t_h^0 = x_j^0 +$$

$$+ \frac{a_i-r}{a_j} x_i^0 \in Z$$

$$\Leftrightarrow x_e = x_e^0,\ e \in \{1,2,\ ...,\ n\} \setminus \{j\}$$

and $t_h = t_h^0$ is a particular integer solution for (3).

<u>Lemma 5</u>. The former algorithm is finite.

<u>Proof</u>

When $r = 1$ the algorithm stops at step 3. We will discuss the case when $r \neq 1$. According to the definition of r, $r \in N^*$. We show that the row of r - s successively obtained by following the algorithm several times is strictly decreasing to 1. Let $r_1$ be the first obtained by following the algorithm one time. $r_1 \neq 1$, go on to steps 4 and, then 5. According to lemma 2 $r_1$ <

< $a_i$ , $\forall_i = 1,n$. Now we shall follow the algorithm a second time, but this time for an equation in which $r_1$ (according to step 5) is substituted for $a_i$. Again, according to lemma 2, the new $r$ written $r_2$ will have the propriety: $r_2 < r_1$ . We will get to $r = 1$ as $r \geq 1$ and $r < \infty$, and if $r \neq 1$, following the algorithm once again we get $r < r_1$ , a.s.o. Hence, the algorhythmus has a finite number of repetitions.

<u>Theorem of Correctness</u>. The former algorithm calculates correctly the general integer solution of the linear equation (2).

<u>Proof</u>

According to lemma 5 the algorithm is finite. From lemma 1 it follows that the set M has a minimum, hence step 2 of the algorithm has meaning. When $r = 1$ it was shown in lemma 3 that step 3 of the algorithm calculates correctly the general integer solution of the respective equation (the equation that appears at step 3). In lemma 4 it is shown that if $r \neq 1$, by the substitutions steps 4 and 5 introduce in the initial equation the general integer solution remains unchanged. That is, we pass from the initial equation to a linear system having the same general solution as the initial equation. The variable h is a counter of the newly introduced variables which are used to successively decompose the system in systems of two linear equations. The variable p is a counter of the substitutions of variables (the relations, at a given moment, between certain variables).

When the initial equation was decomposed to r = 1, we have to follow the reverse way: i.e., to compose its general integer solution. This reverse way is directed by the substeps 3(A), 3(B) and 3(C). The substep 3(D) has only an aesthetic role, i.e., to have the general solution

under the form: $x_i = f_i(k_1, \ldots, k_{n-1})$, $i = \overline{1,n}$, $f_i$ being linear functions with integer number coefficients. This "if possible" shows that substitutions are not always possible. But when they are we have to make all the possible substitutions.

Note 1. The former algorithm is written under a form easy to introduce in the computer.

Note 2. The former algorithm is more "rapid" than that of W. Sierpinski's 1, i.e., the general integer solution is reached after a smaller number of iterations (or, at least, the same) for any linear equation (2). In the first place, both aim at obtaining the coefficient $\pm 1$

for at least one unknown variable. While Sierpinski started only by chance by decomposing the greatest coefficient in the module (writing it under the form of a sum between a multiple of the following smaller coefficient (in the module) and the rest), in our algorithm this decomposition is not accidental but always seeks the smallest $r$ and also chooses the coefficients $a_i$ and $a_j$ for which this minimum is achieved. That is, we test from the beginning the shortest way to the general integer solution. Sierpinski does not attempt at finding the shortest way; he knows that his way will take him to the general integer solution of the equation and is not interested in how long it will be. However, when an algorithm is introduced in a computer, it is preferable that the computer time should be as short as possible.

Example 1

Let us solve in $Z^3$ the equation: $17x - 7y + 10z = -12$. We apply the former algorithm.

1. $h = 1$, $p = 1$
2. $r = 3$, $i = 3$, $j = 2$
3. $3 \neq 1$, go on to step 4.
4. (1)

$y = t_1 = \frac{}{-7} \bullet z = t_1 + z$

5. Consider $y: = t_1$ $h: = 2$

$a_3: = 3$ $p: = 2$

the other coefficients and variables remaining unchanged, go back to step 2.

2. $r = -1, i = 1, j = 3$

3. $-1 = 1$

$x = -1(-3t_2 - (-7t_1) -12) = 3t_2 - 7t_1 - 12$

$$z = -1(17t_2 + (-7t_1) \bullet \frac{17-(-1)}{3} + \frac{-1-17}{3} (-12)) =$$

$$= -17t_2 + 42t_1 - 72$$

(A) We substitute the values of x and z thus determined in the only relation (p) we have:

(1) $y = t_1 + z = -17t_2 + 43t_1 - 72$

(B) The substitution is not possible.

(C) The substitution is not possible.

(D) The general integer solution of the equation is:

$x = 3k_1 - 7k_2 + 12$

$$y = -17k_1 + 43k_2 - 72$$

$$z = -17k_1 + 42k_2 - 72; \; k_1, \; k_2 \in Z$$

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# INTEGER NUMBER SOLUTIONS OF LINEAR SYSTEMS

## Definitions and Properties of the Integer Solution of a Linear System

Let $\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = \overline{1,m}$ (1)

A linear system with all its coefficients' integer numbers (the case with rational coefficients is reduced to the same).

Definition 1. $x_j = x_j^0$, $j = \overline{1,n}$ is a particular integer solution of (1) if $x_j^0 \in Z$, $j = \overline{1,n}$ and $\sum_{j=1}^{n} a_{ij}x_j^0 = b_i$, $i = \overline{1,m}$. Let the functions be $f_j : Z^h \to Z$, $j = \overline{1,n}$, where $h \in N^*$.

Definition 2. $x_j = f_j(k_1, \ldots k_h)$, $j = \overline{1,n}$, is the general integer solution for (1) if:

(a) $\sum_{j=1}^{n} a_{ij}f_j(k_1, \ldots, k_h) = b_i$, $i = \overline{1,m}$ of $(k_1, \ldots, k_h) \in Z$;

(b) For any $x_j = x_j^0$, $j = \overline{1,n}$, particular integer solution of (1), there is $(k_1^0, \ldots, k_h^0) \in Z$ so that $f_j(k_1^0, \ldots, k_h^0) = x_j$, $\forall j = \overline{1,n}$.

(In other words, the general solution is the solution that comprises all the other solutions.)

*

Propriety 1

A general solution of a linear system of m equations with n unknowns, r (A) = m < n is n - m times undetermined.

Proof

We assume by reductio ad absurdum that it is of order r, $1 \leq r < n - m$ (the case r = 0, i.e., the solution is particular, is trivial). It follows that the general solution is of the form:

$$(S_1) \quad \begin{cases} x_1 = u_{11}p_1 + \ldots + u_{1r}p_r + v_1 \\ \vdots \\ x_n = u_{n1}p_1 + \ldots + u_{nr}p_r + v_n, \; u_{ih}, \; v_i \in Z \end{cases}$$

$p_h$ = parameters $\in Z$

We prove that there are n - m times undetermined solutions. The homogenous linear system (1), solved in r admits the solution:

$$x_1 = \frac{D^1_{m+1}}{D} x_{m+1} + \ldots + \frac{D^1_n}{D} x_n$$
$$\vdots$$
$$x_m = \frac{D^m_{m+1}}{D} x_{m+1} + \ldots + \frac{D^m_n}{D} x_n$$

Let $x_i = x^0_i$, $i = 1,n$ be a particular solution of the linear system (1).

Considering

$$x_{m+1} = D \bullet k_{m+1}, \text{ we get a solution}$$
$$\vdots$$
$$x_n = D \bullet k_n$$

$$x_1 = D^1_{m+1}k_{m+1} + \ldots + D^1_n k_n + x^0_1$$
$$\vdots$$
$$x_m = D^m_{m+1}k_{m+1} + \ldots + D^m_n k_n + x^0_m$$
$$x_{m+1} = D \bullet k_{m+1} + x^0_{m+1}$$
$$\vdots$$
$$x_n = D \bullet k_n + x^0_n, \; k_j = $$
$$= \text{parameters} \in Z$$

which depends on the n - m independent parameters, for the system (1). Let the solution be n - m times undetermined:

$$(S_2)\quad \begin{array}{l} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1 \\ . \\ . \\ . \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n \end{array}$$

$c_{ij}$, $d_i \in Z$, $k_j$ = parameters $\in Z$

(There are such solutions, we have proven it before.)

Let the system be:

$$\begin{array}{ll} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 & \\ . & \\ . & \\ . & a_{ij},\ b_i \in Z \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m & x_i = \text{unknowns} \in Z \end{array}$$

I. The case $b_i = 0$, $i = 1,m$ results in a homogenous linear system:

$$a_{il}x_i + \ldots + a_{in} = 0,\ i = 1,m$$

$$(S_2) \Rightarrow \quad a_{i1}(c_{i1}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1) + \ldots a_{in}(c_{n1}k_1 +$$

$$+ \ldots c_{nn-m}k_{n-m} + d_n) = 0$$

$$0 = (a_{i1}c_{11} + \ldots + a_{in}c_{n1})k_1 + \ldots + (a_{i1}c_{1n-m} +$$

$$+ \ldots + a_{in}c_{nn-m}) \bullet k_{n-m} + (a_{i1}d_1 + \ldots +$$

$$+ a_{in}d_n), \ \forall\ k_j \in Z$$

For $k_1 = \ldots = k_{n-m} = 0 \Rightarrow a_{i1}d_1 + \ldots + a_{in}d_n = 0$

For $k_1 = \ldots = k_{h-1} = k_{h+1} = \ldots = k_{n-m} = 0$ and $k_h =$

$$= 1 \Rightarrow (a_{i1}c_{ih} + \ldots + a_{in}c_{nh}) + (a_{i1}d_1 + \ldots +$$

$$+ a_{in}d_dn) = 0 \Rightarrow a_{i1}c_{1h} + \ldots + a_{in}c_{nh} = 0$$

$\forall\ i = 1,m,\ \forall\ h = 1,n-m$

Vect. $V_h = \begin{vmatrix} c_{1h} \\ . \\ . \\ . \\ . \\ c_{nh} \end{vmatrix}$, $h = 1,n-m$, are the particular solutions of the system.

