CHERN-SIMONS (SUPER) GRAVITY AND E_8 YANG-MILLS FROM A CLIFFORD ALGEBRA GAUGE THEORY

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Abstract

It is shown why the E_8 Yang-Mills can be constructed from a $Cl(16)$ algebra Gauge Theory and why the $11D$ Chern-Simons (Super) Gravity theory is a very small sector of a more fundamental theory based on a $Cl(11)$ algebra Gauge theory. These results may shed some light into the origins behind the hidden E_8 symmetry of $11D$ Supergravity and reveal more important features of a Clifford-algebraic structure underlying M, F theory.

1. INTRODUCTION

Ever since the discovery [1] that $11D$ supergravity, when dimensionally reduced to an n-dim torus led to maximal supergravity theories with hidden exceptional symmetries E_n for $n \leq 8$, it has prompted intensive research to explain the higher dimensional origins of these hidden exceptional E_n symmetries [2, 6] . More recently, there has been a lot of interest in the infinite-dim hyperbolic Kac-Moody E_{10} and non-linearly realized E_{11} algebras arising in the asymptotic chaotic oscillatory solutions of Supergravity fields close to cosmological singularities [1,2].

The classification of symmetric spaces associated with the scalars of N extended Supergravity theories (emerging from compactifications of $11D$ supergravity to lower dimensions), and the construction of the U-duality groups as spectrum-generating symmetries for four-dimensional BPS black-holes [6] also involved exceptional symmetries associated with the Jordan algebras $J_3[R, C, H, O]$. The discovery of the anomaly free 10-dim heterotic string for the algebra $E_8 \times E_8$ was another hallmark of the importance of Exceptional Lie groups in Physics.

Exceptional, Jordan, Division and Clifford algebras are deeply related and essential tools in many aspects in Physics [3, 5, 8, 9,14,15,16,17,18,19,20]. In this work we will focus mainly on the Clifford algebraic structures and show how the E_8 Yang-Mills theory can naturally be embedded into a $Cl(16)$ algebra Gauge Theory and why the $11D$ Chern-Simons (Super) Gravity [4] is a very small sector of a more fundamental theory based on the $Cl(11)$ algebra Gauge theory. Polyvector-valued Supersymmetries [11] in Clifford-spaces [3] turned out to be more fundamental than the supersymmetries associated with M, F theory superalgebras [7,10]. For this reason we believe that Clifford structures may shed some light into the origins behind the hidden E_8 symmetry of $11D$ Supergravity and reveal more important features underlying M, F theory.

2. THE E_8 YANG-MILLS FROM A $Cl(16)$ ALGEBRA GAUGE THEORY

It is well known among the experts that the E_8 algebra admits the $SO(16)$ decomposition $\mathbf{248} \rightarrow \mathbf{120} \oplus \mathbf{128}$. The E_8 admits also a $SL(8, R)$ decomposition [6]. Due to the triality property , the $SO(8)$ admits the vector $\mathbf{8}_v$ and spinor representations $\mathbf{8}_s, \mathbf{8}_c$. After a triality rotation, the $SO(16)$ vector and spinor representations decompose as [6]

$$\mathbf{16} \rightarrow \mathbf{8}_s \oplus \mathbf{8}_c. \qquad (2.1a)$$

$$\mathbf{128}_s \rightarrow \mathbf{8}_v \oplus \mathbf{56}_v \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v. \qquad (2.1b)$$

$$\mathbf{128}_c \rightarrow \mathbf{8}_s \oplus \mathbf{56}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_c. \qquad (2.1c)$$

To connect with (real) Clifford algebras [8], i.e. how to fit E_8 into a Clifford structure , start with the 248-dim fundamental representation E_8 that admits a $SO(16)$ decomposition given by the 120-dim bivector representation plus the 128-dim chiral-spinor representations of $SO(16)$. From the modulo 8 periodicity of Clifford algebras one has $Cl(16) = Cl(2 \times 8) = Cl(8) \otimes Cl(8)$, meaning, roughly, that the $2^{16} = 256 \times 256$ $Cl(16)$-algebra matrices can be obtained effectively by replacing each single one of the *entries* of the $2^8 = 256 = 16 \times 16$ $Cl(8)$-algebra matrices by the 16×16 matrices of the second copy of the $Cl(8)$ algebra. In particular, $120 = 1 \times 28 + 8 \times 8 + 28 \times 1$ and $128 = 8 + 56 + 8 + 56$, hence the 248-dim E_8 algebra decomposes into a $120 + 128$ dim structure such that E_8 can be represented indeed within a tensor product of Cl(8) algebras.

At the E_8 Lie algebra level, the E_8 gauge connection decomposes into the $SO(16)$ vector $I, J = 1, 2, ...16$ and (chiral) spinor $A = 1, 2, ...128$ indices as follows

$$\mathcal{A}_\mu = \mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A. \quad X_{IJ} = -X_{JI}. \quad I, J = 1, 2, 3, ..., 16. \quad A = 1, 2, ..., 128. \tag{2.3}$$

where X_{IJ}, Y_A are the E_8 generators. The Clifford algebra ($Cl(8) \otimes Cl(8)$) structure behind the $SO(16)$ decomposition of the E_8 gauge field $\mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A$ can be deduced from the expansion of the generators X_{IJ}, Y_A in terms of the $Cl(16)$ algebra generators. The $Cl(16)$ bivector basis admits the decomposition

