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Feigenbaum Attractor and the Generation Structure of Particle Physics

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The standard model for high-energy physics (SM) describes fundamental interactions between subatomic particles down to a distance scale on the order of $10^{-18}\,m$. Despite its widespread acceptance, SM operates with a large number of arbitrary parameters whose physical origin is presently unknown. Our work suggests that the generation structure of at least some SM parameters stems from the chaotic regime of renormalization group flow. Invoking the universal route to chaos in systems of nonlinear differential equations, we argue that the hierarchical pattern of parameters amounts to a series of scaling ratios depending on the Feigenbaum constant. Leading order predictions are shown to agree reasonably well with experimental data.

Keywords: Feigenbaum attractor, Renormalization Group, Standard Model, Universal transition to chaos.

1. Introduction and overview

The standard model (SM) represents a highly successful framework for the description of elementary particles and their interactions in an energy range bounded by the electroweak scale. Its predictive power rests on the regularization of divergent quantum corrections and the so-called renormalization group equations (RG), which characterize the dependence of observables on the energy scale. Despite its remarkable success, SM has several shortcomings. For instance, it requires about eighteen free parameters that are not derived from first principles and must be put in "by hand" when performing calculations

[Donoghue *et al*, 1994; Altarelli, 2005]. A large number of extensions of SM have been advanced over the years. As of today, a compelling and fully supported resolution of all open questions regarding SM has not been found [Kazakov *et al*, 2006]. It is generally expected that future experiments, slated to begin soon at the Large Hadron Collider and other accelerator facilities, will shed light on how to further develop the theory beyond SM.

The origin of the SM parameters represents a topic of active investigation. Expanding on a series of recent studies centered on the contribution of nonlinear dynamics and complexity in field theory [Damgaard and Thorleifsson 1991; Batunin, 1995; Biro *et al*, 2001; Kogan and Polyakov, 2003; Morozov and Niemi 2003; El Naschie 2006; Goldfain 2002, 2005, 2006], our work suggests that the chaotic behavior of the RG flow is responsible for the generation structure of SM. We start from the Feigenbaum-Sharkovskii-Magnitskii (FSM) scenario describing the universal path to chaos in systems of nonlinear dissipative differential equations [Magnitskii 2006, 2007]. Elaborating from this baseline, we find that the hierarchical pattern of parameters amounts to a series of scaling ratios depending on the Feigenbaum constant.

The approach discussed here is intentionally left informal. We mainly target a qualitative understanding rather than formally rigorous results. Additional research is needed to validate, expand or reject our conclusions.

The outline of the paper is as follows: section 2 presents a short overview of RG flow theory. The implications of the FSM scenario on the RG flow are discussed in section 3. A comparison between actual data and predicted results is detailed in section 4. The last section is devoted to a brief summary and to a list of future challenges.

2. Renormalization flow equations

The renormalization group (RG) flow is a key concept in quantum field theory (QFT) and statistical physics. In the Wilson picture, RG equations describe the trajectories of operators towards or away from a functional attractor set. According to this model, the flow of masses, gauge couplings, fields and mixing angles is given by the corresponding set of β-functions [Fisher 1974; Itzykson and Zuber 1980; Creswick *et al*, 1992; Zinn-Justin 2002; Amit 2005; Christensen and Moloney 2005]. A standard assumption in perturbative QFT is that the attractors of the RG flow consist of a finite number of isolated fixed points [Morozov and Niemi 2003]. There is now preliminary evidence that the end of the RG flow is a limit cycle or an attractor with a more complex structure [Wilson 1971; Bernard and LeClair 2001; Glazek and Wilson 2002]. We generalize below this conjecture by assuming that:

a) RG flow occurs in the presence of residual non-perturbative effects produced by high order quantum corrections.

a) RG flow approaches a singular limit cycle rather than a plain set of isolated fixed points.

The parameters of the Standard Model $\boldsymbol{\sigma} = \left(\sigma_{\boldsymbol{i}}\right)$; $i = 1, 2, ..., n$ evolve according to the free-flow equations [Donoghue *et al* 1994; Amit 2005]

$$\mu \frac{d\sigma_i}{d\mu} = \frac{d\sigma_i}{dt} = \beta_i(\sigma_i) \tag{1}$$

where

$$t = \log(\frac{\mu}{\Lambda}) \tag{2}$$

Here, μ denotes the sliding renormalization scale and Λ the momentum cutoff. In the presence of noise-like perturbations $\lambda_i = \lambda_i(\sigma_i, t)$, these equations may be written as

$$\frac{d\sigma_i}{dt} = \beta_i[\sigma_i, \lambda_i(\sigma_i, t)] \tag{3}$$

For the sake of concision and simplicity, we limit the analysis to the simplest case of stationary perturbations having constant amplitude

$$\lambda_i(\sigma_i, t) = \lambda \tag{4}$$

In addition, we take λ to represent the single control parameter of system (3), which then assumes the form of a generic autonomous system of ordinary differential equations (ODE)

$$\frac{d\sigma_i}{dt} = \beta_i(\sigma_i, \lambda) \tag{5}$$

3. Transition to dynamical chaos

This section relies entirely on arguments developed in [Magnitskii 2006, 2007] and is founded on the following assumptions:

