Where is the polarization energy hidden and where does the force act?

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It is shown that the quadratic dependence of electric energy on electric strength causes an additional electric energy in comparison with the energy corresponding to a macroscopic electric field. The force acting on a dielectric, in general, does not act on the bound polarization charge. The Maxwell stress tensor in dielectric is considered.

I. The standard explanations are incorrect. Results

As is well known, the electrical energy

$$W = \int E^2 dV/2 \tag{1.1}$$

of a capacitor of a constant electric field E is being multiplied by the factor ε when the capacitor is being filled up by a dielectric with dielectric constant ε (we set $\varepsilon_0 = 1$)

$$W = \int \varepsilon E^2 dV/2 . \tag{1.2}$$

We are interested in what is nature of the extra energy?

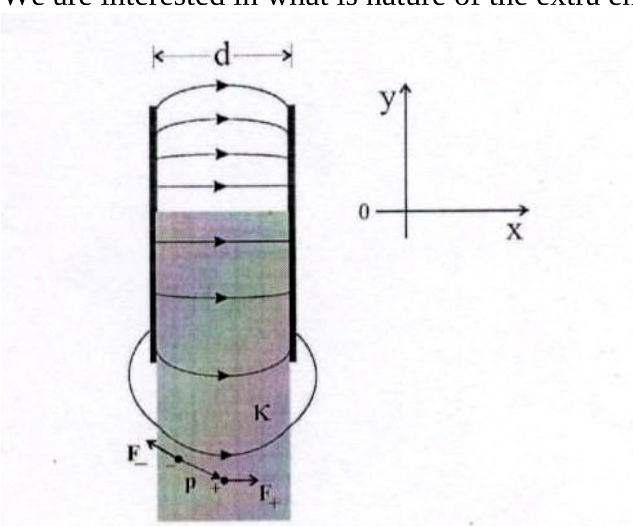


Fig. 1. Linear dielectric slab partially inserted between parallel conducting charged plates. Unbalanced forces $\mathbf{F}_+$ and $\mathbf{F}_-$ are shown acting on a dipole $\mathbf{p}$ in the fringing field region. The $+z$ direction is out of the page.

D. Griffiths [1] answers this subtle question. According to him, the energy corresponds to the work involved in stretching the dielectric molecules. He writes that if we picture the positive and negative charges as held together by tiny springs, the spring energy, $kx^2/2$, associated with polarizing each molecule, must be taken into account.

We will show this explanation is not completely correct. We will show the “spring” energy is zero, or the “spring” energy is located not in the “spring”.

Another subtle question is where does the force exert over a dielectric slab pushed halfway into the gap of the parallel-plate condenser? There are, at the minimum, three different answers.

E. Dietz [2] attributes the force directly to the fringing electrical field (see Fig. 1 from [2]. The force on a single electric dipole p^i within the dielectric is given as $p^i \partial_i E^j$, so at a point where the polarization is P^i, the force on a volume element dV of material is

$$dF^j = P^i \partial_i E^j dV = (\varepsilon - 1) E^i \partial_i E^j dV , \tag{1.3}$$

and the *volume* force

$$F^j = \int (\varepsilon - 1) E^i \partial_i E^j dV \tag{1.4}$$

exerts over the slab. Integrating of (1.4) yields

$$F = (\varepsilon - 1) E^2 a/2 \tag{1.5}$$

where a is the area of the butt-end of the slab. This result coincides with the result of the standard energy method [1],

$$F_y = -\partial_y W(y)\,, \tag{1.6}$$

but the author emphases the dominant contribution to the integral (1.4) is due to the fringing field region, whereas in the standard energy approach, the calculation of $\partial_y W(y)$ during the virtual displacement seems to be on the interface between the air and the dielectric at $y = 0$ in Fig. 1.