$$X^{IJ} = a_{ij}^{IJ} \ (\gamma_{ij} \otimes \mathbf{1}) \ + \ b_{ij}^{IJ} \ (\mathbf{1} \otimes \gamma_{ij}) \ + \ c_{ij}^{IJ} \ (\gamma_i \otimes \gamma_j). \tag{2.4}$$

where γ_i, are the Clifford algebra generators of the $Cl(8)$ algebra present in $Cl(16) = Cl(8) \otimes Cl(8)$; $\mathbf{1}$ is the unit $Cl(8)$ algebra element that can be represented by a unit 16×16 diagonal matrix. The tensor products $\otimes$ of the 16×16 $Cl(8)$-algebra matrices, like $\gamma_i \otimes \mathbf{1}$, $\gamma_i \otimes \gamma_j$, ... furnish a 256×256 $Cl(16)$-algebra matrix, as expected. The $Cl(8)$ algebra basis elements are

$$\gamma_M \ = \ \mathbf{1}, \quad \gamma_i, \quad \gamma_{i_1 i_2} = \gamma_{i_1} \wedge \gamma_{i_2}, \quad \gamma_{i_1 i_2 i_3} = \gamma_{i_1} \wedge \gamma_{i_2} \wedge \gamma_{i_3}, \ ...., \gamma_{i_1 i_2 .... i_8} = \gamma_{i_1} \wedge \gamma_{i_2} \wedge .... \wedge \gamma_{i_8} \tag{2.5}$$

Therefore, the decomposition in (2.4) yields the $28+28+8\times8 = 56+64 = 120$-dim bivector representation of $SO(16)$; i.e. for each *fixed* values of IJ there are 120 terms in the r.h.s of (2.4), that match the number of *independent* components of the E_8 generators $X^{IJ} = -X^{JI}$, given by $\frac{1}{2}(16 \times 15) = 120$. The decomposition of Y_A is more subtle. A spinor Ψ in $16D$ has $2^8 = 256$ components and can be decomposed into a 128 component left-handed spinor Ψ^A and a 128 component right-handed spinor $\Psi^{\dot{A}}$; The 256 spinor indices are $\alpha = A, \dot{A}$; $\beta = B, \dot{B}, ...$ with $A, B = 1, 2, ...128$ and $\dot{A}.\dot{B} = 1, 2, ..., 128$, respectively.

Spinors are elements of right (left) ideals of the $Cl(16)$ algebra and admit the expansion $\Psi = \Psi_\alpha \xi^\alpha$ in a 256-dim spinor basis ξ^α which in turn can be expanded as sums of Clifford polyvectors of *mixed* grade; i.e. into a sum of scalars, vectors, bivectors, trivectors, ... The chiral (left handed, right-handed) 128-component spinors $\Psi^\pm$ are obtained via the projection operators

$$\Psi^\pm = \frac{1}{2}(1 \pm \Gamma_{17})\Psi. \quad \Gamma^{17} = \Gamma^1 \wedge \Gamma^2 \wedge .... \wedge \Gamma^{16}. \tag{2.6}$$

such that $\xi_+^\alpha \equiv \xi^A$; $\xi_-^\alpha \equiv \xi^{\dot{A}}$, so the left-handed (right-handed) spinor basis $\xi_\pm$ can be represented by a column matrix (an element of the left ideal) with 128 non-vanishing upper (lower) components in the Weyl representation as

$$\xi_\pm^\alpha = (\frac{1 \pm \Gamma_{17}}{2})^{\alpha\beta} \ \ [\ (\mathbf{1} \otimes \mathbf{1})^{\beta\delta} \ \mathcal{A}^\delta \ + \ (\gamma_i \otimes \mathbf{1})^{\beta\delta} \ \mathcal{A}_i^\delta \ + \ (\gamma_{i_1 i_2} \otimes \mathbf{1})^{\beta\delta} \ \mathcal{A}_{i_1 i_2}^\delta \ + \ ....$$

$$(\gamma_{i_1 i_2 ..... i_7} \otimes \mathbf{1})^{\beta\delta} \ \mathcal{A}_{i_1 i_2 ..... i_7}^\delta \ + \ (\gamma_{i_1 i_2 ..... i_8} \otimes \mathbf{1})^{\beta\delta} \ \mathcal{A}_{i_1 i_2 ..... i_8}^\delta \ ] \tag{2.7}$$

where the numerical tensor-spinorial coefficients in the r.h.s of (2.7) are constrained to satisfy all the conditions imposed by the definition of an ideal element of the $Cl(16)$ algebra; namely that *any* element of the ideal upon a multiplication from the left by *any* Clifford algebra element yields another element of the left ideal. Similar definitions apply to the right ideal elements upon multiplication from the right by any Clifford algebra element. The row matrix (an element of the right ideal) with 128 non-vanishing components is just given by $(\xi^\pm)^\dagger$.