A1) (5) is a smooth family of nonlinear autonomous systems of ODE in three-dimensional phase space M that is dependent on the single control parameter λ $(\sigma \in M \subset \mathrm{R}^3,\ \lambda \in \mathrm{I} \subset \mathrm{R})$.

A2) (5) are analytic functions of λ.

A3) the limit cycle $\sigma_0(t, \lambda)$ of period $T(\lambda)$ represents a solution of (5) for all $\lambda \in \mathrm{I}$.

A4) the limit cycle $\sigma_0(t, \lambda)$ is stable for $\lambda < 0$ and it becomes unstable at $\lambda = 0$ after a period-doubling bifurcation created as a result of crossing the imaginary axis by one of the Floquet exponents.

According to the theorem 4.4 of [Magnitskii 2007], the *first* stage of the transition to chaos driven by the continuous variation of $\lambda > 0$ represents a Feigenbaum cascade of period-doubling bifurcations for $\sigma_0(t,\lambda)$. Numerous examples of this scenario [Magnitskii 2007] show that the sequence of critical values λ_n, $n \in \mathrm{N}$, leading to the onset of super-stable orbits, satisfies the geometric progression

$$\lambda_n - \lambda_\infty \approx K\bar{\delta}^{-n} \tag{6}$$

Here, K is a multiplicative factor and $\bar{\delta}$ a scaling constant that is, in general, different than the standard $\delta = 4.669...$ for quadratic maps.

Based on A2), we expand $\sigma_0(t,\lambda)$ around the critical value of $\lambda = \lambda_\infty$ that leads to fully developed chaos

$$\sigma_0(t,\lambda_n) = \sigma_0(t,\lambda_\infty) + (\lambda_n - \lambda_\infty)\left.\frac{\partial\sigma_0(t,\lambda)}{\partial\lambda_n}\right|_{\lambda_\infty} + \frac{(\lambda_n - \lambda_\infty)^2}{2}\left.\frac{\partial^2\sigma_0(t,\lambda)}{\partial\lambda_n^2}\right|_{\lambda_\infty} + ... \tag{7}$$

This yields

$$\sigma_0(t,\lambda_n) = \sigma_0(t,\lambda_\infty) + (\bar{\delta})^{-n}\left.\frac{\partial\sigma_0(t,\lambda)}{\partial\lambda_n}\right|_{\lambda_\infty} + \frac{(\bar{\delta})^{-2n}}{2}\left.\frac{\partial^2\sigma_0(t,\lambda)}{\partial\lambda_n^2}\right|_{\lambda_\infty} + ... \tag{8}$$

For $n = 2^p$, $p \geq 1$ the ratio of two consecutive terms in the series then takes the form

$$\frac{\Delta\sigma_{0,n}}{\Delta\sigma_{0,n+1}} = \frac{\sigma_0(t,\lambda_n) - \sigma_0(t,\lambda_\infty)}{\sigma_0(t,\lambda_{n+1}) - \sigma_0(t,\lambda_\infty)} = \frac{\sum_k c_k[K\bar{\delta}^{-n}]^k}{\sum_k c_k[K\bar{\delta}^{-(n+1)}]^k} \tag{9}$$

Under the assumption $c_1 \neq 0$ and $\bar{\delta}^{-n} \propto O(\varepsilon)$ corresponding to $p \gg 1$, we obtain

$$\boxed{\frac{\Delta\sigma_{0,2^{p+1}}}{\Delta\sigma_{0,2^p}} \approx \bar{\delta}^{-(2^p)}} \tag{10}$$