C. Utreras-Diaz [3] argues, quite the contrary, the force acts on bound polarization charges located on dielectric *surfaces bordering upon the condenser plates*. He does not use eqns (1.3), (1.4) and calculates the force between the condenser and the slab as the force between the charge distributions, as shown in Fig. 2 from [3] where $\mathbf{F}_a$ are the (attractive) forces between the upper/lower conducting plate and upper/ lower dielectric surface; and $\mathbf{F}_r$ are the (repulsive) forces between the upper/lower conducting plate and lower/upper dielectric surface. His result is the same, (1.5).

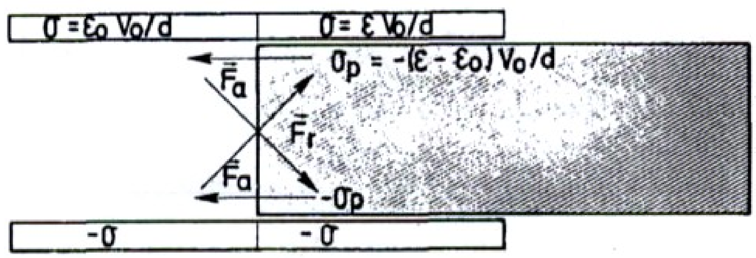


Fig. 2. The charge on the density on the upper conducting surface is $\sigma = \epsilon_0 V_0/d$, which produces an attractive force $\mathbf{F}_a$ on the charge $\sigma_p = -(\epsilon - \epsilon_0)V_0/d$ on the upper dielectric surface; and a repulsive force $\mathbf{F}_r$ on the charge $-\sigma_p$ on the lower dielectric surface.

S. Margulies [4] interprets eqn. (1.4) in term of bound polarization charges located on the slab surface,

$$F^j = \int_V (\varepsilon - 1) E^i \partial_i E^j dV = \oint_{\partial V} E^j P^i da_i = \oint_{\partial V} E^j dq_p\,, \tag{1.7}$$

where da_i is a surface element of the boundary ∂V of the slab volume V, E^j is the field inside the boundary, and $dq_p = P^i da_i$ is the bound charge at da_i. According to (1.7), the author claims the force (1.5) arises from the interaction of the electric field in the dielectric with the bound polarization charge at the surface of the dielectric. But, according to [4], the force exerts on the *surface of the slab outside the gap* of the condenser, in the region I of Fig. 3 from [4], rather than in the region II and III, as previous author insists.

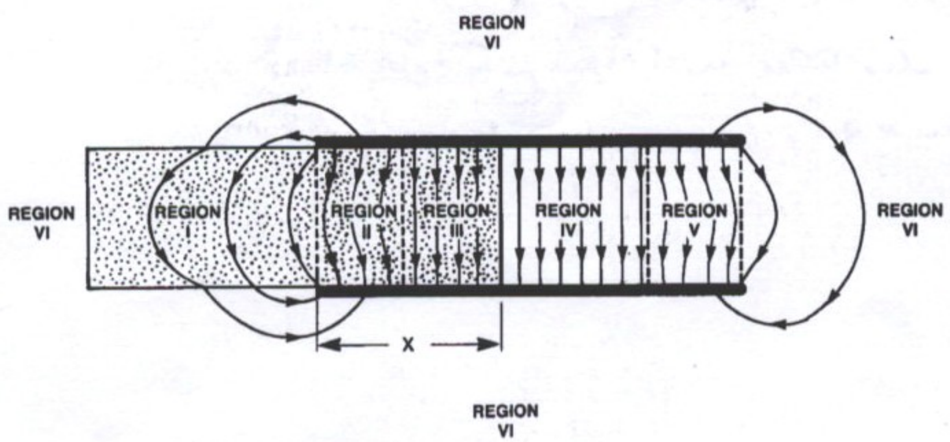


Fig. 3. Division of all space into six regions for the purpose of evaluating the total field energy. Regions I, II, and III are within the dielectric; regions IV, V, and VI are in vacuum.