$V_h$, $h = 1,n-m$ are also linearly independent because the solution is n - m times undetermined. $\{V_1, \ldots, V_{n-m}\} + d$ is a linear variety that includes the solutions of the system obtained from $(S_2)$. Similarly, for $(S_1)$ we deduce

that $\begin{matrix} U_{1s} \\ . \\ . \end{matrix}$ , $s = 1,r$ are particular solutions of

.
$U_{ns}$

the given system and are linearly independent because $(S_1)$ = r - times undetermined solution and

$$V = \begin{matrix} v_1 \\ . \\ . \\ . \\ v_n \end{matrix} = \text{a solution of the given system.}$$

<u>The case (a)</u>. $U_1, \ldots, U_r, \forall$ = linearly dependent, it follows that $\{U_1, \ldots, U_r\}$ is a free submodule of order $r < n - m$ of solutions of the given system, then, it follows that there are solutions that belong to $\{V_1, \ldots, V_{n-m}\} + d$ and which do not belong to $\{U_1, \ldots, U_r\}$, a fact which contradicts the assumption that $(S_1)$ is the general solution.

<u>The case (b)</u>. $U_1, \ldots, U_r$, V = linearly independent. $\{U_1, \ldots, U_r\} + V$ is a linear variety that comprises the solutions of the given system, which were obtained from $(S_1)$. It follows that the solution belongs to $\{V_1, \ldots, V_{n-m}\} + d$ and does not belong to $\{U_1, \ldots, U_r\} + V$, a fact which is in contradiction with the assumption that $(S_1)$ is the general solution.

II. When there is an $i \in 1,m$, $b_i \neq 0 \Rightarrow$ nonhomogeneous linear system

$$a_{i1}x_1 + ... + a_{in}x_n = b_i, \quad i = 1,m$$

$$(S_2) \qquad \Rightarrow a_{i1}\,(c_{11}k_1 + ... + c_{1n-m}k_{n-m} + d_1) + ... +$$

$$+ a_{1n}\,(c_{n1}k_1 + ... + c_{nn-m}k_{n-m} + d_n = b_i;$$

it follows that

$$\Rightarrow (a_{i1}c_{11} + ... + a_{in}c_{nl})\ k_1 + ... + (a_{i1}c_{1n-m} +$$

$$+ ... + a_{in}c_{nn-m})\ k_{n-m} + (a_{i1}d_1 + ... + a_{in}d_n) =$$

$$= b_i;$$

for $k_1 = ... = k_{n-m} = 0 \Rightarrow a_{i1}d_1 + ... + a_{in}d_n = b_1;$

for $k_1 = ... = k_{j-1} = k_{j+1} = ... = k_{n-m} = 0$ and $k_j = 1$
$\Rightarrow$

$$\Rightarrow (a_{i1}c_{1j} + ... + a_{in}c_{nj} + (a_{in}d_1 + ... +$$

$+ a_{1n}\ d_n) = b_i$ it follows that

$$a_{i1}c_{1j} + ... + a_{in}c_{nj} = 0$$

$$a_{i1}d_1 + ... + a_{in}d_n = b_i,\ \forall\ i = 1,m,\ \forall\ j = 1,n-m;$$

$$V_j = \begin{matrix} c_{1j} \\ \cdot \\ \cdot \\ \cdot \\ c_{nj} \end{matrix}, \quad j = 1,n-m$$ are linearly independent because the solution ($S_2$) in n - m times is undetermined.

?! $V_j$, $j = 1,n-m$ and $d = \begin{matrix} d_1 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{matrix}$ are linearly independent.

We assume that they are not linearly independent. It follows that

$$d = s_1V_1 + \ldots + s_{n-m}V_{n-m} =$$

$$\begin{matrix} s_1c_{11} + \ldots + s_{n-m}c_{1n-m} \\ \cdot \\ \cdot \\ \cdot \\ s_1c_{n1} + \ldots + s_{n-m}c_{nn-m} \end{matrix}$$

Irrespective of $i = 1,m$:

$$b_1 = a_{i1}d_1 + \ldots + a_{in}d_n = a_{i1}\,(s_1c_{11} + \ldots + s_{n-m}c_{1n-m}) +$$

$$+ \ldots + a_{in}\,(s_1c_{n1} + \ldots + s_{n-m}c_{nn-m}) = (a_{i1}c^{11} + \ldots +$$

$$+ a_{in}c_{n1})\,s_1 + \ldots + (a_{i1}c_{1n-m} + \ldots + a_{in}c_{nn-m}),\ s_{n-m} =$$

0

Then, $b_i = 0$, irrespective of $i = 1,m$, contradicts the hypothesis (that if there is an $i \in 1,m$, $b_i \neq 0$). It follows that $V_1, \ldots, V_{n-m}$, d are linearly independent.

$\{V_1, ..., V_{n-m}\} + d$ is a linear variety that contains the solutions of the nonhomogeneous system, solutions obtained from $(S_2)$. Similarly, from $(S_1)$ it follows that $\{G_1, ..., G_r\} + V$ is a linear variety containing the solutions of the nonhomogeneous system, obtained from $(S_1)$.

$n-m > r$ follows that there are solutions of the system that belong to $\{V_1, ..., V_{n-m}\} + d$ and which do not belong to $\{G_1, ..., G_r\} + V$ (it contradicts the fact that $(S_1)$ is the general solution). Then, it results that the general solution depends on the n-m independent parameters.

Theorem 1. The general solution of a nonhomogeneous linear system is equal to the general solution of an associated linear system plus a particular solution of the nonhomogeneous system.

Proof

Let the homogeneous linear solution:

$$\begin{matrix} a_{11}x_1 + ... + a_{1n}x_n = 0 \\ . \\ . \\ . \\ a_{m1}x_1 + ... + a_{mn}x_n = 0 \end{matrix} \quad , (AX = 0)$$

with the general solutions:

$$x_1 = c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + d_1$$
$$\vdots$$
$$x_n = c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + d_n$$

and

$$x_1 = x_1^0$$
$$\vdots$$
$$x_n = x_n^0$$

a particular solution of the nonhomogeneous linear system $AX = b$;

?!

$$x_1 = c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + d + x_1^0$$
$$\vdots$$
$$x_n = c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + d_n + x_n^0$$

is a solution of the nonhomogeneous linear system.

We have written

$$A = \begin{matrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{matrix}, \quad X = \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix}, \quad b = \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix}, \quad 0 = \begin{matrix} o \\ \vdots \\ o \end{matrix}$$

(vector of dimension m),

$$K = \begin{pmatrix} k_1 \\ \vdots \\ k_{n-m} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \dots & c_{1n-m} \\ \vdots & & \\ c_{n1} & \dots & c_{nn-m} \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \quad x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix};$$

$$AX = A(Ck + d + x^0) = A(Ck + d) + AX^0 = b + 0 = b$$

We will prove that irrespective of $x_1 = y_1^0, \dots, x_n = y_n^0$

there is a particular solution of the nonhomogeneous system

$$k_1 = k_1^0 \in Z, \dots, k_{n-m} = k_{n-m}^0 \in Z$$

with the propriety:

$$x_1 = c_{11}k_1^0 + \dots + c_{1n}k_{n-m}^0 + d_1 + x_1^0 = y_1^0$$
$$\vdots$$
$$x_n = c_{n1}k_1^0 + \dots + c_{nn-m}k_{n-m}^0 + d_1 + x_n^0 = y_n^0 \quad ;$$

We write $y^0 = \begin{pmatrix} y_1^0 \\ \vdots \end{pmatrix}$

$\dot{y}_n^0$

We demonstrate that those $k_j^0 \in Z$, $j = 1,n-m$ are those for which $A(CX^0 + d) = 0$ (there are such $X_j^0 \in Z$ because

$$x_1 = 0$$
$$\vdots$$
$$x_n = 0$$

is a particular solution of the homogenous linear system and $X = CK + d$ is a general solution of the nonhomogeneous linear system) $A(CK^0 + d + X^0 - Y^0) = A(CK^0 + d) + AX^0 - AY^0 = 0 + b - b = 0$.

<u>Propriety 2</u>. The general solution of a homogenous linear system can be written under the form:

$$\text{(SG)}\ (2)\quad \begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m}, \end{cases}$$

$k_j$ = a parameter belonging to Z (with $d_1 = d_2 = \ldots = d_n = 0$)

<u>Proof</u>

(SG) = general solution. It results that (SG) is (n-m) times undetermined.

Let (SG) of the form be

$$\begin{aligned} x_1 &= c_{11}p_1 + \ldots + c_{1n-m}p_{n-m} + d_1 \\ &\vdots \\ x_n &= c_{n1}p_1 + \ldots + c_{nn-m}p_{n-m} + d_n \end{aligned}$$

with not all $d_i = 0$; we demonstrate that it can be written under the form (2); the system admits the trivial solution

$$\begin{aligned} x_1 &= 0 \in Z \\ &\vdots \qquad ; \\ x_n &= 0 \in Z \end{aligned}$$

it results that there are $p_j \in Z$, $j = 1,n-m$

$$(4) \qquad \begin{aligned} x_1 &= c_{11}\overset{0}{p_1} + \ldots + c_{1n-m}\overset{0}{p}_{n-m} + d_1 = 0 \\ &\vdots \\ x_n &= c_{n1}\overset{0}{p_1} + \ldots + c_{nn-m}\overset{0}{p}_{n-m} + d_n = 0 \end{aligned}$$

Substituting $p_j = k_j + \overset{0}{p_j}$, $j = 1,n-m$ in (3)

$$\left.\begin{array}{l} k_j \in Z \\ \overset{0}{p}_j \in Z \end{array}\right\} \Rightarrow p_j \in Z$$

$$\left.\begin{array}{l} p_j \in Z \\ \overset{0}{p}_j \in Z \end{array}\right\} \Rightarrow k_j = p_j - \overset{0}{p}_j \in Z$$

which means that they do not make any restrictions.

It results that

$$\begin{array}{l} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + (c_{11}\overset{0}{p}_1 + \ldots + c_{1n-m}\overset{0}{p}_{n-m} + d_1) \\ . \\ . \\ . \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + (c_{n1}\overset{0}{p}_1 + \ldots + c_{nn-m}\overset{0}{p}_{n-m} + d_n) \end{array}$$

But $c_{h1}\overset{0}{p}_1 + \ldots + c_{hn-m}\overset{0}{p}_{n-m} + d_h = 0$, $h = \overline{1,n}$ (from (4)).

Then the general solution is of the form:

$$\begin{array}{l} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ . \\ . \\ . \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} \end{array}$$

$k_j$ = parameters $\in Z$, $j = \overline{1,n-m}$; it results that $d_1 = d_2 =$

$= \ldots = d_n = 0$

<u>Theorem 2</u>. Let the homogenous linear system be:

$$a_{11}x_1 + \ldots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n = 0,\ r(A) = m;$$

$(a_{h1}, \ldots, a_{hn}) = 1$, $h = 1,m$ and the general solution

$$x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m}$$
$$\vdots$$
$$x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m}$$

then $(a_{h1}, \ldots, a_{hi-1}, a_{hi+1}, \ldots, a_{hn})\ (c_{i1}, \ldots, c_{in-m})$

irrespective of $h = 1,m$ and $i = 1,n$.