<u>**4. Predictions versus experimental data**</u>

It is apparent that (10) provides only a *leading-order approximation* if the iteration index is not large enough, that is, if $p \approx O(1)$. Numerical results derived from (10) are displayed in the table below. This table contains a side-by-side comparison of estimated versus actual mass ratios for charged leptons and quarks and a similar comparison of gauge coupling ratios. Fermion masses are reported in MeV and evaluated at the energy scale set by the top quark mass. Using the most recent results issued by the Particle Data Group [Particle Data Group, 2005], we take

$$m_u = 2.12\,, \quad m_d = 4.22\,, \quad m_s = 80.9$$

$$m_c = 630, \quad m_b = 2847, \quad m_t = 170{,}800$$

Coupling strengths are evaluated at the scale set by the mass of the Z^0 boson, namely

$$\alpha_{EM} = 1/128\,, \quad \alpha_W = 0.0338\,, \quad \alpha_s = 0.123$$

where subscripts denote the electromagnetic, weak and strong interactions, respectively. Tab. 1 and Fig. 1 are based on taking $\overline{\delta} = 4.669$ whereas Tab. 2 and Fig. 2 are built using the best-fit numerical value for $\overline{\delta}$, that is $\overline{\delta} = 3.9$. Data on the horizontal axis is partitioned in ascending order according to the following representation:

$$1 = m_\mu / m_\tau\,, \quad 2 = (\alpha_{EM}/\alpha_W)^2, \quad 3 = m_d / m_s\,, \quad 4 = m_s / m_b$$

$$5 = m_e / m_\mu\,, \quad 6 = (\alpha_{EM}/\alpha_s)^2, \quad 7 = m_c / m_t\,, \quad 8 = m_u / m_c$$

It is interesting to note that $\overline{\delta} = 3.9$ falls close to the average value of the period-doubling constant corresponding to actual hydrodynamic flows [Peitgen *et al*, 1992].

5. Summary and concluding remarks

Motivated by recent advances in the study of complex systems, our investigation has led to the conclusion that the pattern of particle masses and gauge couplings might emerge from the chaotic dynamics of RG flow equations. We have found that the observed hierarchies of some of the SM parameters amount to a series of scaling ratios depending on the Feigenbaum constant. As pointed out in section 1, the analysis presented here is far from being either entirely rigorous or formally complete. Although leading-order predictions match reasonably well the existing experimental database, follow-up efforts are required to provide all the necessary clarifications. The list of open questions includes (but is not limited to) the following items:

a) how does the Higgs mechanism of generating masses fit into the picture?

b) can the hierarchy of mixing angles be consistently derived from this approach? [Wolfenstein 1983; Caso *et al*.1998; Goldfain 2007]

c) is there experimental evidence for additional fermion and gauge boson states that fit the same pattern? [Goldfain 2007]

APPENDIX A: The generation structure of SM parameters

Quark and lepton masses exhibit the following generation structure [Kielanowski 2000]

$$\frac{m_u}{m_c} \sim \frac{m_c}{m_t} \approx \zeta^4 \qquad \frac{m_e}{m_\mu} \approx \zeta^4 \qquad \frac{m_\tau}{m_t} \approx \zeta^3 \tag{A1}$$

$$\frac{m_d}{m_s} \sim \frac{m_s}{m_b} \sim \frac{m_b}{m_t} \approx \zeta^2 \qquad \frac{m_\mu}{m_\tau} \approx \zeta^2 \tag{A2}$$

where $\zeta \approx 0.22$ is numerically close to the so-called Cabibbo angle [Donoghue *et al*, 1994]. A similar structure shows up in the composition of the CKM matrix that describes the pattern of quark mixing angles [Wolfenstein 1983; Donoghue *et al*, 1994; Caso *et al*, 1998]

$$\begin{bmatrix} 1-\frac{\zeta^2}{2} & \zeta & \zeta^3 A(\rho - i\eta) \\ -\zeta & 1-\frac{\zeta^2}{2} & \zeta^2 A \\ \zeta^3 A(1-\rho - i\eta) & -\zeta^2 A & 1 \end{bmatrix} \quad \text{(A3)}$$