We will show all three answers are incorrect. We will show that the force acts on the surface of the slab, but on the butt-end of the slab where there are no bound charges. In general, bound charges cannot be used for obtaining of a force. Besides this, Eqns.

$$dF^j = E^j dq_p \tag{1.8}$$

and (1.7) are incorrect because E^j is not defined on the boundary surface, i.e. E^j has a discontinuity on the surface of the dielectric. The wrong formulae (1.7), (1.8) give the right result in[4] because only the tangential component of E^j, which is continuous at the surface, gives a contribution in (1.7) due to symmetry of the condenser.

Note, the place of applying the force affects internal stresses of the dielectric material and is determinable experimentally. And another argument against the forces (1.7), (1.8) exists. A liquid cannot bear tangential load. So, tangential forces from [3, 4] lead to a paradox if the attractive force pulls a liquid dielectric into vertical parallel plates dipped into the dielectric liquid. Forces, acting on a piece of dielectric of constant ε, are surface forces, normal to the surface of the piece, according to the fundamental formula for the volume force [5]

$$\mathbf{f} = -E^2 \mathrm{grad}\varepsilon / 2 . \tag{1.9}$$

II. Griffiths' spring energy.

Here we calculate the energy of the stretching molecules or atoms. First of all it is clear that the picture of the positive and negative charges as held together by the Coulomb force $q^2/(4\pi r^2)$ is unreal because the dipole moment of such a molecule is not proportional to an external electric field. Thus we adopt, according to Feynman [6], that an atom has a positive charge on the nucleus, which is surrounded by negative electrons. In an electric field, the nucleus will be attracted in one direction and the electron in the other. The orbits or wave patterns of the electrons (or whatever picture is used in quantum mechanics) will be distorted to some extent, as shown in Fig. 10-4; the center of gravity of the negative charge will be displaced and will no longer coincide with the positive charge of the nucleus. If we look from a distance, such a neutral configuration is equivalent, to a first approximation, to a little dipole. We will show that energy of such an atom does not change with its polarization.

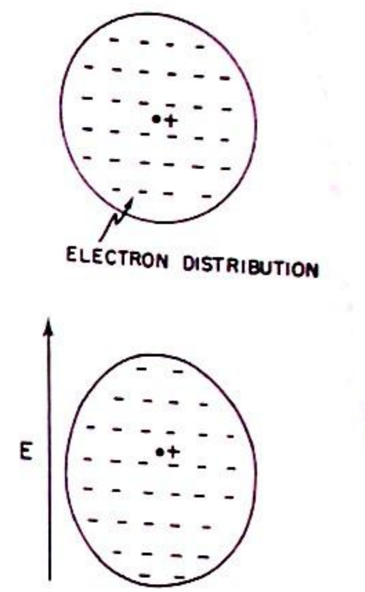


Fig. 10–4. An atom in an electric field has its distribution of electrons displaced with respect to the nucleus.

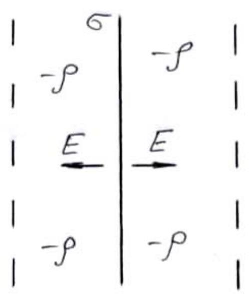
For simplicity, we consider the nucleus as a plate with a charge density $\sigma > 0$ in the center of a negative charged layer of a thickness l (Fig. 4). The volume charge density of the layer is $-\rho < 0, \quad \rho l = \sigma$. The relationships of electric field $E(x)$ and potential $\phi(x)$ to the coordinate are plotted in Fig. 4.

$$E(x) = -\sigma x / l \quad \text{if} \quad 0 < x < l/2 , \quad E(x) = \sigma(1 - x/l) \quad \text{if} \quad l/2 < x < l . \tag{2.1}$$

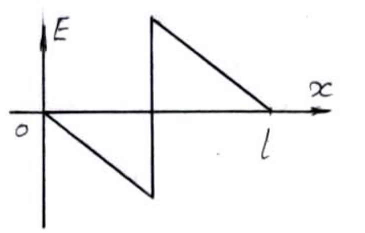
$$\phi(x) = \int_x^0 E dx, \quad \phi(0) = \phi(l) = 0 . \tag{2.2}$$