<u>Proof</u>

Let some arbitrary be $h \in 1,m$ and some arbitrary $i \in 1,n$; $a_{h1}x_1 + \ldots + a_{hi-1}x_{i-1} + a_{hi+1}x_{i+1} + \ldots + a_{hn}x_n = a_{hi}x_i$. Because $(a_{h1}, \ldots, a_{hi-1}, a_{hi+1}, \ldots, a_{hn})\ a_{hi}$ it results that $d = (a_{h1}, \ldots, a_{hi-1}, a_{hi+1}, \ldots, a_{hn})\ x_i$, irrespective of the value of $x_i$ in the vector of particular solutions; for $k_2 = k_3 = \ldots = k_{n-m} = 0$ and $k_1 = 1$ we get the particular solution:

$$x_1 = c_{11}$$
$$\vdots$$
$$x_i = c_{i1} \Rightarrow d\ c_{i1} \text{ and so on}$$
$$\vdots$$
$$x_n = c_{n1}$$

for $x_1 = k_2 = \ldots = k_{n-m-1} = 0$ and $k_{n-m} = 1$ the following particular solution results:

$$x_1 = c_{1n-m}$$
$$\vdots$$
$$x_i = c_{in-m}$$
$$\vdots$$
$$x_n = c_{nn-m}$$

it results that $d\ c_{in-m}$; hence, $d\ c_{ij}$, $j = 1,n-m \Rightarrow$
$\Rightarrow d\ (c_{i1}, \ldots, c_{in-m})$.

Theorem 3.

$$x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \qquad c_{ij} \in Z \text{ being given}$$

If $\quad \vdots$

.
$x_n = c_{n1}k_1 + ... + c_{nn-m}k_{n-m}$, $k_j$ = parameters $\in Z$

is the general solution of the homogenous linear system

$a_{11}x_1 + ... + a_{1n}x_n = 0$   $m < n$;
.
.
.
$a_{m1}x_1 + ... + a_{mn}x_n = 0$   $r(A) = m$;

then $(c_{1j}, ..., c_{nj}) = 1$, $\forall j = \overline{1,n-m}$.

Proof

We assume, by reductio ad apsurdum, that there is

$j_0 \in \overline{1,n-m}$: $(c_{1j_0}, ..., c_{nj_0}) = d \neq 1$, we consider the maximal co-divisor > 0; we reduce the case when the maximal co-divisor is -d to the case when it is equal to d (nonrestrictive hypothesis); then the general solution can be written under the form:

$$(5) \qquad \begin{aligned} x_1 &= c_{11}k_1 + ... + c'_{1j_0}\, dk_{j_0} + ... + c_{1n-m}k_{n-m} \\ &\;\;\vdots \\ x_n &= c_{n1}k_1 + ... + c'_{nj_0}\, dk_{j_0} + ... + c_{nn-m}k_{n-m} \end{aligned}$$

' '

where $d = (c_{ij_0}, \ldots, c_{nj_0})$, $c_{ij_0} = d \bullet c_{ij_0}$ and $(c_{ij_0}, \ldots,$ $\ldots, c'_{nj_0}) = 1$.

We prove that

$$
\begin{array}{l}
x_1 = c'_{1j_0} \\
\vdots \\
x_n = c'_{nj_0}
\end{array}
$$

is a particular solution of the homogenous linear system. We write

$$
C = \begin{array}{ccccc}
c_{11} & \ldots & c'_{ij_0}\,d & \ldots & c_{1n-m} \\
\vdots & & \vdots & & \vdots \\
c_{n1} & \ldots & c_{nj_0}\,d & \ldots & c_{nn-m}
\end{array}
\qquad
k = \begin{array}{c}
k_1 \\ \vdots \\ k_{j_0} \\ \vdots \\ k_{n-m}
\end{array}
\;;
$$

$x = c \bullet k$ the general solution;

$$
\text{We know } AX = 0 \Rightarrow A(CK) = 0,\ A = \begin{array}{ccc}
a_{11} & \ldots & a_{1n} \\
\vdots & & \\
a_{n1} & \ldots & a_{mn}
\end{array}
\;;
$$

We assume that the principal variants are $x_1, \ldots, x_m$ (if not we have to renumber). It follows that $x_{m+1}, \ldots x_n$ is the secondary variant.

For $k_1 = \ldots = k_{j_0-1} = k_{j_0+1} = \ldots = k_{n-m} = 0$ and $k_{j_0} = 1$ we get a particular solution of the system

$$\begin{array}{l} x_1 = c'_{1j_0} d \\ \vdots \\ x_n = c'_{nj_0} d \end{array} \Rightarrow 0 = A \begin{pmatrix} c'_{1j_0} d \\ \vdots \\ c'_{nj_0} d \end{pmatrix} = d \bullet A \begin{pmatrix} c'_{1j_0} \\ \vdots \\ c'_{nj_0} \end{pmatrix} \Rightarrow$$

$$\Rightarrow A \begin{pmatrix} c'_{1j_0} \\ \vdots \\ c'_{nj_0} \end{pmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = c'_{1j_0} \\ \vdots \\ x_n = c'_{nj_0} \end{array}$$

is the particular solution of the system.

We demonstrate that this particular solution cannot be obtained by

$$(6) \quad \begin{array}{l} x_1 = c_{11}k_1 + \ldots c'_{1j_0} dk_{j_0} + \ldots + c_{1n-m}k_{n-m} = c'_{1j_0} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c'_{nj_0} dk_{j_0} + \ldots + c_{nn-m}k_{n-m} = c'_{nj_0} \end{array}$$

$$(7)\quad \begin{cases} x_{m+1} = c_{m+1}k_1 + \dots + c'_{m+1}dk_{j_0} + \dots + c_{m+1,n-m}k_{n-m} = c'_{m+1j_0} \\ \vdots \\ x_n = c_{n1}k_1 + \dots + c'_{nj_0}dk_{j_0} + \dots + c_{nn-m}k_{n-m} = c'_{nj_0} \end{cases}$$

$$\Rightarrow\quad k_{j_0} = \frac{\begin{vmatrix} c_{m+1,1} & \dots & c_{m+1j_0} & \dots & c_{m+1n-m} \\ \vdots & & \vdots & & \vdots \\ c_{h1} & \dots & c_{nj} & \dots & c_{n,n-m} \end{vmatrix}}{\begin{vmatrix} c_{m+1,1} & \dots & c'_{m+1,j_0}d & \dots & c_{m+1,n-m} \\ \vdots & & & & \\ c_{n1} & \dots & c'_{nj}d & \dots & c_{n,n-m} \end{vmatrix}} = \frac{1}{d} \notin Z \text{ (because } d \neq 1).$$

It is important to point out the fact that those $k_j = k_j^0$, $j = 1,n-m$ that satisfy system (7) also satisfy system (6), because, otherwise (6) would not satisfy the definition of the solution of a linear system of equations (i.e., considering system (7) the hypothesis was not restrictive). From $X_{j_0} \in Z$ it follows that (6) is not the general solution of the homogenous linear system contrary to the hypothesis); then $(c_{1j} \dots c_{nj}) = 1$, irrespective of $j = 1,n-m$.

<u>Propriety 3</u>. Let the linear system be

$$a_{11}x_1 + \dots a_{1n}x_n = b_1$$

.
.
.
$a_{m1}x_1 + \ldots + a_{mn}x_n = b_m$

$a_{ij}, b_i \in Z$, $r(A) = m < n$, $x_j$ = unknowns $\in Z$

Solved in R, we get

$$\begin{cases} x_1 = f_1(x_{m+1}, \ldots, x_n) \\ \vdots \\ x_m = f_m(x_{m+1}, \ldots, x_n) \end{cases}$$ ; $x_1, \ldots, x_m$ are the

main variants where $f_i$ are linear functions of the form:

$$f_i = \frac{c^i_{m+1}x_{m+1}+\ldots+c^i_n x_n+e_i}{d_i}$$ where $c^i_{m+j}, d_i, e_i \in Z$; $i = \overline{1,m}$;

$j = \overline{1,n-m}$.

If $\frac{e_i}{d_i} \in Z$ irrespective of $i = \overline{1,m}$ then the linear system admits integer solution.

<u>Proof</u>

For $1 \le i \le m$, $x_i \in Z$, then $f_j \in Z$. Let:

$$\text{Solution}\quad \begin{cases} x_{m+1} = u_{m+1}k_{m+1} \\ \vdots \\ x_n = u_n k_n \\ \vdots \\ x_1 = v^1_{m+1}k_{m+1} + \ldots + v^1_n k_n + \dfrac{e_1}{d_1} \\ \vdots \\ x_m = v^m_{m+1}k_{m+1} + \ldots + v^m_n k_n + \dfrac{e_m}{d_m} \end{cases}$$

where $u_{m+1}$ is the maximal co-divisor of the denominators of the fractions $\dfrac{c^i_{m+j}}{d_i}$, $i = \overline{1,m}$, $j = \overline{1,n-m}$ calculated after their simplification when they were irreducible.

$v^i_{m+j} = \dfrac{c^i_{m+j}u_{m+j}}{d_i} \in Z$; this is a solution n-m times

undetermined (depends on n-m independent parameters:

$k_{m+n}, \ldots, k_n$) but is not a general solution.

<u>Propriety 4</u>. Under the conditions of propriety 3, if

there is an $i_0 \in \overline{1,m}$: $f_{i_0} = u^{i_0}_{m+1}x_{m+1} + \ldots + u^{i_0}_n x_n + \dfrac{e_{i_0}}{d_{i_0}}$

with $u^{i_0}_{m+j} \in Z$, $j = \overline{1,n-m}$ and $\frac{e_{i_0}}{d_{i_0}} \notin Z$ then the system does not admit integer solution.

Proof

$\forall x_{m+1}, \ldots, x_n$ in Z it results in $x_{i_0} \notin Z$.

Theorem 4. Let the linear system be

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$a_{ij}, b_i \in Z$, $x_j$ = unknowns $\in Z$, $r(A) = m < n$. If there are indices $1 \leq i_1 < \ldots < i_m \leq n$, $i_h \in \{1, 2, \ldots, n\}$, $h = \overline{1,m}$ with the propriety:

$$\Delta = \begin{vmatrix} a_{1i_1} & \ldots & a_{1i_m} \\ \vdots & & \vdots \\ a_{mi_1} & \ldots & a_{mi_m} \end{vmatrix} \quad \text{and}$$

$$\Delta x_{i_1} = \begin{vmatrix} b_1 & a_{1i_2} & \dots & a_{1i_m} \\ \vdots & \vdots & & \vdots \\ b_m & a_{mi_2} & \dots & a_{mi_m} \end{vmatrix} \text{ is divided by}$$

$$\vdots$$

$$\Delta x_{i_m} = \begin{vmatrix} a_{1i_1} & \dots & a_{1i_{m-1}} & b_1 \\ \vdots & & \vdots & \\ a_{mi_1} & \dots & a_{mi_{m-1}} & b_m \end{vmatrix} \text{ is divided by}$$

then the system admits integer number solutions.