The rigorous procedure to construct spinors as elements of right/left ideals of Clifford algebras using primitive idempotents can be found in [5] and references therein. The final outcome is the same as performing the expansion (2.7) and solving for the coefficients. In this fashion one can construct the 128-dim left handed (right handed) chiral spinor representations of $SO(16)$ that match the number of 128 generators Y_A. Hence, the total number of E_8 generators is then $120 + 128 = 248$. What remains to be done is to enforce the E_8 commutation relations that in conjunction with the defining relations of a primitive ideal element of the $Cl(16)$ algebra will fix the values of the coefficients appearing in eqs-(2.4, 2.7) . Based on the fact that the Clifford algebra commutators of even and odd grade satisfy the relations

$$[Even\ , Even] = Even. \quad [Odd\ , Odd] = Even. \quad [Even\ , Odd] = [Odd\ , Even] = Odd. \tag{2.8}$$

which are similar to the E_8 commutation relations described below, one can immediately choose to expand the spinor basis elements in (2.7) as sums of Polyvectors of *odd* grade only, meaning that for each fixed value of δ, there are only 128 terms in the r.h.s of (2.7) given by the number of odd-grade elements of the $Cl(8)$ algebra $8 + 56 + 56 + 8 = 128$. This is consistent with the fact that a chiral spinor in $16D$ has 128 non-vanishing components in a Weyl representation. Therefore, the generators $Y^A \equiv Y^\alpha_+$; $Y^{\dot{A}} = Y^\alpha_-$ must involve *odd* grade elements of the form

$$Y^\alpha_\pm = (\frac{1 \pm \Gamma_{17}}{2})^{\alpha\beta}[(\gamma_i \otimes \mathbf{1})^{\beta\delta}\mathcal{A}^\delta_i + (\gamma_{i_1 i_2 i_3} \otimes \mathbf{1})\beta\delta\ \mathcal{A}^\delta_{i_1 i_2 i_3} + (\gamma_{i_1 i_2 ....i_5} \otimes \mathbf{1})^{\beta\delta}\mathcal{A}^\delta_{i_1 i_2 ...i_5} + (\gamma_{i_1 i_2 .....i_7} \otimes \mathbf{1})^{\beta\delta}\mathcal{A}^\delta_{i_1 i_2 .....i_7}] \tag{2.9}$$

The commutation relations of E_8 are [6]

$$[X^{IJ}, X^{KL}] = 4(\delta^{IK}\ X^{LJ} - \delta^{IL}\ X^{KJ} + \delta^{JK}\ X^{IL} - \delta^{JL}\ X^{IK})$$

$$[X^{IJ}, Y^A] = -\frac{1}{2}\Gamma^{IJ}_{AB} Y^B; \quad [Y^A, Y^B] = \frac{1}{4}\Gamma^{IJ}_{AB} X^{IJ} \tag{2.10}$$

The combined E_8 indices are denoted by $\mathcal{A} \equiv [IJ],\ A$ ($120 + 128 = 248$ indices in total) that yield the Killing metric and the structure constants

$$\eta^{\mathcal{AB}} = \frac{1}{60}\ Tr\ T^{\mathcal{A}} T^{\mathcal{B}} = -\frac{1}{60} f^{\mathcal{A}}_{\mathcal{CD}}\ f^{\mathcal{BCD}} \tag{2.11a}$$

.

$$f^{IJ,KL,MN} = -8\delta^{IK}\ \delta^{LJ}_{MN} +\ permutations; \quad f^{IJ,A,B} = -\frac{1}{2}\Gamma^{IJ}_{AB}; \quad \eta^{IJKL} = -\frac{1}{60} f^{IJ}_{\mathcal{CD}}\ f^{KL,\mathcal{CD}} \tag{2.11b}$$

Therefore, the *odd* grade expansion in (2.9) and the bivector grade expansion in (2.4) is consistent with the commutation relations of E_8. We shall proceed with the construction of a novel $Cl(16)$ gauge theory that encodes the exceptional Lie algebra E_8 symmetry from the start. The E_8 gauge theory in $D = 4$ is based on the E_8-valued field strengths

$$F^{IJ}_{\mu\nu} X_{IJ} = (\partial_\mu \mathcal{A}^{IJ}_\nu - \partial_\nu \mathcal{A}^{IJ}_\mu)\ X_{IJ} + \mathcal{A}^{KL}_\mu\ \mathcal{A}^{MN}_\nu\ [X_{KL}, X_{MN}] + \mathcal{A}^A_\mu\ \mathcal{A}^B_\nu\ [Y_A, Y_B]. \tag{2.12}$$

$$F^A_{\mu\nu} Y_A = (\partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\mu)\ Y_A + \mathcal{A}^A_\mu\ \mathcal{A}^{IJ}_\nu\ [Y_A, X_{IJ}]. \tag{2.13}$$

The E_8 actions are

$$S_{Topological}[E_8] = \int\ d^4x\ \frac{1}{60} Tr\ [\ F^{\mathcal{A}}_{\mu\nu}\ F^{\mathcal{B}}_{\rho\tau}\ T_{\mathcal{A}} T_{\mathcal{B}}\ ]\ \epsilon^{\mu\nu\rho\tau} = \int\ d^4x\ F^{\mathcal{A}}_{\mu\nu}\ F^{\mathcal{B}}_{\rho\tau}\ \eta_{\mathcal{AB}}\ \epsilon^{\mu\nu\rho\tau} =$$

$$\int\ d^4x\ [\ F^{IJ}_{\mu\nu} F^{KL}_{\rho\tau}\ \eta_{IJKL}\ + F^A_{\mu\nu} F^B_{\rho\tau}\ \eta_{AB}\ + 2F^{IJ}_{\mu\nu} F^B_{\rho\tau}\ \eta_{IJB}\ ]\ \epsilon^{\mu\nu\rho\tau}. \tag{2.14}$$

where $\epsilon^{\mu\nu\rho\tau}$ is the covariantized permutation symbol and

$$S_{YM}[E_8] = \int d^4x \sqrt{g} \frac{1}{60} Tr [F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} T_{\mathcal{A}} T_{\mathcal{B}}] g^{\mu\rho} g^{\nu\tau} = \int d^4x \sqrt{g} F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} \eta_{\mathcal{AB}} g^{\mu\rho} g^{\nu\tau} =$$

$$\int d^4x \sqrt{g} [F^{IJ}_{\mu\nu} F^{KL}_{\rho\tau} \eta_{IJKL} + F^{A}_{\mu\nu} F^{B}_{\rho\tau} \eta_{AB} + 2 F^{IJ}_{\mu\nu} F^{B}_{\rho\tau} \eta_{IJB}] g^{\mu\rho} g^{\nu\tau}. \quad (2.15)$$