Electrical energy per unite surface of such an atom can be readily calculated

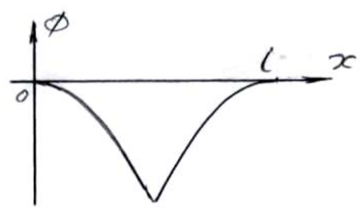
$$W = \int_0^l E^2 dx / 2 = \sigma^2 l / 24 . \tag{2.3}$$



Plane atom



Electric field
in the plane atom



Electric potential
in the plane atom

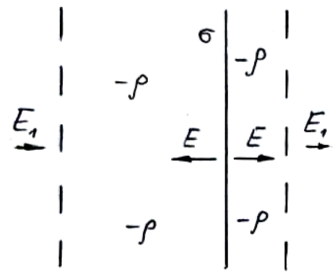
Fig. 4. Plane model of an atom

When the atom is placed into an external electric field E_1, the positive charged plate is displaced at the distance E_1/ρ as is shown in Fig. 5 where the electric field and the potential is plotted as well,

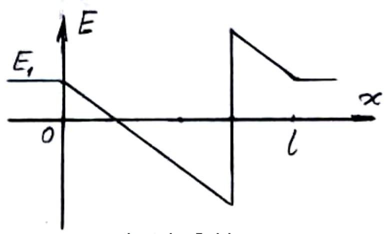
$$E = E_1 - \rho x \quad \text{if} \quad 0 < x < l/2 + E_1/\rho\,,$$

$$E = E_1 + \sigma - \rho x \quad \text{if} \quad l/2 + E_1/\rho < x < l\,. \qquad (2.4)$$

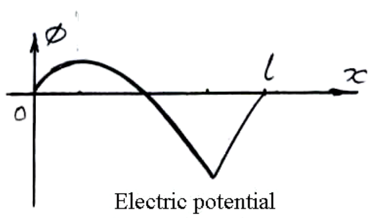
Substituting (2.4) into (2.3) yields the same result (2.3) independently on value of E_1. Thus there is no polarization energy inside the atom.



Plane atom



Electric field
in the plane atom

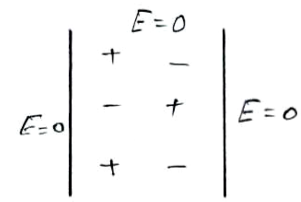


Electric potential
in the plane atom

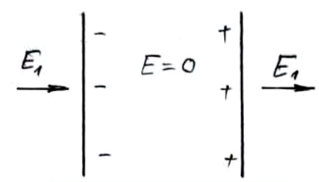
Fig. 5. Model of the plane atom in the presence of external electric field

Absence of the polarization energy inside the atom can be confirmed by another model of an atom. Consider an atom consists of positive and negative substances, which are put one over other. For this atom inner electrical field and energy is zero. When the external field appears, these substances are displaced one from other. In the new equilibrium state, there are surface charges on the sides of the atom, but the inner field and energy are conserved to be zero. This is depicted in Fig. 6. Such an atom behaves as a piece of electric conductor.

Accordingly, Feynman suggested a simple model for what happens with dielectrics – that inside the material there are many little sheets of conductive material, or conducting spheres separated from each other by insulation, as shown in Fig. 10-3. The phenomenon of the dielectric constant is explained by the effect of the charges which would be induced on each sphere. The dielectric constant ε would depend on the proportion of space which was occupied by the conducting sheets.



External field is absent



The atom in external field

Fig. 6.
The polarization of the plane atom consisting of a mix of negative and positive substances

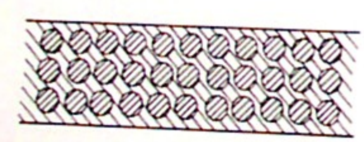


Fig. 10–3. A model of a dielectric: small conducting spheres embedded in an idealized insulator.