<u>Proof</u>

We use propriety 3

$$d_i = \Delta,\ i = \overline{1,m};\ e_{i_h} = \Delta x_{i_h},\quad h = \overline{1,m}$$

<u>Note 1</u>. Conversely, it is not true.

<u>Consequence 1</u>. Any homogenous linear system admits integer number solutions (beside the trivial one); $r(A) = m < n$.

<u>Proof</u>

$$\Delta x_{i_h} = 0 \vdots \Delta, \text{ irrespective of } h = \overline{1,m}.$$

$i_h$

Consequence 2. If $\Delta = \pm 1$, it follows that the linear system admits integer number solutions.

Proof

$\Delta x_{i_h} \vdots (\pm 1)$, irrespective of $h = 1,m$;

$\Delta x_{i_h} \in Z$.

# FIVE INTEGER NUMBER ALGORITHMS TO SOLVE LINEAR SYSTEMS

This chapter further extends the results obtained in 4 and 5 (from linear equations to linear systems). Each algorithm is strictly demonstrated and then an example is given.

Five integer number algorithms to solve linear systems are further given.

Algorithm 1 (method of substitution)

(Although simple, this algorithm requires complex calculus but is, nevertheless, advantageous in introducing it in the computer).

Some integer number equations are initially solved (which is usually simpler) by means of one of the algorithms 4 or 5. (If there is an equation of the system which does not admit integer systems, then the integer system does not admit integer systems. Stop.) the general integer solution of the equation will depend on n-1 integer number parameters (see 5):

$(p_1)\ x_{i_1} = f_{i_1}^{(1)}\ (k_1^{(1)}, \ldots, k_{n-1}^{(1)}),\ i\ = 1,n$ where all the functions $f_{i_1}^{(1)}$ are linear with integer number coefficients.

This general integer number system $(p_1)$ is introduced in the other m-1 equations of the system. We get a new system of m-1 equations with n-1 unknown variables: $k_{i_1}^{(1)}$, $i_i = 1,n-1$, which is also to be solved in integer numbers. The procedure is similar. Solving a new equation, we obtain its general integer solution:

$$(p_2)\ k_{i_2}^{(1)} = f_{i_2}^{(2)} (k_1^{(2)}, \ldots, k_{n-2}^{(2)}),\ i_2 = 1,n-1$$

where all the functions $f_{i_2}^{(2)}$ are linear, with integer number coefficients. (If, along this algorithm we come across an equation which does not admit integer solutions, then, the initial system does not admit integer solution. Stop.)

In the case that all the solved equations admitted integer systems at step (j), $1 \leq j \leq r$, (r being of the same rank as the matrix associated to the system) then:

(j-1) (j) (j) (j)

$$(p_j)\ k_{i_j} = f_{i_j}\ (k_1, \ldots, k_{n-j}),\ i_j = \overline{1,n-j+1},$$

$f_{i_j}^{(j)}$ are linear functions with integer number coefficients.

Finally, after r steps, and if all the solved equations admitted integer solutions, we get to the integer solution of one equation with n-r+1 unknown variables.

The system will accept integer solutions if and only in this last equation will have integer solutions. If it does, let the general integer solution of it be:

$$(p_r)\ k_{i_r}^{(r-1)} = f_{i_r}^{(r)}\ (k_1^{(r)}, \ldots, k_{n-1}^{(r)}),\ i_r = \overline{1,n-r+1},$$

where all $f_{i_r}^{(r)}$ are linear functions with integer number coefficients.

Now the reverse way follows.

We introduce the values of $k_{i_r}^{(r-1)}$, $i_r = \overline{1,n-r+1}$

at step $(p_r)$ in the values of $k_{i_{(r-1)}}^{(r-2)}$, $i_{r-1} = \overline{1,n-r+2}$ from step $(p_{r-1})$.

It follows:

$$= \; k_{i_{r-1}}^{(r-2)} = f_{i_{r-1}}^{(r-1)}(f_1^{(r)}(k_1^{(r)},\ldots,k_{n-r}^{(r)},\ldots,f_{n-r+1}^{(r)}(k_1^{(r)},\ldots,k_{n-r}^{(r)}))$$

$$= g_{i_{r-1}}^{(r-1)}(k_1^{(r)},\ldots,k_{n-r}^{(r)}),\ i_{r-1} = 1,n-r-1$$

from which it follows that $g_{i_r}^{(r-1)}$ are linear functions with integer number coefficients.

Then follow those from $(p_{r-2})$ in $(p_{r-e})$, and so on, until we introduce the values obtained at step $(p_2)$ in those from the step $(p_1)$. It will follow:

$$x_{i_1} = g_i^{(1)}(k_1^{(r)},\ \ldots,\ k_{n-r}^{(r)}\ \underline{\underline{\text{notation}}}\ g_{i_1}(k_1,\ \ldots,\ k_{n-r}),$$

$$i = 1,n$$

with all $g_{i_1}$, most obviously, linear functions with integer number coefficients (the notation was made for simplicity and aesthetical aspects). This is, thus, the general integer solution, of the initial system.

<u>The correctness of algorithm 1</u>. The algorithm is finite because it has r steps on the first way and r-1

steps on the reverse. ($r < +\infty$). Obviously, if one equation of one system does not accept (integer number) solutions then the system does not have either.

Writing S for the initial system and $S_j$ the system resulted from step ($p_j$), $1 \leq j \leq r-2$, it follows that passing from ($p_j$) to ($p_{j+1}$) we pass from a system $S_j$ to a system $S_{j+1}$ equivalent from the viewpoint of the integer number solution, i.e., $k_{i_j}^{(j-1)} = t_{i_j}^{0}$ , $i_j = 1,n-j+1$ which is a particular integer solution of the system $S_j$ if and only if $k_{i_{j+1}}^{(j)} = h_{i_{j+1}}^{0}$, $i_{j+1} = 1,n-j$ is a particular integer solution of the system $S_{j+1}$ where $h_{i_{j+1}}^{0} = f_{i_{j+1}}^{(j+1)}(t_1^0,...,t_{n-j+1}^0)$, $i_{j+1} = 1,n-j$. Hence, their general integer solutions are also equivalent (considering these substitutions). So that, in the end, the solving of the initial system S is equivalent with the solving of the equation (of the system consisting of one equation) $S_{r-1}$ with integer number coefficients. It follows that the system S admits integer number solution if and only if all the systems $S_j$ admit integer number solution, $1 \leq j \leq r-1$.

Example 1. By means of algorhythmus 1, let us calculate the integer number solution of the system:

$$(S)\quad \begin{cases} 5x - 7y - 2z + 6w = 6 \\ -4x + 6y - 3z + 11w = 0 \end{cases}$$

Solution: We solve the first integer number equation. We obtain the general integer solution (see [4] or [5]):

$$(p_1)\quad \begin{aligned} x &= t_1 + 2t_2 \\ y &= t_1 \\ z &= -t_1 + 5t_2 + 3t_3 - 3 \\ w &= t_3 , \end{aligned}$$

where $t_1$, $t_2$, $t_3 \in Z$.

Substituting in the second, we get the system:

$$(S_1)\quad 5t_1 - 23t_2 + 2t_3 + 9 = 0$$

Solving this integer equation we obtain its general integer solution:

$$(p_2)\quad \begin{aligned} t_1 &= k_1 \\ t_2 &= k_1 + 2k_2 + 1 \\ t_3 &= 9k_1 + 23k_2 + 7 , \end{aligned}$$

where $k_1$, $k_2 \in Z$.

The reverse way. Substituting ($p_2$) in ($p_1$) we obtain:

$$x = 3k_1 + 4k_2 + 2$$
$$y = k_1$$
$$z = 31k_1 + 79k_2 + 23$$
$$w = 9k_1 + 23k_2 + 7 ,$$

where $k_1, k_2 \in Z$ which is the general integer solution of the initial system (S). Stop.

Algorithm 2

Input

A linear system (1) without all $a_{ij} = 0$.

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. $t = 1, h = 1, p = 1$
2. (A) Divide each equation by the maximal co-divisor of the coefficients of the unknown variables. If you do not get an integer

quotient for at least one equation, then the system does not admit integer solutions. Stop.

(B) If there is an inequality in the system, then the system does not admit integer solutions. Stop.

(C) If the repetition of more equations occurs, keep one and if an equation is an identity, remove it from the system.

3. If there is $(i_0, j_0)$ so that $a_{i_0 j_0} = 1$, then obtain the value of the variable $x_{j_0}$ from the equation $i_0$; relation $(T_t)$. Substitute this relation (where possible) in the other equations of the system and in the relations $(T_{t-1})$, $(H_h)$ and $(P_p)$ for all i, h and p. Consider t: = t + 1, remove equation $(i_0)$ from the system. If there is not such a pair, go on to step 5.

4. Does the system (left) have at least one unknown variable? If it does, consider the new date and go on to step 2. If it does not, write the general integer solution of the system substituting $k_1$; $k_2$, ... for all the variables from the right term of each expression which gives the value of the unknowns of the initial

system. Stop.

5. Calculate $a = \min_{i,j_1,j_2} \{ r , a_{ij_1} \equiv r \pmod{a_{ij_2}},$ $0 < r < a_{ij_2} \}$, and determine the indices $i, j_1, j_2$ as well as the r for which this minimum can be calculated. (If there are more variants, choose one, arbitrarily.)

6. Write: $$x_{j_2} = t_h \frac{a_{ij_1} - r}{a_{ij_2}} x_{ij_1},$$ relation ($H_h$).

   Substitute this relation (where possible) in all the equations of the system and in the relations ($T_t$), ($H_h$) and ($P_p$) for all t, h and p.

7. (A) If $a \neq 1$, consider $x_{j_2} := t_h$, $h := h + 1$ and go on to step 2.

   (B) If $a = 1$, then obtain the value of $x_{j_1}$ from from the equation (i); relation ($P_p$).

   Substitute this relation (where possible) in the other equations of the system and in the relations ($T_t$), ($H_h$) and ($P_{p-1}$) for all t, h and p. Remove the equation (i) from the system. Consider $h := h + 1$, $p := p + 1$ and go back to step 4.

<u>The correctness of algorithm 2</u>. Let the system (1) be.