The above E_8 actions (are part of) can be embedded onto more general $Cl(16)$ actions with a much larger number of terms given by

$$S_{Topological}[Cl(16)] = \int d^4x < F^{\mathcal{M}}_{\mu\nu} F^{\mathcal{N}}_{\rho\tau} \Gamma_{\mathcal{M}} \Gamma_{\mathcal{N}} > \epsilon^{\mu\nu\rho\tau} = \int d^4x F^{\mathcal{M}}_{\mu\nu} F^{\mathcal{N}}_{\rho\tau} G_{\mathcal{MN}} \epsilon^{\mu\nu\rho\tau}. \quad (2.16)$$

and

$$S_{YM}[Cl(16)] = \int d^4x \sqrt{g} < F^{\mathcal{M}}_{\mu\nu} F^{\mathcal{N}}_{\rho\tau} \Gamma_{\mathcal{M}} \Gamma_{\mathcal{N}} > g^{\mu\rho} g^{\nu\tau} = \int d^4x \sqrt{g} F^{\mathcal{M}}_{\mu\nu} F^{\mathcal{N}}_{\rho\tau} G_{\mathcal{MN}} g^{\mu\rho} g^{\nu\tau}. \quad (2.17)$$

where $< \Gamma_{\mathcal{M}} \Gamma_{\mathcal{N}} > = G_{\mathcal{MN}} \mathbf{1}$ denotes the *scalar* part of the Clifford geometric product of the gammas. Notice that there are a total of 65536 terms in

$$F^{\mathcal{M}}_{\mu\nu} F^{\mathcal{N}}_{\rho\tau} G_{\mathcal{MN}} = F_{\mu\nu} F_{\rho\tau} + F^{I}_{\mu\nu} F^{I}_{\rho\tau} + F^{I_1 I_2}_{\mu\nu} F^{I_1 I_2}_{\rho\tau} + ... + F^{I_1 I_2 .... I_{16}}_{\mu\nu} F^{I_1 I_2 ...... I_{16}}_{\rho\tau}. \quad (2.18)$$

where the indices run as $I = 1, 2, ...16$. The Clifford algebra $Cl(16)$ has the graded structure (scalars, bivectors, trivectors,....., pseudoscalar) given by

$$1 \ 16 \ 120 \ 560 \ 1820 \ 4368 \ 8008 \ 11440 \ 12870$$

$$11440 \ 8008 \ 4368 \ 1820 \ 560 \ 120 \ 16 \ 1. \quad (2.19)$$

consistent with the dimension of the $Cl(16)$ algebra $2^{16} = 256 \times 256 = 65536$. The possibility that one can acommodate another copy of the E_8 algebra within the $Cl(16)$ algebraic structure warrants further investigation by working with the duals of the bivectors X_{IJ} and recurring to the remaining $Y_{\dot{A}}$ generators. The motivation is to understand the full symmetry of the $E_8 \times E_8$ heterotic string from this Clifford algebraic perspective. A clear embedding is, of course, the following

$$E_8 \times E_8 \subset Cl(8) \otimes Cl(8) \otimes Cl(8) \otimes Cl(8) \subset Cl(16) \otimes Cl(16) = Cl(32). \quad (2.20)$$

where $SO(32) \subset Cl(32)$ and $SO(32)$ is also an anomaly free group of the heterotic string that has the same dimension and rank as $E_8 \times E_8$.

3. CHERN-SIMONS-GRAVITY IN 11D FROM A CLIFFORD ALGEBRA GAUGE THEORY

The $11D$ Chern-Simons Supergravity action is based on the smallest Anti de Sitter $OSp(32|1)$ superalgebra. The Anti de Sitter group $SO(10, 2)$ must be embedded into a larger group $Sp(32, R)$ to accomodate the fermionic degrees of freedom associated with the superalgebra $OSp(32|1)$. The bosonic sector involves the connection [4]

$$\mathbf{A}_\mu = A^a_\mu \Gamma_a + A^{ab}_\mu \Gamma_{ab} + A^{a_1 a_2 ....a_5}_\mu \Gamma_{a_1 a_2 ....a_5} = e^a_\mu \Gamma_a + \omega^{ab}_\mu \Gamma_{ab} + A^{a_1 a_2 ....a_5}_\mu \Gamma_{a_1 a_2 ....a_5} \quad (3.1)$$

with $11 + 55 + 462 = 528$ generators. A Hermitian complex 32×32 matrix has a total of $32 + 2(\frac{32 \times 31}{2}) = 992 + 32 = 1024 = 32^2 = 2^{10}$ independent real components (parameters), the same number as the real parameters of the anti-symmetric and symmetric real 32×32 matrices $496 + 528 = 1024$. The dimension of $Sp(32) = (1/2)(32 \times 33) = 528$. Notice that $2^{10} = 1024$ is also the number of independent generators of the $Cl(11)$ algebra since out of the 2^{11} generators, only half of them 2^{10}, are truly independent due to the duality conditions valid in *odd* dimensions only :

$$\epsilon^{a_1 a_2 .....a_{2n+1}} \Gamma_{a_1} \wedge \Gamma_{a_2} \wedge ..... \wedge \Gamma_{a_p} \sim \Gamma^{a_{p+1}} \wedge \Gamma^{a_{p+2}} \wedge ..... \wedge \Gamma^{a_{2n+1}}. \tag{3.2}$$