III. Electric energy is proportional to square of electric field strength

So-called "macroscopic" electric field E is a smoothing of a real, "microscopic" field E_1 in a dielectric. E is a fictitious field. The spatial variations of the real field E_1 occur over distances of the order of 10^{-10} m, and the sense of the smoothing, or averaging of the field is a satisfaction of the equality

$$\int E_1 dx = \int E dx\,, \qquad (3.1)$$

which provides the right potential difference between boundaries of a dielectric piece. Because electric energy is proportional to square of electric field strength, real energy W_1 is larger than fictitious expression W:

$$W_1 = \int E_1^2 dV/2 \quad > \quad W = \int E^2 dV/2\,. \qquad (3.2)$$

Consider this equation in details by the use of the Feynman's model with sheets of conducting material of thickness l separated by insulation of thickness b. This model is presented in Fig. 7 where E_1 is an external field generated e.g. by plates of a condenser. The field penetrates into material of the dielectric to first conducting sheet. Eqn. (3.1) gives a value of the dielectric constant for such a model,

$$E_1 b = E(b+l), \quad E_1 = \frac{b+l}{b}E, \quad \varepsilon = \frac{b+l}{b} = 1 + l/b, \quad E_1 = \varepsilon E\,, \tag{3.3}$$

whereas eqn. (3.2) gives the real energy density, i.e. the time component of Maxwell energy-momentum tensor

$$w_1 = \frac{W_1}{V} = T_e^{00} = \frac{1}{2(b+l)}\int_0^{b+l} E_1^2 dx = \frac{E_1^2 b}{2(b+l)} = \frac{E_1^2}{2\varepsilon} = \frac{\varepsilon E^2}{2}\,. \tag{3.4}$$

This is a solution of the polarization energy problem.