<u>Lemma 1</u>. We consider the algorithm at step 5. Also,

let $M = \{ r , a_{ij_1} \equiv r \pmod{a_{ij_2}}, 0 < r < a_{ij_2}, i, j_1, j_2 = 1, 2, 3, \ldots\}$. Then $M \neq \emptyset$ .

<u>Proof</u>

Obviously, M is finite and $M \subset N^*$. Then, M has a minimum if and only if $M \neq \emptyset$ . We suppose, conversely, that $M = \emptyset$ . Then $a_{ij_2} \equiv 0 \pmod{a_{ij_2}}$, $\forall$ i, $j_1$, $j_2$. It follows conversely as well that $a_{ij_2} \equiv 0 \pmod{a_{ij_1}}$, $\forall$ i, $j_1$, $j_2$. That is $a_{ij_1} = a_{ij_2}$ , $\forall$ i, $j_1$, $j_2$. We consider a $i_0$ arbitrary but fixed. It is clear that $(a_{i_0 1}, \ldots, \ldots, a_{i_0 n}) \sim a_{i_0 j} \neq 0$, $\forall$ j (because the algorithm has passed through the substeps 2(B) and 2(C)). But, as it has also passed through step 3, it follows that $a_{i_0 j} \neq 1$, $\forall$ j, but as it previously passed through step 2(A), it would result that $a_{i_0 j} = 1$, $\forall$ j. This contradiction shows that the assumption is false.

<u>Lemma 2</u>. Let $a_{i_0 j_1} \equiv r \pmod{a_{ij_2}}$. Substitute $x_{j_2} = $

$$= t_h - \frac{a_{i_0 j_1} - r}{a_{i_0 j_2}} x_{j_1}$$

in the system (A) obtaining the system (B). Then, $x_j = x_j^0$, $j = \overline{1,n}$ is the particular integer solution of (A) if and only if $x_j = x_j^0$, $j \neq j_2$ and $t_h =$

$$= x^0_{j_2} + \frac{a_{i_0 j_1} - r}{a_{i_0 j_2}}$$ is the particular integer solution of (B).

Lemma 3. Let $a_1 \neq 1$ and $a_2$ be obtained at step 5. Then $0 < a_2 < a_1$.

Proof

It is sufficient to show that $a_1 < a_{ij}$, $\forall$ i,j because in order to get $a_2$ step 6 is obligatory, when the coefficients of the new system are calculated, $a_1$ being equal to a coefficient from the new system (equality of modules), the coefficient on $(i_0 j_1)$.

Let $a_{i_0 j_0}$ with the propriety $a_{i_0 j_0} \leq a_1$. Hence, $a_1 \geq \geq a_{i_0 j} = \min \{ a_{i_0 j} \}$. Let $a_{i_0 j_s}$ with $a_{i_0 j_s} > a_{i j_m}$ there is such an element because $a_{i_0 j_m}$ is the minimum of the coefficients in the module and not all $a_{i_0 j}$, $j = \overline{1,n}$ are equal (conversely, it would result that $(a_{i_0 j}, \ldots, a_{i_0 n}) \sim \sim a_{i_0 j}$, $\forall j \in \overline{1,n}$; the algorithm passed through substep 2(A) has simplified each equation by the maximal co-divisor of its coefficients; hence, it would follow that $a_{i_0 j} = 1$, $\forall j = \overline{1,n}$ which, again, cannot be real because the algorithm has also passed through step 3). Of the coefficients $a_{i_0 j_m}$ we choose one with the propriety $a_{i_0 j_{s_0}} \neq \neq M a_{i_0 j_m}$ there is such an element (contrary, it would result $(a_{i_0 1}, \ldots a_{i_0 n}) \sim a_{i_0 j_m}$, but the algorithm has also passed through step 2(A) and it would mean that $a_{i_0 j_m} = 1$ which contradicts step 3 through which the algorithm has also passed).

Considering $q_0 = \left[ a_{i_0 j_{s_0}} / a_{i_0 j_m} \right] \in Z$ and $r = a_{i_0 j_{s_0}} - q_0 a_{i_0 j_m} \in Z$, we have $a_{i_0 j_{s_0}} \equiv r_0 \pmod{a_{i_0 j_m}}$ and $0 <$ $< r_0 < a_{i_0 j_m} < a_{i_0 j_0} \le a_1$. We have, thus, obtained an $r_0$ with $r_0 < a_1$, which is in contradiction with the very definition of $a_1$. Thus, $a_1 < a_{ij}$, V i,j.

<u>Lemma 4</u>. Algorithm 2 is finite.

<u>Proof</u>

The functioning of the algorithm is meant to transform a linear system of m equations and n unknowns into one of $m_1 \times n_1$, with $m_1 < m$, $n_1 \le n$ and, thus, successively into a final linear equation with $n - r + 1$ unknowns (where r is the rank of the associated matrix). This equation is solved by means of the same algorithm (which works as [5]). The general integer solution of the system will depend on the $n - 1$ integer number independent parameters (see [6]--similar proprieties can be established also for the general integer solution of the linear system). The reduction of equations occurs at steps 2, 3 and substep 7(B). Steps 2 and 3 are obvious and, hence, trivial; they can reduce the equations of the system (or even put an end to it) but only under particular conditions. The most important case finds its solution at step 7(B), which always reduces one equation of the system. As the number of equations is finite, we come to solve a single integer number equation. We also have to show that the transfer from one system $m_i$x $n_i$ to another $m_{i+1}$x$n_{i+1}$ is made in a finite interval of time: by steps 5 and 6 permanent substitution of variables are made until we get to a = 1 (we get to a = 1 because, according to lemma 3, all a - s are positive integer numbers and form a strictly decreasing row).

Theorem of correctness. Algorithm 2 correctly calculates the general integer solution of the linear system.

Proof

Algorithm 2 is finite according to lemma 4. Steps 2 and 3 are obvious (see also [4], [5]). Their part is to simplify calculus as much as possible. Step 4 tests the finality of the algorithm; the substitution with the parameters $k_1$, $k_2$, ... has systemization and aesthetic reasons. The variables t, h, p are counter variables (started at step 1) and they are meant to count the relations of the type T, H, P (numbering required by the substitutions at steps 3, 6 and substep 7(B); h also counts the new (auxiliary) variables introduced in the moment of decomposition of the system. The substitution from step 6 does not affect the general integer solution of the system (it follows from lemma 2). Lemma 1 shows that at step 5 there is always a, because $\emptyset \neq M \subset N^*$.

The algorithm performs the transformation of a system $m_i x n_i$ into another, $a_{i+1} x n_{i+1}$, equivalent to it, preserving the general solution (taking into account, however, the substitutions) (see also lemma 2).

Example 2. Calculate the integer solution of:

$$\begin{aligned} -12x - 7y + 9z \phantom{+ 10w} &= 12 \\ -5y + 8z + 10w &= 0 \\ 0z + 0w &= 0 \\ 15x + 21z + 69w &= 3 \end{aligned}$$

Solution

We apply algorithm 2 (we purposely looked for an example to be passed through all the steps of this algorithm:

1. $t = 1$, $h = 1$, $p = 1$
2. (A) The fourth equation becomes: $5x + 7z +$

$23w = 1$

(B) --

(C) Equation 3 is removed.

3. No; go on to step 5.

5. $a = 2$ and $i = 1$, $j_1 = 2$, $j_2 = 3$ and $r = 2$.

6. $z = t_1 + y$, the relation ($H_1$). Substituting it in the system:

$$\begin{aligned} -12x + 2y + 9t_1 \qquad &= 12 \\ 3y + 8t_1 + 10w &= 0 \\ 5x + 7y + 7t_1 + 23w &= 1 \end{aligned}$$

7. $a \neq 1$; consider $z := t_1$, $h := 2$ and go back to step 2.

2. --

3. No. Step 5.

5. $a = 1$ and $i = 2$, $j_1 = 4$, $j_2 = 2$ and $r = 1$.

6. $y = t_2 - 3w$, the relation ($H_2$). Substituting in the system:

$$\begin{aligned} -12x + 2t_2 + 9t_1 - 6w &= 12 \\ 3t_2 + 8t_1 + w &= 0 \\ 5x + 7t_2 + 7t_1 + 2w &= 1 \end{aligned}$$

Substituting it in relation to ($H_1$), we get:

$z = t_1 + t_2 - 3w$, relation $(H_1)'$.

7. $w = -3t_2 - 8t_1$, relation $(P_1)$.

Substituting it in the system, we get:

$$- 12x + 20t_2 + 57t_1 = 12$$
$$5x + t_2 - 9t_1 = 1$$

Substituting it in the other relations, we get:

$z = 10t_2 + 25t_1$ , $(H_1)''$;

$y = 10t_2 + 24t_1$ , $(H_2)''$;

h: = 3, p: = 2 and go back to step 4.

4. Yes

2. --

3. $t_2 = 1 - 5x + 9t_1$, relation $(T_1)$.

Substituting it (where possible) we get:

$\{- 112x + 237t_1 = -8$ (the new system);

$z = 10 - 50x + 115t_1$, $(H_1)'''$

$y = 10 - 50x + 114t_1$, $(H_2)''$

$w = -3 + 15x - 35t_1$, $(P_1)'$

Consider t: = 2; go on to step 4.

4. Yes. Go back to step 2. (From now on algorithm 2 works similarly with that from [5], with the

only difference that the substitutions must also be made in the relations obtained up to this point).

2. --

3. No. Go on to step 5.

5. $a = 13$ (one three) and $i = 1$, $j_1 = 2$, $j_2 = 1$ and $r = 13$.

6. $x = t_3 + 2t_1$, relation ($H_3$).

   After substitution we get:

$$-112t_3 + 13t_1 = -8 \qquad \text{(the system)}$$

$$z = 10 - 50t_3 + 15t_1 \ , \quad (H_1)^{IV} \ ;$$

$$y = 10 - 50t_3 + 14t_1 \ , \quad (H_2)'''$$

$$w = -3 + 15t_3 - 5t_1 \ , \quad (P_1)''$$

$$t_2 = 1 - 5t_3 - t_1 \ , \quad (T_1)'$$

7. $x: = t_3$, $h: = 4$ and go on to step 2.

2. --

3. No, go on to step 5.

5. $a = 5$ and $i = 1$, $j_1 = 1$, $j_2 = 2$ and $r = 5$

6. $t_1 = t_4 + 9t_3$, relation ($H_4$).