This counting of components is the underlying reason why the $Cl(11)$ algebra appears in this section. The generators of the $Cl(11)$ algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbf{1}$ and the unit element $\mathbf{1}$ generate the Clifford polyvectors (including a scalar, pseudoscalar) of different grading

$$\Gamma^A = \mathbf{1},\ \Gamma^a,\ \Gamma^{a_1} \wedge \Gamma^{a_2},\ \Gamma^{a_1} \wedge \Gamma^{a_2} \wedge \Gamma^{a_3},\ ....,\ \Gamma^{a_1} \wedge \Gamma^{a_2} \wedge\ .... \wedge \Gamma^{a_{11}}. \tag{3.3}$$

obeying the conditions (3.2). The commutation relations (see eqs-(3.4) below) involving the generators $\Gamma_a, \Gamma_{ab}, \Gamma_{a_1 a_2 ....a_5}$ do in fact *close* due to the duality conditions (3.2). The $Cl(11)$ algebra commutators, up to numerical factors, are

$$[\Gamma^a, \Gamma^b] = \Gamma^{ab}. \quad [\Gamma^a, \Gamma^{bc}] = 2\eta^{ab}\Gamma^c - 2\eta^{ac}\Gamma^b \tag{3.4a}$$

$$[\Gamma^{a_1 a_2}, \Gamma^{b_1 b_2}] = -\eta^{a_1 b_1}\Gamma^{a_2 b_2} + \eta^{a_1 b_2}\Gamma^{a_2 b_1} - .... \tag{3.4b}$$

$$[\Gamma^{a_1 a_2 a_3}, \Gamma^{b_1 b_2 b_3}] = \Gamma^{a_1 a_2 a_3 b_1 b_2 b_3} - (\eta^{a_1 b_1 a_2 b_2}\Gamma^{a_3 b_3} + ...). \tag{3.4c}$$

$$[\Gamma^{a_1 a_2 a_3 a_4}, \Gamma^{b_1 b_2 b_3 b_4}] = -(\eta^{a_1 b_1}\Gamma^{a_2 a_3 a_4 b_2 b_3 b_4} + ...) - (\eta^{a_1 b_1 a_2 b_2 a_3 b_3}\Gamma^{a_4 b_4} + ...). \tag{3.4d}$$

$$[\Gamma^{a_1 a_2}, \Gamma^{b_1 b_2 b_3 b_4}] = -\eta^{a_1 b_1}\Gamma^{a_2 b_2 b_3 b_4} + ... \tag{3.4e}$$

$$[\Gamma^{a_1}, \Gamma^{b_1 b_2 b_3}] = \Gamma^{a_1 b_1 b_2 b_3}. \quad [\Gamma^{a_1 a_2}, \Gamma^{b_1 b_2 b_3}] = -2\eta^{a_1 b_1}\Gamma^{a_2 b_2 b_3} + ... \tag{3.4f}$$

$$[\Gamma^{a_1}, \Gamma^{b_1 b_2 b_3 b_4}] = -\eta^{a_1 b_1}\Gamma^{b_2 b_3 b_4} + ... \tag{3.4g}$$

$$[\Gamma^{a_1 a_2 ....a_5}, \Gamma^{b_1 b_2 ....b_5}] = \Gamma^{a_1 a_2 ...a_5 b_1 b_2 ....b_5} + (\eta^{a_1 b_1 a_2 b_2}\Gamma^{a_3 a_4 a_5 b_3 b_4 b_5} + ...) + (\eta^{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}\Gamma^{a_5 b_5} + ...) =$$
$$\epsilon^{a_1 a_2 ...a_5 b_1 b_2 ....b_5 c}\ \Gamma_c + (\eta^{a_1 b_1 a_2 b_2}\epsilon^{a_3 a_4 a_5 b_3 b_4 b_5 c_1 c_2 .....c_5}\Gamma_{c_1 c_2 ....c_5} + ...) + (\eta^{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}\Gamma^{a_5 b_5} + ...). \tag{3.4h}$$

etc....... with

$$\eta_{a_1 b_1 a_2 b_2} = \eta_{a_1 b_1}\eta_{a_2 b_2} - \eta_{a_2 b_1}\eta_{a_1 b_2} \tag{3.5a}$$

$$\eta_{a_1 b_1 a_2 b_2 a_3 b_3} = \eta_{a_1 b_1}\eta_{a_2 b_2}\eta_{a_3 b_3} - \eta_{a_1 b_2}\eta_{a_2 b_1}\eta_{a_3 b_3} + ... \tag{3.5b}$$

$$\eta_{a_1 b_1 a_2 b_2 ......a_n b_n} = \frac{1}{n!}\ \epsilon_{i_1 i_2 ......i_n}\ \epsilon_{j_1 j_2 ......j_n}\ \eta_{a_{i_1} b_{j_1}}\ \eta_{a_{i_2} b_{j_2}} ....\ \eta_{a_{i_n} b_{j_n}}. \tag{3.5c}$$