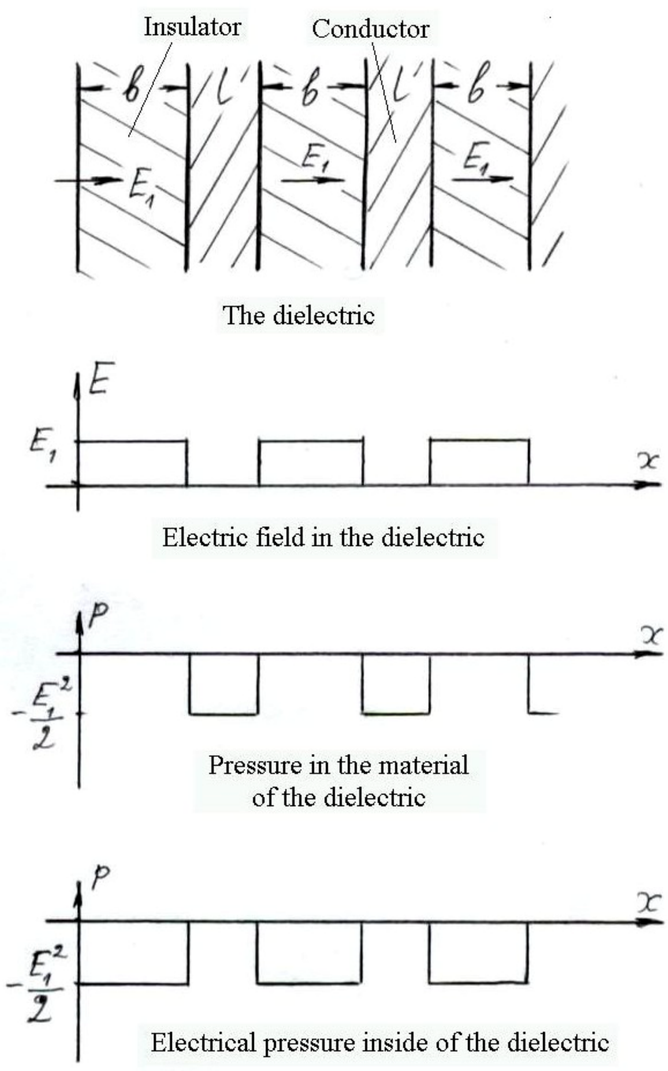


Fig. 7.
The schistose dielectric in external electric field

IV. Mechanical stresses in dielectric. Field is normal to the surface

If a dielectric is in a uniform external electric field E_1, in the frame of using model, this field stretches the conducting sheets, i.e. causes a negative pressure along the *x*-axis which corresponds to the component of the Maxwell stress tensor

$$T_1^{xx} = -E_1^2/2 = -\varepsilon^2 E^2/2\,. \tag{4.1}$$

This field does not produce strength in the isolating sheets. Thus, the average mechanical stress in the material, i.e. pressure, i.e. the corresponding component of the mechanical stress tensor, is

$$p_m = T_m^{xx} = \frac{l}{b+l}T_1^{xx} = -\frac{\varepsilon^2 E^2 l}{2(b+l)} = -\frac{\varepsilon(\varepsilon-1)E^2}{2}\,. \tag{4.2}$$

Since surface polarization charge density is

$$\sigma_p = -(\varepsilon - 1)E\,, \tag{4.3}$$

the pressure (4.2) cannot be found as the product $\sigma_p E_f$, where $E_f = (E_1 + E)/2$ is the free field (which will remain if the polarization charges σ_p is removed). We have the wrong result, according to this paradigm,

$$dF/da = \sigma_p E_f = -(\varepsilon^2 - 1)E^2/2\,; \tag{4.4}$$

instead of (4.2), and according to eqn. (1.7) we have a wrong result as well:

$$dF/da = \sigma_p E = -(\varepsilon - 1)E^2/2\,. \tag{4.5}$$

At the same time, the negative electric pressure (4.1) is contained in the isolating sheets. Thus, the average electrical stress in the dielectric along the *x*-axis, i.e. the corresponding component of the average Maxwell stress tensor in a dielectric, is

$$T_e^{xx} = \frac{b}{b+l}T_1^{xx} = -\frac{\varepsilon^2 E^2 b}{2(b+l)} = -\frac{\varepsilon E^2}{2}\,. \tag{4.6}$$

Naturally, this quantity is coincided with energy density (3.4) in a magnitude. The total pressure in the dielectric,

$$\underset{m}{T}^{xx} + \underset{e}{T}^{xx} = -E_1^2/2 = -\varepsilon^2 E^2/2\,, \tag{4.7}$$

equals the Maxwell tensor component for the external field E_1 (4.1). So we have

$$\underset{m}{T}^{xx} + \underset{e}{T}^{xx} = T_1^{xx} \tag{4.8}$$

at a free surface of a dielectric.

If the field E_1 is produced by condenser plates, the plates experience mutual attractive forces corresponding to the component of Maxwell tensor (4.1). If the plates lean on the dielectric, the plates compress the dielectric with the pressure $E_1^2/2 = \varepsilon^2 E^2/2$, and a bonded strain gage will show the corresponding force. Under this condition the total pressure in the dielectric is zero, according to the conducting sheets, which are stretched in the absence of the condenser plate, have no stress, and the insulation is mechanically compressed but contains stretching of the electric field.

V. Maxwell stress tensor in dielectric

Average pressure in the *y*-direction (and *z*-direction) of the electric field inside the dielectric is, in accordance with (4.6),

$$\underset{e}{T}^{yy} = \varepsilon E^2/2\,. \tag{5.1}$$

So, energy-momentum tensor of the electric field in dielectric has the components (3.4), (4.6), (5.1):

$$\underset{e}{T}^{00} = \varepsilon (E^x)^2/2,\quad \underset{e}{T}^{xx} = -\varepsilon (E^x)^2/2,\quad \underset{e}{T}^{yy} = \varepsilon (E^x)^2/2,\quad \underset{e}{T}^{zz} = \varepsilon (E^x)^2/2\,, \tag{5.2}$$

where E^x is the fictitious, average electric field, which is x-directed. Spatial components of (5.2) can be combined into the Maxwell stress tensor in dielectric:

$$\underset{e}{T}^{ij} = \varepsilon(-E^i E^j + g^{ij} E^2/2)\,. \tag{5.3}$$

It is remarkable that the simple Feynman's model of dielectric gives this important expression.