   Substituting it, we get: $5t_3 + 13t_4 = -8$ (the system).

$$z = 10 + 85t_3 + 15t_4 \ , \quad (H_1)^{V} \ ;$$

$$y = 10 + 76t_3 + 14t_4 \ , \quad (H_2)^{IV} \ ;$$

$$x = 19t_3 + 2t_4 \;, \quad (H_3)' \;;$$
$$w = -3 - 30t_3 - 5t_4 \;, \quad (P_1)''' \;;$$
$$t_2 = 1 - 14t_3 - t_4 \;, \quad (T_1)'' \;;$$

7. $t_1 := t_4$, $h := 5$ and go back to step 2.

2. --

3. No; step 5.

5. $a = 2$ and $i = 1$, $j_1 = 2$, $j_2 = 1$ and $r \equiv -2$.

6. $t_3 = t_5 - 3t_4$, relation $(H_5)$. After substitution, we get:

$$5t_5 - 2t_4 = -8 \qquad \text{(the system)}$$
$$z = 10 + 85t_5 - 240t_4 \;, \quad (H_1)^{VI} \;;$$
$$y = 10 + 76t_5 - 214t_4 \;, \quad (H_2)^{V} \;;$$
$$x = 19t_5 - 55t_4 \;, \quad (H_3)^{IV} \;;$$
$$w = -3 - 30t_5 + 85t_4 \;, \quad (P_1)^{IV} \;;$$
$$t_2 = -1 - 14t_5 + 41t_4 \;, \quad (T_1)''' \;;$$
$$t_1 = 9t_5 + 26t_4 \;, \quad (H_4)' \;;$$

7. $t_3 := t_6$, $h := 6$ and go back to step 2.

2. --

3. No; step 5.

5. $a = 1$ and $i = 1$, $j_1 = 1$, $j_2$, $r = 1$.

6. $t_4 = t_6 + 2t_5$, relation $(H_6)$. After substitution, we get:

$$
\begin{array}{llr}
t_5 - 2t_6 = -8 & & \text{(the system)} \\
z = 10 - 395t_5 - 240t_6 , & & (H_1)^{VII} ; \\
y = 10 - 392t_5 - 214t_6 , & & (H_2)^{VI} ; \\
x = \quad - 91t_5 - 55t_6 , & & (H_3)''' ; \\
w = -3 + 140t_5 + 85t_6 , & & (P_1)^{V} ; \\
t_2 = 1 + 68t_5 + 41t_6 , & & (T_1)^{IV} ; \\
t_1 = \quad - 43t_5 - 26t_6 , & & (H_4)'' ; \\
t_3 = \quad - 5t_5 - 3t_6 , & & (H_5)' ;
\end{array}
$$

7. $t_5 = 2t_6 - 8$, relation $(P_2)$. Substituting it in the system, we get: $0 = 0$.

Substituting it in the other relations, it follows:

$$
\begin{array}{ll}
z & = -1030t_6 + 3170 \\
y & = -918t_6 + 2826 \\
x & = -237t_6 + 728 \\
w & = 365t_6 - 1123 \\
t_2 & = 177t_6 - 543 \\
t_1 & = 112t_6 + 344 \\
t_3 & = 13t_6 + 40 \\
t_4 & = 5t_6 - 16
\end{array}
$$

relations of no importance

Consider $h := 7$, $p := 3$ and go back to step 4.

$t_6 \in Z$

4. No. The general integer solution of the system is:

$x = -237k_1 + 728$

$y = -918k_1 + 2826$

$z = 1030k_1 + 3170$

$w = 365k_1 - 1123$

where $k_1$ is an integer number parameter.

Stop.

Algorithm 3

Input

A linear system (1).

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. Solve the system in $R^n$. If it does not have solutions in $R^n$, it does not have solutions in $Z^n$ either. Stop.
2. $f = 1$, $t = 1$, $h = 1$, $g = 1$
3. Write the value of each main variable $x_i$ under

the form:

$$(E_{f,i})_i: \quad x_i = \sum_j q_{ij} x_j' + q_i + (\sum_j r_{ij} x_j' + r_i)/\Delta_i,$$

with all $q_{ij}$, $q_i$, $r_{ij}$, $r_i$, $\Delta_i$ in Z so that all $|r_{ij}| < |\Delta_i|$, $\Delta_i \neq 0$, $|r_i| < |\Delta_i|$ (where all $x_j'$ of the right term are integer number variables: either of the secondary variables of the system or other new variables introduced with the algorhythmus). For all i, we write $r_{ij_f} \equiv \Delta_i$.

4. $(F_{f,i})_i$: $\sum_j r_{ij} x_j' - r_{i,j_f} y_{f,i} + r_i = 0$ where $(Y_{f,i})_i$ are auxiliary integer number variables. We remove all the equations $(F_{f,i})$ which are identities.

5. Does at least one equation $(F_{f,i})$ exist? If it does not, write the general integer solution of the system substituting $k_i$, $k_2$, ... for all the variables from the right term of each expression representing the value of the initial unknowns of the system. Stop.

6. (A) Divide each equation $(F_{f,i})$ by the maximal co-divisor of the coefficients of their unknowns. If the quotient is not an

integer number for at least one $i_0$ then the system does not admit integer solutions. Stop.

(B) Simplify--as in m--all the fractions from the relations $(E_{f,i})_i$.

7. Does it exist $r_{i_0 j_0}$ with the module 1?
If it does not, go on to step 8.
If it does, find the value of $x'_j$ from the equation $(F_{f,i_0})$; write $(T_t)$ for this relation and substitute it (where it is possible) in the relations $(E_{f,i})$, $({}_T t\text{-}1)$, $(H_h)$, $(G_g)$ for all i, t, h and g. Remove the equation $(F_{f,i_0})$. Consider f: = f + 1, t: = t + 1 and go back to step 3.

8. Calculate $a = \min_{i,j_1,j_2} \{ r,\ r_{ij_1} \equiv r(\text{mod } r_{ij_2}),\ 0 < r < r_{ij_2} \}$ and determine the indices $i_m$, $j_1$, $j_2$ as well as the r for which this minimum can be obtained. (When there are more variants, choose only one).

9. (A) Write $x'_{j_2} = z_h - \dfrac{a_{i_m j_1} - r}{a_{j_m j_2}} x'_{j_1}$, where $z_h$ is a new integer variable; relation $(H_h)$.

(B) Substitute the letter (where possible) in the relations $(E_{f,i})$, $(F_{f,i})$, $(T_t)$, $(H_{h-1})$, $(G_g)$ for all i, t, h and g.

(C) Consider h: = h + 1.

10. (A) If $a \neq 1$, go back to step 4.

(B) If a = 1, calculate the value of the variable $x'_j$ from the equation $(F_{f,i})$; relation $(G_g^1)$. Substitute it (where possible) in the relations $(E_{f,i})$, $(T_t)$, $(H_h)$, $(G_{g-1})$ for all i, t, h and g. Remove the equation $(F_{f,i})$. Consider g: = g + 1, f: = f + 1 and go back to step 3.

The correctness of algorithm 3

Lemma 5. Let i be fixed. Then $\left(\sum_{j=n_1}^{n_2} r_{ij}x'_j + r_i\right)/\Delta_i$ (with all $r_{ij}$, $r_i$, $\Delta_i$, $n_1$, $n_2$ being integers, $n_1 \leq n_2$, $\Delta_i \neq 0$ and all $x'_j$ being integer variables) can have integer values if and only if $(r_{in_1}, \dots, r_{in_2}, \Delta_i)/r_i$.

Proof

The fraction from the enunciation can have integer values if and only if there is a $z \in Z$ so that $\left(\sum_{j=n_1}^{n_2} r_{ij}x'_j +\right.$

$+ r_i)/\Delta_i = z \Leftrightarrow \sum_{j=n_1}^{n_2} r_{ij}x_j' - \Delta_i z + r_i = 0$ which is a linear equation. This equation admits integer solution $\Leftrightarrow (r_{in_1}, \ldots, r_{in_2}, \Delta_i) | r_i$.

Lemma 6. The algorithm is finite. It is true. The algorithm can stop at steps 1, 5 or substep 6(A). (It rarely happens to stop at step 1). An equation after another are gradually eliminated at step 4 and especially 7 and 10(B) ($F_{f,i}$)--the number of equations is finite. If at steps 4 and 7 the elimination of equation may occur only in special cases, elimination of equations at the substep 10(B) is always true because, through steps 8 and 9 we get to a = 1 (see [5]) or even lemma 4 (from the correctness of algorithm 2). Hence, the algorithm is finite.

Theorem of Correctness. The algorithm 3 correctly calculates the general integer solution of the system (1).

Proof

The algorithm is finite according to lemma 6. It is obvious that if the system does not have solutions in $R^n$ it does not have in $Z^n$ either, because $Z^n \subset R^n$ (step 1). The variables f, t, h, g are counter variables and are meant to number the relations of the type E, F, t, H and G, respectively. They are used to distinguish between the relations and make the necessary substitutions (step 2). Step 3 is obvious. All the coefficients of the unknowns being integers, each main variable $x_i$ will be written:

$$x_i = (\sum_j c_{ij}x_j' + c_i)/\Delta_i ,$$

which can assume the form and conditions required in this step. Step 4 is obtained from 3 by writing each fraction equal to an integer variable $y_{f,i}$ (this being $x_i \in Z$). Step 5 is very close to the end. If there is no fraction among all ($E_{f,i}$) it means that all the main variables $x_i$ already have values in Z, while the secondary variables of the system can be arbitrary in Z, or can be obtained from the relations T, H or G (but these have only integer expressions because of their definition and because only integer substitutions are made). The second assertion of this step is meant to systematize the parameters and renumber; it could be left out but aesthetic reasons dictate its presence. According to lemma 5 the step 6(A) is correct. (If a fraction depending on certain parameters (integer variables) cannot have values in Z, then the main variable which has in the value of its expression such a friction cannot have values in Z either; hence, the system does not admit integer systems). This 6(A) also has a simplifying role. The correctness of step 7, trivial as it is, also results from [4], and the steps 8-10 from [5] or even from algorhythmus 2 (lemma 4).

The initial system is equivalent to the "system" from step 3 (in fact, $(E_{f,i})$ as well, can be considered a system). So, the general integer solution is preserved (the changes of variables do not prejudice it (see [4], [5], and also lemma 2 from the correctness of algorithm 2)). From a system $m_i x n_i$ we form an equivalent system $m_{i+1} x n_{i+1}$ with $m_{i+1} < m_i$ and $n_{i+1} < n_i$. This algorithm works similarly to algorithm 2.