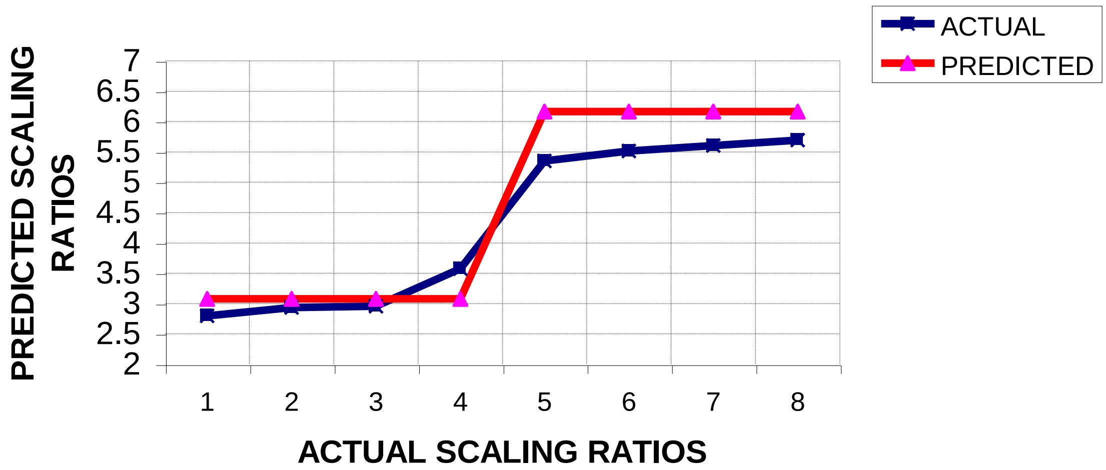

Fig 1: Actual versus predicted scaling ratios ($\overline{\delta} = 4.669$) (abs. log. scale)

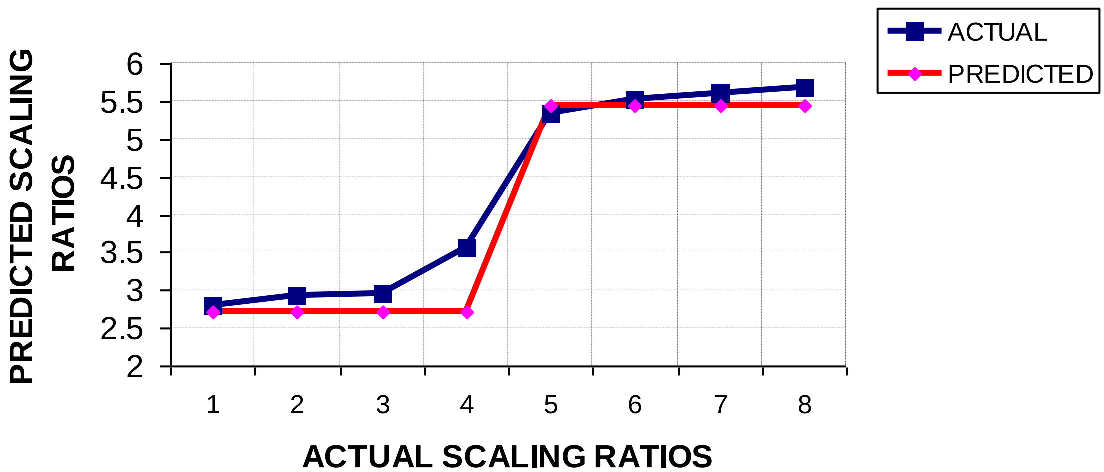

Fig 2: Actual versus predicted scaling ratios ($\overline{\delta} = 3.9$) (abs. log. scale)

Parameter ratio	Behavior	Actual	Predicted
m_u / m_c	$\overline{\delta}^{-4}$	3.365×10^{-3}	2.104×10^{-3}
m_c / m_t	$\overline{\delta}^{-4}$	3.689×10^{-3}	2.104×10^{-3}
m_d / m_s	$\overline{\delta}^{-2}$	0.052	0.046
m_s / m_b	$\overline{\delta}^{-2}$	0.028	0.046
m_e / m_μ	$\overline{\delta}^{-4}$	4.745×10^{-3}	2.104×10^{-3}
m_μ / m_τ	$\overline{\delta}^{-2}$	0.061	0.046
$(\alpha_{EM} / \alpha_W)^2$	$\overline{\delta}^{-2}$	0.053	0.046
$(\alpha_{EM} / \alpha_s)^2$	$\overline{\delta}^{-4}$	4.034×10^{-3}	2.104×10^{-3}

Tab. 1: Actual versus predicted scaling ratios for $\overline{\delta} = 4.669$

Parameter ratio	Behavior	Actual	Predicted
m_u / m_c	$\bar{\delta}^{-4}$	3.365×10^{-3}	4.323×10^{-3}
m_c / m_t	$\bar{\delta}^{-4}$	3.689×10^{-3}	4.323×10^{-3}
m_d / m_s	$\bar{\delta}^{-2}$	0.052	0.066
m_s / m_b	$\bar{\delta}^{-2}$	0.028	0.066
m_e / m_μ	$\bar{\delta}^{-4}$	4.745×10^{-3}	4.323×10^{-3}
m_μ / m_τ	$\bar{\delta}^{-2}$	0.061	0.066
$(\alpha_{EM} / \alpha_W)^2$	$\bar{\delta}^{-2}$	0.053	0.066
$(\alpha_{EM} / \alpha_s)^2$	$\bar{\delta}^{-4}$	4.034×10^{-3}	4.323×10^{-3}

Tab. 2: Actual versus predicted scaling ratios for $\bar{\delta} = 3.9$

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