If our dielectric is bounded by the surface $y = 0$, as in Fig. 1, then the outside pressure of the external field E, T_1^{yy}, is less then (5.1),

$$T_1^{yy} = E^2/2\,. \tag{5.4}$$

So, material of our dielectric must be stretched in the *y*-direction, i.e. the pressure T^{yy} exerts on the surface, although there are no bound charges at the surface:

$$\underset{m}{T}^{yy} = T_1^{yy} - \underset{e}{T}^{yy} = -(\varepsilon - 1)E^2/2 \tag{5.5}$$

This result gives the force (1.5) acting on the slab.

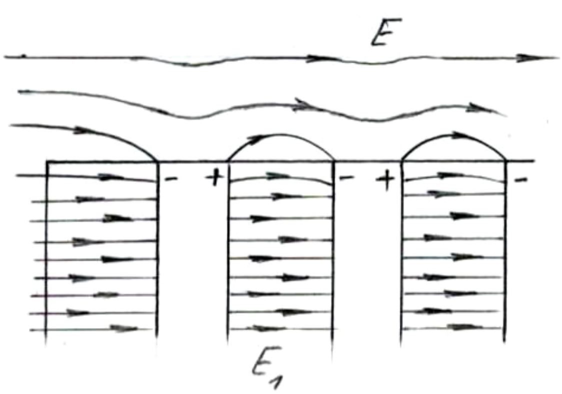


Fig. 8.
Electric field near the butt-end of the slab

The beginnings of the pressure (5.5) are explained in Fig. 8. Lines of electric field become bent near the butt-end of the slab and pull the material upwards.

The relations (5.5), (4.8) for the stress tensor $\underset{m}{T}^{ij}$ can be obtained in a general form. If ε has a jump at a surface of dielectric, we encircle an element da_i of the surface by a closed tablet-like surface which bounds an infinitesimal volume $dV = da_i dl^i$. Since minus divergence of a stress tensor is a volume force, the force acting on the volume dV is

$$dF^i = -\int \partial_j \underset{e}{T}^{ij} dV = -\oint \underset{e}{T}^{ij} da_j = -T_1^{ij} d\underset{1}{a}_j - \underset{e}{T}^{ij} da_j = (T_1^{ij} - \underset{e}{T}^{ij}) da_j = \underset{m}{T}^{ij} da_j \tag{5.6}$$

because $d\underset{1}{a}_j = -da_j$. (Here $d\underset{1}{a}_j$ denotes the surface element in vacuum, and T_1^{ij} denotes the stress tensor in vacuum, when $\varepsilon = 1$).

Now we prove that the force (5.6) acting on a surface of dielectric is normal to the surface. We start from an expression ρE^i where ρ is free charge density. We have

$$\rho E^i = \partial_j(\varepsilon E^j)E^i = \partial_j \varepsilon E^j E^i + \varepsilon \partial_j E^j E^i . \qquad (5.7)$$

But

$$\partial_j E^j E^i = \partial_j (E^j E^i) - E^j \partial_j E^i = \partial_j (E^j E^i) - E^k \partial_k E_j g^{ij} = \partial_j (E^j E^i) - E^k \partial_j E_k g^{ij} = \partial_j (E^j E^i - g^{ij} E^2 / 2) \qquad (5.8)$$