Example 3. Employing algorithm 3, find an integer solution of the following system:

$$\begin{aligned} 3x_1 + 4x_2 \quad & + 22x_4 - 8x_5 = 25 \\ 6x_1 + \quad & + 46x_4 - 12x_5 = 2 \\ 4x_2 + 3x_3 & - x_4 + 9x_5 = 26 \end{aligned}$$

Solution

1. Common solving in $R^3$, it follows:

$23x_4-6x_5-1$

$x_1 =$
$- \quad -3$

$$x_2 = \frac{x_4+2x_5+24}{4}$$

$$x_3 = \frac{11x_5+2}{3}$$

2. $f = 1$, $t = 1$, $h = 1$, $g = 1$

3.

$$x_1 = -7x_4+2x_5 + \frac{2x_4-1}{-3} \qquad (E_{1,1})$$

$$x_2 = \quad 6 + \frac{x_4+3x_5}{4} \qquad (E_{1,2})$$

$$x_3 \quad -4x_5 + \frac{x_5+2}{3} \qquad (E_{1,3})$$

4. $2x_4 \quad + \quad 3y_{11} \quad - 1 = 0 \qquad (F_{1,1})$

$x_4 + 2x_5 \quad - 4y_{12} \quad = 0 \qquad (F_{1,2})$

$x_5 \quad - 3y_{13} \quad + 2 = 0 \qquad (F_{1,3})$

5. Yes.

6. --

7. Yes: $r_{35} = 1$. Then $x_5 = 3y_{13} - 2$, the relation

$(T_1)$. Substituting it in the others, we get:

$$x_1 = -7x_4 + 6y_{13} - 4 + \frac{2x_4-1}{-3} \qquad (E_{1,1})$$

$$x_2 = \quad 6 + \frac{x_4+6y_{13}-4}{4} \qquad (E_{1,2})$$

$$x_3 = \quad -12y_{13}+8 + \frac{3y_{13}-2+2}{3} \qquad (E_{1,3})$$

Remove the equation $(F_{1,3})$.

Consider f: = 2, t: = 2; go back to step 3.

3. $$x_1 = -7x_4 + 6y_{13} - 4 + \frac{2x_4-1}{-3} \qquad (E_{2,1})$$

$$x_2 = \quad y_{13}+5 + \frac{x_4+2y_{13}}{4} \qquad (E_{2,2})$$

$$x_3 = \quad -11y_{13}+8 \qquad (E_{2,3})$$

4. $2x_4 + 3y_{21} - 1 = 0$ $(F_{2,1})$

   $x_4 + 2y_{13} - 4y_{22} = 0$ $(F_{2,2})$

5. Yes.

6. --

7. Yes. $r_{24} = 1$. We obtain $x_4 = -2y_{13} + 4y_{22}$, relation $(T_2)$. Substituting it in the others we get:

$$x_1 = -28y_{22} + 20y_{13}-4 + \frac{-4y_{13}+8y_{22}-1}{-3} \qquad (E_{2,1})'$$

$$x_2 = \quad y_{22} + \quad y_{13}+5 \qquad (E_{2,2})'$$

$$x_3 = \quad - 11y_{13}+8 \qquad (E_{2,3})'$$

Remove the equation $(F_{22})$.

Consider f: = 3, t: = 3 and go back to step 3.

3.

$$x_1 = -22y_{13} - 30y_{22}-4 + \frac{-2y_{13}+2y_{22}-1}{-3} \qquad (E_{3,1})$$

$$x_2 = \quad y_{13} + \quad y_{22}+5 \qquad (E_{3,2})$$

$$x_3 = -11y_{13} \quad +8 \qquad (E_{3,3})$$

4. $2y_{13} + 2y_{22} + 3y_{31} - 1 = 0$ $(F_{3,1})$

5. Yes.

6. --

7. No.

8. $a = 1$, and $i_m = 1$, $j_1 = 31$, $j_2 = 22$ and $r = 1$.

9. (A) $y_{22} = z_1 - y_{31}$, relation $(H_1)$.

(B) Substituting it in the others we get:

$$x_1 = -22y_{13} - 30z_1 + 30y_{31} - 4 + \frac{-2y_{13} + 2z_1 - 2y_{31} - 1}{-3} \qquad (E_{3,1})'$$

$$x_2 = y_{13} + z_1 - y_{31} + 5 \qquad (E_{3,2})'$$

$$x_3 = -11y_{13} + 8 \qquad (E_{3,3})'$$

$$2y_{13} + 2z_1 + y_{31} - 1 = 0 \qquad (F_{3,1})'$$

$$x_4 = -2y_{13} + 4z_1 - 4y_{13} \qquad (T_2)'$$

(C) Consider h: = 2

10. (B) $y_{31} = 1 - 2y_{13} - 2z_1$, relation $(G_1)$.

Substituting it in the others we get:

$$x_1 = -40y_{13} - 92z_1 + 27 \qquad (E_{3,1})''$$

$$x_2 = 3y_{13} + 3z_1 + 4 \qquad (E_{3,2})''$$

$$x_3 = -11y_{13} + 8 \qquad (E_{3,3})''$$

$$x_4 = 6y_{13} + 12z_1 - 4 \qquad (T_2)''$$

$$y_{22} = 2y_{13} + 3z_1 - 1 \qquad (H_1)'$$

Remove the equation $(F_{3,1})$.

Consider g: = 2, f: = 4 and go back to step 3.

3. $x_1 = -40y_{13} - 92z_1 + 27$ $(E_{4,1})$

$x_2 = 3y_{13} + 3z_1 + 4$ $(E_{4,2})$

$x_3 = -11y_{13} + 8$ $(E_{4,3})$

4. --

5. No. The general integer solution of the initial system is:

$x_1 = -40k_1 - 92k_2 + 27,$ from $(E_{4.1})$

$x_2 = 3k_1 + 3k_2 + 4,$ from $(E_{4,2})$

$x_3 = -11k_1 + 8,$ from $(E_{4,3})$

$x_4 = 6k_1 + 12k_2 - 4,$ from $(T_2)''$

$x_5 = 3k_1 - 2,$ from $(T_1)$

where $k_1, k_2 \in Z$.

Algorithm 4

Input

A linear system (1) with not all $a_{ij} = 0$.

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. $h = 1$, $v = 1$.

2. (A) Divide every equation i by the maximal co-divisor of the coefficients of the unknowns. If the quotient is not an integer for at least one $i_o$, then the system does not admit integer solutions. Stop.

   (B) If there is an inequality in the system, then it does not admit integer solutions.

   (C) In case of repetition, retain only one equation of that kind.

   (D) Remove all the equations which are identities.

3. Calculate $a = \min_{i,j} \{ a_{ij} , a_{ij} \neq 0\}$ and determine the indices $i_0$, $j_0$ for which this minimum can be obtained. (If there are more variants, choose one, at random.)

4. If $a \neq 1$, go on to step 6.

   If $a = 1$, then:

   (A) Calculate the value of the variable $x_{j_0}$ from the equation $i_0$; write this relation $(V_v)$.

   (B) Substitute this relation (where possible) in all the equations of the system as well as in the relations $(V_{v-1})$, $(H_h)$ for all v and h.

(C) Remove the equation $i_0$ from the system.

(D) Consider v: = v+1.

5. Does at least one equation exist in the system?

(A) If it does not, write the general integer solution of the system substituting $k_1, k_2, \ldots$ for all the variables from the right term of each expression representing the value of the initial unknowns of the system.

(B) If it does, considering the new data, go back to step 2.

6. Write all $a_{i_0 j}$, $j \neq j_0$ and $b_{i_0}$ under the form:

$$a_{i_0 j} = a_{i_0 j_0} \, q_{i_0 j} + r_{i_0 j}, \text{ with } r_{i_0 j} < a_{i_0 j} \;;$$

$$b_{i_0} = a_{i_0 j_0} \, q_{i_0} + r_{i_0}, \text{ with } r_{i_0} < a_{i_0 j_0} \;.$$

7. Write $x_{j_0} = - \sum_{j \neq j_0} q_{i_0 j} \, x_j + q_{i_0} + t_h$, relation $(H_h)$. Substitute (where possible) this relation in all the equations of the system as well as in the relations $(V_v)$, $(H_h)$ for all v and h.

8. Consider

$x_{j_0} := t_h,\ h := h + 1,$

$a_{i_0 j} := r_{i_0 j},\ j \neq j_0,$

$a_{i_0 j_0} := \pm a_{i_0 j_0},\ b_{i_0} := +r_{i_0},$

and go back to step 2.

The Correctness of Algorithm 4

This algorithm extends the algorithm from [4] (integer solutions of equations to integer solutions of linear systems). The algorithm was strictly demonstrated in our previous article; the present one introduces a new cycle--having as cycling variable the number of equations of the system--the rest remaining unchanged; hence, the correctness of algorithm 4 is obvious.

Discussion

1. The counter variables h and v count the relations H and V, respectively, differentiating them (to enable the substitutions);
2. Step 2 (A + B) + (C)) is trivial and is meant to simplify the calculus (as algorithm 2);
3. Substep 5(A) has aesthetic function (as all the

algorithms described). Everything else has been proven in the previous chapters (see [4], [5], and algorithm 2).

Example 4. Let us use algorithm 4 to calculate the integer solution of the following linear system:

$$3x_1 - 7x_3 + 6x_4 = -2$$
$$4x_1 + 3x_2 + 6x_4 - 5x_5 = 19$$

Solution

1. $h = 1$, $v = 1$
2. --
3. $a = 3$ and $i = 1$, $j = 1$
4. $3 \neq 1$. Go on to step 6.
6. So,

$$-7 = 3 \bullet (-3) + 2$$
$$6 = 3 \bullet 2 + 0$$
$$-2 = 3 \bullet 0 - 2$$

7. $x_1 = 3x_3 - 2x_4 + t_1$, relation $(H_1)$. Substituting it in the second equation we get:

$$4t_1 + 3x_2 + 12x_3 - x_4 - 5x_5 = 19$$

8. $x_1 := t_1$, $h := 2$, $a_{12} := 0$, $a_{13} := +2$, $a_{14} := 0$,

$a_{11}: = +3$, $b: = -2$

Go back to step 2.

2. The equivalent system was written:

$$+ 3t_1 + 3x_3 = -2$$
$$4t_1 + 3x_2 + 12x_3 - x_4 - 5x_5 = 19$$

3. $a = 1$, $i = 2$, $j = 4$

4. $1 = 1$

   (A) Then: $x_4 = 4t_1 + 3x_2 + 12x_3 - 5x_5 - 19$, relation $(V_1)$.

   (B) Substituting it in $(H_1)$, we get:

$$x_1 = - 7t_1 - 6x_2 - 21x_3 + 10x_5 + 38, \qquad (H_1)$$

   (C) Remove the second equation of the system.

   (D) Consider: $v: = 2$.