The $Cl(11)$ algebra gauge field is

$$\mathbf{A}_\mu = \mathcal{A}^A_\mu = \mathcal{A}_\mu \mathbf{1} + \mathcal{A}^a_\mu \Gamma_a + \mathcal{A}^{a_1 a_2}_\mu \Gamma_{a_1 a_2} + \mathcal{A}^{a_1 a_2 a_3}_\mu \Gamma_{a_1 a_2 a_3} + ... + \mathcal{A}^{a_1 a_2 ....a_{11}}_\mu \Gamma_{a_1 a_2 ....a_{11}}. \tag{3.6}$$

and the $Cl(11)$-algebra-valued field strength

$$\mathcal{F}^A_{\mu\nu}\ \Gamma_A = \partial_{[\mu}A_{\nu]}\ \mathbf{1}\ +\ [\ \partial_{[\mu}A^a_{\nu]} + A^{b_2}_{[\mu}A^{b_1 a}_{\nu]}\eta_{b_1 b_2}\ + ...\ ]\ \Gamma_a\ +$$
$$[\ \partial_{[\mu}A^{ab}_{\nu]} + A^a_{[\mu}A^b_{\nu]} - A^{a_1 a}_{[\mu}A^{b_1 b}_{\nu]}\eta_{a_1 b_1} - A^{a_1 a_2 a}_{[\mu}A^{b_1 b_2 b}_{\nu]}\eta_{a_1 b_1 a_2 b_2} - A^{a_1 a_2 a_3 a}_{[\mu}A^{a_1 b_2 b_3 b}_{\nu]}\eta_{a_1 b_1 a_2 b_2 a_3 b_3}\ + ...\ ]\ \Gamma_{ab}\ +$$
$$[\ \partial_{[\mu}A^{abc}_{\nu]} + A^{a_1 a}_{[\mu}A^{b_1 bc}_{\nu]}\eta_{a_1 b_1}\ + ...\ ]\ \Gamma_{abc}\ +\ [\ \partial_{[\mu}A^{abcd}_{\nu]} - A^{a_1 a}_{[\mu}A^{b_1 bcd}_{\nu]}\eta_{a_1 b_1}\ + ...\ ]\ \Gamma_{abcd} + ...$$
$$[\ \partial_{[\mu}A^{a_1 a_2 ....a_5 b_1 b_2 .....b_5}_{\nu]} + A^{a_1 a_2 ...a_5}_{[\mu}A^{b_1 b_2 ....b_5}_{\nu]}\ + ...\ ]\ \Gamma_{a_1 a_2 ....a_5 b_1 b_2 .....b_5} + ... \tag{3.7}$$

The Chern-Simons actions rely on Stokes theorem

$$\int_{M^{12}} \epsilon^{\mu_1\mu_2....\mu_{11}\mu_{12}} \partial_{\mu_{12}} (A_{\mu_1\mu_2....\mu_{11}}) = \int_{\partial M^{12}=\Sigma^{11}} \epsilon^{\mu_1\mu_2....\mu_{11}\mu_{12}} A_{\mu_1\mu_2....\mu_{11}} d\Sigma^{11}_{\mu_{12}}. \qquad (3.8)$$

which in our case reads

$$d (\mathcal{L}_{Clifford}) =< \mathcal{F} \wedge \mathcal{F} \wedge .... \wedge \mathcal{F} > = < \mathcal{F}^{A_1} \wedge \mathcal{F}^{A_2} \wedge .... \wedge \mathcal{F}^{A_6} \Gamma_{A_1}\Gamma_{A_2}....\Gamma_{A_6} > \qquad (3.9)$$

where the bracket $< ...... >$ means taking the scalar part of the Clifford geometric product among the gammas. It involves products of the d_{ABC}, f_{ABC} structure constants corresponding to the (anti) commutators $\{\Gamma_A, \Gamma_B\} = d_{ABC}\Gamma^C$ and $[\Gamma_A, \Gamma_B] = f_{ABC}\Gamma^C$.

One of the main results of this work is that the $Cl(11)$ algebra based action (3.9) contains a *vast number* of *terms* among which is the Chern-Simons action of [4] $\mathcal{L}^{11}_{CS}(e, \omega, A_5)$

$$\mathcal{L}_{Clifford}(\mathcal{A}^A_\mu \Gamma_A) = \mathcal{L}^{11}_{CS}(\omega, e, A_5) + EXTRA\ TERMS. \qquad (3.10)$$

$$S_{CS}(\omega, e, A_5) = \int_{\partial M^{12}} \mathcal{L}^{11}_{CS} = \int_{\Sigma^{11}} \mathcal{L}^{11}_{CS}. \qquad (3.11)$$

$$\mathcal{L}^{11}_{CS}(\omega, e, A_5) = \mathcal{L}^{11}_{Lovelock}(\omega, e) + \mathcal{L}^{11}_{Pontryagin}(\omega, e) + \mathcal{L}^{11}(A_5, \omega, e) \qquad (3.12)$$