Substituting (5.8) into (5.7) yields

$$\rho E^i = \partial_j \varepsilon E^j E^i + \varepsilon \partial_j (E^j E^i - g^{ij} E^2/2) = \partial_j \varepsilon E^j E^i + \partial_j [\varepsilon (E^j E^i - g^{ij} E^2/2)] - \partial_j \varepsilon (E^j E^i - g^{ij} E^2/2)$$
$$= \partial_j [\varepsilon (E^j E^i - g^{ij} E^2/2)] + \partial^i \varepsilon E^2/2 . \qquad (5.9)$$

Thus the volume force acting on dielectric is

$$f^i = -\partial_j \underset{e}{T}^{ij} = -\partial_j [\varepsilon(-E^j E^i + g^{ij} E^2/2)] = -\partial^i \varepsilon E^2/2 + \rho E^i , \qquad (5.10)$$

or (see (1.9))

$$\mathbf{f} = -E^2 \mathrm{grad}\varepsilon/2 + \rho \mathbf{E} . \qquad (5.11)$$

Since $\mathrm{grad}\varepsilon$ is normal to a surface of dielectric and $\rho = 0$ at the surface, $\mathbf{f}$ from (5.11) and dF^i from (5.6) are normal to the surface, Q.E.D.

Importantly! In reality, electric field in a neighborhood of a dielectric surface determines the Maxwell tensor (5.3) in vacuum $(\varepsilon = 1)$ and in the dielectric, but does not determine material stress tensor $\underset{m}{T}^{ij}$. We will break the relations (5.5), (4.8) by our hands if we apply mechanical stress to the dielectric. But the surface force (5.6) is determined by the electric field; the transvection of $\underset{m}{T}^{ij}$ and da_j is determined by (5.6).

VI. Force acting on a surface of dielectric

Now we calculate the pressure on a surface of dielectric in the case when the exterior field $\mathbf{E}_1$ enters dielectric at angle α with the normal, which is parallel to *x*-axis. The components of the field near and inside of the dielectric are

$$E_1^x = E_1 \cos\alpha, \quad E_1^y = E_1 \sin\alpha , \quad E^x = (E_1 \cos\alpha)/\varepsilon, \quad E^y = E_1 \sin\alpha . \qquad (6.1)$$

Thus, components of the Maxwell tensors, according to (5.3), are

$$T_1^{xx} = E_1^2(-\cos^2\alpha + \sin^2\alpha)/2, \quad T_1^{yx} = -E_1^2(\cos\alpha \sin\alpha),$$
$$\underset{e}{T}^{xx} = E_1^2(-\cos^2\alpha + \varepsilon^2 \sin^2\alpha)/\varepsilon, \quad \underset{e}{T}^{yx} = -E_1^2(\cos\alpha \sin\alpha). \qquad (6.2)$$

Substituting of (6.2) into

$$T_1^{ij} - \underset{e}{T}^{ij} = \underset{m}{T}^{ij} \qquad (6.3)$$

yields

$$\underset{m}{T}^{xx} = E_1^2[-\cos^2\alpha(1-1/\varepsilon) + \sin^2\alpha(1-\varepsilon)]/2, \quad \underset{m}{T}^{yx} = 0 . \qquad (6.4)$$

Thus

$$dF^x = \underset{m}{T}^{xx} da_x = E_1^2[-\cos^2\alpha(1-1/\varepsilon) + \sin^2\alpha(1-\varepsilon)]da/2, \quad dF^y = \underset{m}{T}^{yx} da_x = 0 , \qquad (6.5)$$

and the pressure on a surface of dielectric is

$$p_m = \underset{m}{T}^{xx} = E_1^2[-\cos^2\alpha(1-1/\varepsilon) + \sin^2\alpha(1-\varepsilon)]/2 . \qquad (6.6)$$

If $\alpha = 0$, $p_m = E_1^2(1/\varepsilon - 1)/2$ in accordance with (4.2). If $\alpha = 90^\circ$, $p_m = E_1^2(1-\varepsilon)/2$ in accordance with (5.5) because E_1 stands here for E.

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