5. Yes. Go back to step 2.

2. The equation $+ 3t_1 + 2x_3 = -2$ is left.

3. $a = 2$ and $i = 1$, $j = 3$

4. $2 \neq 2$, go to step 6.

6. $+ 3 = + 2 \bullet 2 - 1$

   $- 2 = + 2 (-1) + 0$

7. $x_3 = - 2t_1 + t_2 - 1$, relation $(H_2)$.

   Substituting it in $(H_1)'$, $(V_1)$, we get:

$$x_1 = \ \ 35t_1 - 6x_2 - 21t_2 + 10x_5 + 59 \qquad (H_1)''.$$

$$x_4 = -\ 20t_1 + 3x_2 + 12t_2 - \ 5x_5 - 31 \qquad (V_1)'.$$

8. $x_3 := t_2$, $h := 3$, $a_{11} := -1$, $a_{13} := +2$, $b_1 := 0$ (the others being all = 0). Go back to step 2.

2. The equation $-5_1 + 2t_2 = 0$ was obtained.
3. $a = 1$, and $i = 1$, $j = 1$
4. $1 = 1$
   (A) Then, $t_1 = 2t_2$, relation $(V_2)$.
   (B) After substitution, we get:

$$x_1 = \ 49t_2 - 6x_2 + 10x_5 + 59 \qquad (H_1)''';$$

$$x_4 = -28t_2 + 3x_2 - \ 5x_5 - 31 \qquad (V_1)'';$$

$$x_3 = -\ 3t_2 \qquad\qquad - \ 1 \qquad (H_2)' ;$$

(C) Remove the first equation from the system.
   (D) $v := 3$
5. No. The general integer solution of the initial system is:

$$x_1 = \ 49k_1 - 6k_2 + 10k_3 + 59$$

$$x_2 = \qquad k_2$$

$$x_3 = -\ 3k_1 \qquad\qquad - \ 1$$

$$x_4 = -28k_1 + 3k_2 - \ 5k_3 - 31$$

$x_5 =$ $k_3$

where $(k_1, k_2, k_3) \in Z^3$.

Stop.

<u>Algorithm 5</u>

<u>Input</u>

A linear system (1).

<u>Output</u>

We decide on the possibility of a integer solution of this system. If it is possible, we obtain its general integer solution.

<u>Method</u>

1. We solve the common system in $R^n$. If it does not have solutions in $R^n$, then it does not have solutions in $Z^n$ either. Stop.
2. f = 1, v = 1, h = 1
3. Write the value of each main variable $x_i$ under the form:

$$(E_{f,i})_i: x_i = \sum_j q_{ij}x_j + q_i + (\sum_j r_{ij}x_j + r_i)/\Delta_i,$$

with all $q_{ij}$, $q_i$, $r_{ij}$, $r_i$, $\Delta_i$ from Z, so that all

$r_{ij} < \Delta_i$ , $r_i < \Delta_i$ , $\Delta_i \neq 0$ (where all $x_j$'s of the right term are integer variables: either from the secondary variables of the system or the new variables introduced with the algorithm). For all i, we write $r_{i,j_f} \equiv \Delta_i$.

4. $(F_{f,i})_i$: $\sum_j r_{ij}x'_j - r_{i,j_f}\, y_{f,i} + r_i = 0$, where $(y_{f,i})$ are auxiliary integer variables. Remove all the equations $(F_{f,i})$ which are identities.

5. Does it exist at least one equation $(F_{f,i})$? If it does not, write the general integer solution of the system substituting $k_1$, $k_2$, ... for all the variables of the right member of each expression representing the value of the initial unknowns of the system. Stop.

6. (A) Divide each equation $(F_{f,i})$ by the maximal co-divisor of the coefficients of their unknowns. If the quotient is not an integer for at least one $i_0$, then the system does not admit integer solutions. Stop.

   (B) Simplify--as previously ((A)) all the fractions in the relations $(E_{f,i})_i$.

7. Calculate $a = \min_{i,j} \{ r_{ij} , r_{ij} \neq 0\}$, and determine

the indices $i_0$, $j_0$ for which this minimum is obtained.

8. If $a \neq 1$, go on to step 9.

   If $a = 1$, then:

(A) Calculate the value of the variable $x'_{j_0}$ from the equation $(F_{f,i})$; write $(V_v)$ for this relation.

(B) Substitute this relation (where possible) in the relations $(E_{f,i})$, $(V_{v+1})$, $(H_h)$ for all i, v and h.

(C) Remove the equation $(F_{f,i})$.

(D) Consider v: = v + 1, f: = f + 1 and go back to step 3.

9. Write all $r_{i_0 j}$, $j \neq j_0$ and $r_{i_0}$ under the form:

$$r_{i_0 j} = \Delta_{i_0} \bullet q_{i_0 j} + r'_{i_0 j}, \text{ with } r'_{i_0 j} < \Delta_i ;$$

$$r_{i_0 j} = \Delta_{i_0} \bullet q_{i_0} + r'_{i_0} \text{ with } r'_{i_0} < \Delta_i .$$

10. (A) Write $x'_{j_0} = - \sum_{j \neq j_0} q_{i_0 j} \bullet x'_j + q_{i_0} + t_h$, relation $(H_h)$.

(B) Substitute this relation (where possible) in all the relations $(E_{f,i})$, $(F_{f,i})$, $(V_v)$, $(H_{h-1})$.

(C) Consider h: = h + 1 and go back to step 4.

The correctness of the algorithm is obvious. It consists of the first part of algorithm 3 and the end part of algorithm 4. Then, steps 1-6 and their correctness were discussed in the case of algorithm 3. The situation is similar with steps 7-10. (After calculating the real solution in order to calculate the integer solution, we resorted to the procedure from 5 and algorithm 5 was obtained.) It means that all these insertions were proven previously.

<u>Example 5</u>

Using algorithm 5, let us obtain the general integer solution of the system:

$$\begin{aligned} 3x_1 \quad + 6x_3 + 2x_4 \quad &= 0 \\ 4x_2 - 2x_3 \quad - 7x_5 &= -1 \end{aligned}$$

<u>Solution</u>

1. Solving in $R^5$, we get:

$$x_1 = \frac{-6x_3-2x_4}{3}$$

$$x_2 = \frac{2x_3+7x_5-1}{4}$$

2. $f = 1$, $v = 1$, $h = 1$

3. $(E_{1,1})$ : $x_1 = 2x_3 + \frac{-2x_4}{3}$

$(E_{1,2})$ : $x_2 = x_5 + \frac{2x_3+3x_5-1}{4}$

4. $(F_{1,1})$ : $-2x_4 - 3y_{11} = 0$

$(F_{1,2})$ : $2x_3 + 3x_5 - 4y_{12} - 1 = 0$

5. Yes

6. --

7. $i = 2$ and $i_0 = 2$, $j_0 = 3$

8. $2 \neq 1$

9. $3 = 2 \bullet 1 + 1$

$-4 = 2 \bullet (-2)$

$-1 = 2 \bullet 0 - 1$

10. $x_3 = -x_5 + 2y_{12} + t_1$, relation $(H_1)$. After substitution:

$(E_{1,1})'$ : $x_1 = 2x_5 - 4y_{12} - 2t_1 + \frac{-2x_4}{3}$

$(E_{1,2})'$ : $x_2 = x_5 + \frac{x_5+4y_{12}+2t_1-1}{4}$

$(F_{1,2})'$ : $x_5 + 2t_1 - 1 = 0$

Consider h: = 2 and go back to step 4.

4. $(F_{1,1})'$ : $-2x_4 - 3y_{11} = 0$

$(F_{1,2})''$: $2t_1 + x_5 - 1 = 0$

5. Yes

6. --

7. $a = 1$ and $i_0 = 2$, $j_0 = 5$

   (A) $x_5 = -2t_1 + 1$, relation $(V_1)$

   (B) Substituting it, we get:

$$(E_{1,1})'': x_1 = -6t_1 + 2 - 4y_{12} + \frac{-2x_4}{3}$$

$$(E_{1,2})'': x_2 = -2t_1 + 1 + y_{12}$$

$$(H_1)' : x_3 = 3t_1 - 1 + 2y_{12}$$

   (C) Remove the equation $(F_{1,2})$.

   (D) Consider $v = 2$, $f = 2$ and go back to step 3.

3. $(E_{2,1})$ : $x_1 = -6t_1 - 4y_{12} + 2 + \frac{-2x_4}{3}$

   $(E_{2,2})$ : $x_2 = -2t_1 + y_{12} + 1$

4. $(F_{2,1})$ : $-2x_4 - 3y_{21} = 0$

5. Yes

6. --

7. $a = 2$ and $i_0 = 1$, $j_0 = 4$

8. $2 \neq 1$

9. $-3 = -2 \bullet (1) - 1$

10. (A) $x_4 = -y_{21} + t_2$, relation $(H_2)$.

    (B) After substitution, we get:

$$(E_{2,1})' : x_1 = -6t_1 - 4y_{12} + 2 + \frac{2y_{21}-2t_2}{3}$$

$$(F_{2,1})' : -y_{21} - 2t_2 = 0$$

Consider h: = 3 and go back to step 4.

4. $(F_{2,1})' : -y_{21} - 2t_2 = 0$

5. Yes

6. --

7. a = 1 and $i_0 = 1$, $j_0 = 21$ (two, one).

    (A) $y_{21} = -2t_2$, relation $(v_2)$.

    (B) After substitution, we get:

$$(E_{2,1})'' : x_1 = -6t_1 - 4y_{12} - 2t_2 + 2$$

$$(H_2)' : x_4 = 3t_2$$

    (C) Remove the equation $(F_{2,1})$.

    (D) Consider v = 3, f = 3 and go back to step 3.

3. $(E_{3,1}) : x_1 = -6t_1 - 4y_{12} - 2t_2 + 2$

    $(E_{3,2}) : x_2 = -2t_1 + y_{12} + 1$

4. --

5. No. The general integer solution of the system is:

$$
\begin{aligned}
x_1 &= -6k_1 - 4k_2 - 2k_3 + 2, && \text{from } (E_{3,1}); \\
x_2 &= -2k_1 + k_2 + 1, && \text{from } (E_{3,2}); \\
x_3 &= 3k_1 + 2k_2 - 1, && \text{from } (H_1)'; \\
x_4 &= 3k_3, && \text{from } (H_2)'; \\
x_5 &= -2k_1 + 1, && \text{from } (V_1);
\end{aligned}
$$

where $(k_1, k_2, k_3) \in Z$.

Stop.

Note 1. Algorithms 3, 4 and 5 can be applied in the calculation of the integer solution of a linear equation.

Note 2. The algorithms, because of their form, are easy to introduce in the computer.

Note 3. It is up to the reader to decide on the algorithm to use. Good luck!

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