In odd dimensions $D = 2n - 1$, the Lanczos-Lovelock Lagrangian is

$$\mathcal{L}^D_{Lovelock} = \sum_{p=0}^{n-1} a_p\ L_p(D). \quad a_p = \kappa \frac{(\pm 1)^{p+1} l^{2p-D}}{(D-2p)} C^{n-1}_p; \quad p = 1, 2, ..., n-1 \qquad (3.13)$$

C^{n-1}_p is the binomial coefficient. The constants κ, l are related to the Newton's constant G and to the cosmological constant Λ through $\kappa^{-1} = 2(D-2)\Omega_{D-2}G$ where Ω_{D-2} is the area of the $D-2$-dim unit sphere and $\Lambda = \pm(D-1)(D-2)/2l^2$ for de Sitter (Anti de Sitter) spaces [4] . A derivation of the vacuum energy density of Anti de Sitter space (de Sitter) as the geometric mean between an upper and lower scale was obtained in [17] based on a BF-Chern-Simons-Higgs theory. Upon setting the lower scale to the Planck scale L_P and the upper scale to the Hubble radius (today) R_H, it yields the observed value of the cosmological constant $\rho = L_P^{-2} R_H^{-2} = L_P^{-4}(L_P/R_H)^2 \sim 10^{-120} M_P^4$.

The terms inside the summand of (3.13) are

$$L_p(D) = \epsilon_{a_1 a_2 ....a_D} R^{a_1 a_2} R^{a_3 a_4} ....R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} ....e^{a_D} \qquad (3.14)$$

where we have omitted the space-time indices $\mu_1, \mu_2, ...$ Despite the higher powers of the curvature (after eliminating the spin connection ω^{ab}_μ in terms of the e^a_μ field) the $\mathcal{L}^D_{Lovelock}$ furnishes equations of motion for the e^a_μ field containing at most derivatives of *second* order, and not higher, due to the Topological property of the Lovelock terms

$$d (\mathcal{L}^{11}_{Lovelock}) = \epsilon_{a_1 a_2 .....a_{11}} (R^{a_1 a_2} + \frac{e^{a_1} e^{a_2}}{l^2}) ......(R^{a_9 a_{10}} + \frac{e^{a_9} e^{a_{10}}}{l^2})\ T^{a_{11}} = Euler\ density\ in\ 12D. \quad (3.15)$$

The exterior derivative of the Lovelock terms can be rewritten compactly as

$$d (\mathcal{L}^{11}_{Lovelock}) = \epsilon_{A_1 A_2 ....A_{12}} F^{A_1 A_2} ......F^{A_{11} A_{12}} \qquad (3.16)$$

where $F^{A_1 A_2}$ is the curvature field strength associated with the $SO(10, 2)$ connection $\Omega^{A_1 A_2}_\mu$ in $12D$ and which can be decomposed in terms of the fields $e^a_\mu, \omega^{ab}_\mu,\ a, b = 1, 2, ..., 11$ by identifying $\Omega^{aD}_\mu = \frac{1}{l} e^a_\mu$ and $\Omega^{ab}_\mu = \omega^{ab}_\mu$ so that the Torsion and Lorenz curvature 2-forms are

$$T^a(\omega, e) = F^{aD} = d\Omega^{aD} + \Omega^a_b \wedge \Omega^{bD} = \frac{1}{l}(de^a - \omega^a_b \wedge e^b).$$

$$F^{ab} = (d\Omega^{ab} + \Omega^a_c \wedge \Omega^{cb}) + (\Omega^a_D \wedge \Omega^{Db}) = R^{ab}(\omega) + \frac{1}{l^2} e^a \wedge e^b. \quad R^{ab}(\omega) = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \tag{3.17}$$

where a length parameter l must be introduced to match dimensions since the connection has units of $1/l$. This l parameter is related to the cosmological constant.

$\mathcal{L}^{11}_{Pontryagin}(\omega, e)$ is the Chern-Simons 11-form whose exterior derivative

$$d\ (\mathcal{L}_{Pontryagin}) = F^{A_1}_{A_2} F^{A_2}_{A_3} ....F^{A_5}_{A_6} F^{A_6}_{A_1} \tag{3.18}$$

is the (one of the many) Pontryagin 12-form (up to numerical factors) for the $SO(10,2)$ connection in $12D$. As mentioned above, the $SO(10,2)$ connection Ω^{AB}_μ can be broken into the e^a_μ field and the $SO(10,1)$ spin connection ω^{ab}_μ such that the number of components is $11 + \frac{1}{2}(11 \times 10) = 66 = \frac{1}{2}(12 \times 11)$. Finally, the exterior derivative of $\mathcal{L}^{11}(A_5, \omega, e)$ is the 12-form (we are omitting space-time indices $\mu_1, \mu_2, ..., \mu_{12}$)

$$d\mathcal{L}^{11}(A_5, \omega, e) = (\epsilon_{a_1 a_2 .....a_{11}} R^{a_1 a_2 .....a_5} R^{a_6 a_7 ....a_{10}} T^{a_{11}})(\epsilon_{b_1 b_2 .....b_{11}} R^{b_1 b_2 .....b_5} R^{b_6 b_7 ....b_{10}} T^{b_{11}}) \tag{3.19}$$

the curvature 2-form associated with the field $A^{c_1 c_2 .....c_5}_\mu$, after recurring to the duality conditions of eq-(3.2), is

$$R^{c_1 c_2 .....c_5}_{\mu\nu} = \partial_{[\mu} A^{c_1 c_2 .....c_5}_{\nu]} + A^{a_1 a_2 .....a_5}_{[\mu} A^{b_1 b_2 .....b_5}_{\nu]} f_{a_1 a_2 ...a_5\ b_1 b_2 ....b_5\ d_1 d_2 ....d_6}\ \epsilon^{d_1 d_2 ....d_6 c_1 c_2 ....c_5} \tag{3.20}$$

where the structure constants f_{ABC} in (3.18) are obtained from the $Cl(11)$ algebra commutation relations in (3.4h).

The $Cl(11)$ algebra based action (3.9) can in turn be embedded into a more general expression in C-space (Clifford Space) which is a generalized tensorial spacetime of coordinates $\mathbf{X} = \sigma, x^\mu, x^{\mu\nu}, x^{\mu\nu\rho}$.... [3] involving antisymetric tensor (and scalar) gauge fields $\Phi(\mathbf{X}), A_\mu(\mathbf{X}), A_{\mu\nu}(\mathbf{X}), A_{\mu\nu\rho}(\mathbf{X})$..... of higher rank (higher spin theories) [13]. The most general action onto which the action (3.9) itself can be embedded requires a tensorial gauge field theory [12, 13] (a Generalization of Yang-Mills theories) and an integration w.r.t the Clifford-valued coordinates $\mathbf{X} = X^M \Gamma_M$ corresponding to the C-space associated with the underlying $Cl(2n)$-algebra in $D = 2n$ dimensions

$$S = \int [d^{2^n} X]\ <\ (\mathcal{F} \wedge \mathcal{F} \wedge ..... \wedge \mathcal{F})\ >. \quad [d^{2^n} X] = (d\sigma)(dx^\mu)(dx^{\mu\nu})(dx^{\mu\nu\rho})...... \tag{3.21}$$

A Generalized Polyvector-valued Supersymmetry [10] based on a Grassmanian extension $\theta, \theta^\alpha, \theta^{\alpha\beta}, \theta^{\alpha\beta\delta}, ...$ of the bosonic C-space coordinates $\mathbf{X}$ was undertaken in [11]. Such C-space Generalized Supersymmetry is based on an *extension* and *generalizations* of the M, F Theory Superalgebras [7] that we will briefly discuss below.

A Chern-Simons Supergravity (CS-SUGRA) in $D = 11$ involves the symplectic supergroup $OSp(32|1)$ and the connection [4]

$$\mathbf{A}_\mu = e^a_\mu \Gamma_a + \omega^{ab}_\mu \Gamma_{ab} + A^{a_1 a_2 ....a_5}_\mu \Gamma_{a_1 a_2 ....a_5} + \bar{\Psi}^\alpha_\mu Q_\alpha. \tag{3.22}$$

whereas the M theory superalgebra involve 32-component spinorial supercharges Q_α whose anticommutators are [7]

$$\{Q_\alpha, Q_\beta\} = (\mathcal{A}\Gamma_\mu)_{\alpha\beta}\ P^\mu + (\mathcal{A}\Gamma_{\mu_1\mu_2})_{\alpha\beta}\ Z^{\mu_1\mu_2} + (\mathcal{A}\Gamma_{\mu_1\mu_2 ....\mu_5})_{\alpha\beta}\ Z^{\mu_1\mu_2 .....\mu_5}. \tag{3.23}$$

there are 32×32 symmetric real matrices with at most $\frac{1}{2}(32 \times 33) = 528$ independent components that match the number of degrees of freedom associated with the translations P^μ and the antisymmetric rank $2, 5$ abelian tensorial central charges $Z^{\mu_1\mu_2}, Z^{\mu_1\mu_2 .....\mu_5}$ in the r.h.s since $11 + 55 + 462 = 528$. The matrix $\mathcal{A}$ plays the role of the timelike γ^0 matrix in Minkowski spacetime and is used to introduce barred-spinors [7]

The F theory $12D$ super-algebra involves the Majorana-Weyl spinors with 32 components whose anti-commutators are [7]

$$\{Q_\alpha, Q_\beta\} = (\mathcal{A}\Gamma_{\mu\nu})_{\alpha\beta}\ Z^{\mu\nu} + (\mathcal{A}\Gamma_{\mu_1\mu_2 ....\mu_6})_{\alpha\beta}\ Z^{\mu_1\mu_2 ....\mu_6}. \tag{3.24}$$

and the counting of components in $D = 12$ yields also $\frac{32\times 33}{2} = 528 = 66 + 462$. In $13D$ it requires the superalgebra $OSp(64|1)$ which is connected to a membrane, a 3-brane and a 6-brane, respectively, since antisymmetric tensors of ranks $2, 3, 6$ in $13D$ have a total of $\frac{64\times 65}{2} = 78 + 286 + 1716 = 2080$ components.

For this reason we believe that Polyvector-valued Supersymmetries in C-spaces [11] deserve to be investigated further since they are more fundamental than the supersymmetries associated with M, F theory superalgebras and also span well beyond the N-extended Supersymmetric Field Theories involving superalgebras, like $OSp(32|N)$ for example, which are related to a $SO(N)$ Gauge Theory coupled to matter fermions (besides the gravitinos). Finally, the results of this work may shed some light into the origins behind the hidden E_8 symmetry of $11D$ Supergravity , the hyperbolic Kac-Moody algebra E_{10} and the non-linearly realized E_{11} algebra related to Chaos in M theory and oscillatory solutions close to cosmological singularities [1,2,6].

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