Inevitability of the electrodynamics' spin tensor

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Theoretical reasons and results of the works: Phys. Rev. **A68** 013806 (2003), Opt. Lett. **22** 52 (1997), Optics Express **14** 6963 (2006), Phys. Rev. Lett. **92** 198104 (2004), Phys. Rev. Lett. **91** 093602 (2003), Phys. Rev. Lett. **88** 053601 (2002) prove that the angular momentum flux carried by a circularly polarized light beam with plane phase front equals two power of the beam divided by the frequency. This fact contradicts the standard electrodynamics, which predicts the beam's angular momentum flux equals power of the beam divided by frequency, and means the electrodynamics is incomplete. To correct the electrodynamics, a spin tensor is used.

1. Does electrodynamics' spin tensor exist?

As is well known, photons carry spin, energy, momentum and angular momentum that is a moment of the momentum relative to a given point or to a given axis. Energy and momentum of electromagnetic waves are described by the Maxwell energy-momentum tensor (density)

$$T^{\lambda\mu} = -g^{\lambda\alpha}F_{\alpha\nu}F^{\mu\nu} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4\,, \qquad (1.1)$$

where $F^{\mu\nu} = -F^{\nu\mu}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor. For example, $P^i = \int_V T^{i0}dV$ is the momentum of a waves inside of the volume *V*, and $dW = \int_a T^{0i}da_i dt$ is the energy that has flowed through the area *a* in the time dt. The angular momentum that is a moment of the momentum can be defined as[1]

$$L^{ij} = \int_V 2x^{[i}T^{j]0}dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV\,, \qquad (1.2)$$

and this construction is known as an orbital angular momentum. But the modern electrodynamics has no describing of spin. Sometimes physicists consider the canonical spin tensor

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \underset{c}{\mathsf{L}}}{\partial(\partial_\nu A_\alpha)} = -2A^{[\lambda}F^{\mu]\nu}\,, \qquad (1.3)$$

where $\underset{c}{\mathsf{L}} = -F_{\mu\nu}F^{\mu\nu}/4$ is the canonical Lagrangian, and A^λ is the magnetic vector potential, $2\partial_{[\mu}A_{\nu]} = F_{\mu\nu}$. But spin tensor (1.3) is invalid, and physicists eliminate it by the Belinfante-Rosenfeld procedure[2,3]. As a result, the electrodynamics has no spin tensor, or rather the modern classical electrodynamics spin tensor equals zero.

Nevertheless, physicists understand they cannot shut eyes on existence of the electrodynamics spin. And they proclaim spin is *in* the moment of the momentum (1.2). I.e., the moment of momentum represents the total angular momentum, orbital angular momentum plus spin:[4 – 8]

$$J^{ij} = L^{ij} + S^{ij} = \int_V 2x^{[i}T^{j]0}dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV\,, \qquad (1.4)$$

Contrary to this paradigm, we introduce a spin tensor $\Upsilon^{\lambda\mu\nu}$ into the modern electrodynamics,[9 – 12] i.e. we complete the electrodynamics by introducing the spin tensor, i.e. we claim the total angular momentum consists of the moment of momentum (1.2) *and* a spin term, i.e. we claim equation (1.4) is wrong, i.e. we state the moment of momentum does not contain spin at all:

$$J^{ij} = L^{ij} + S^{ij} = \int_V (2x^{[i}T^{j]0} + \Upsilon^{ij0})dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV + \int_V \Upsilon^{ij0}dV\,, \qquad (1.5)$$

The sense of the spin tensor $\Upsilon^{\lambda\mu\nu}$ is as follows. The component Υ^{ij0} is a volume density of spin. This means that $dS^{ij} = \Upsilon^{ij0}dV$ is the spin of electromagnetic field inside the spatial element dV. The component Υ^{ijk} is a flux density of spin flowing in the direction of the x^k axis. For example, $dS_z/dt = dS^{xy}/dt = d\tau^{xy} = \Upsilon^{xyz}da_z$ is the *z*-component of spin flux passing through the surface element da_z per unit time, i.e. the torque acting on the element. The explicit expression for the spin tensor is

$$\Upsilon^{\lambda\mu\nu} = A^{[\lambda}\partial^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]}\,, \qquad (1.6)$$

where A^{λ} and Π^{λ} are magnetic and electric vector potentials which satisfy $2\partial_{[\mu}A_{\nu]} = F_{\mu\nu}$, $2\partial_{[\mu}\Pi_{\nu]} = -e_{\mu\nu\alpha\beta}F^{\alpha\beta}$, where $F^{\alpha\beta} = -F^{\beta\alpha}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor of a free electromagnetic field. A relation between Π and F can be readily obtained in the vector form as follows. If $\mathrm{div}\mathbf{E} = 0$, then $\mathbf{E} = \mathrm{curl}\Pi$. And if $\partial\mathbf{E}/\partial t = \mathrm{curl}\mathbf{H}$, then $\partial\Pi/\partial t = \mathbf{H}$. This reasoning is analogous to the common: if $\mathrm{div}\mathbf{B} = 0$, then $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$. And if $\partial\mathbf{B}/\partial t = -\mathrm{curl}\mathbf{E}$, then $\partial\mathbf{A}/\partial t = -\mathbf{E}$.

The difference between our statement (1.5) and the common equation (1.4) is verifiable. The cardinal question is, what angular momentum flux, i.e. torque τ, does a circularly polarized light beam of power P without an azimuth phase structure carry? The common answer, according to (1.4), is

$$\tau = dJ/dt = \mathsf{P}/\omega; \tag{1.7}$$

our answer, according to (1.5), is

$$\tau = dJ/dt = 2\mathsf{P}/\omega. \tag{1.8}$$

Statements (1.5) & (1.8) are also valid in the case of plane waves or a beam which is much larger than the particle under action if P is the power absorbed by the particle.

To verify our statements (1.5), (1.8), we use the angular momentum conservation law. We have calculated the torque acting on a dielectric absorbing the beam. We use the standard formula

$$\tau = \int[\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B}) + \mathbf{P}\times\mathbf{E}]dV \tag{1.9}$$

[see, for example,[7] eqns. (5.1) & (7.18)]. Here $\mathbf{P} = (\varepsilon - 1)\mathbf{E}$ is the polarization, $\mathbf{j} = \partial_t\mathbf{P}$ is the displacement current, $\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})$ is the moment of the total Lorentz force per unit volume, and $\mathbf{P}\times\mathbf{E}$ is the torque on electric dipoles per unit volume.[13] The point is the accurate calculation gives the torque (1.8).[12] At that, we have had for the first two terms and for the last term

$$\int[\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})]dV = \int\mathbf{P}\times\mathbf{E}\,dV = \mathsf{P}/\omega. \tag{1.10}$$

Loudon[7] calculated the torque exerted by a light beam on a dielectric as well. He used the formula (1.9) as well, and he obtained

$$\int[\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})]dV = \mathsf{P}/\omega \tag{1.11}$$

[see his formulae (7.19) – (7.24)]. But he omitted $\mathbf{P}\times\mathbf{E}$ term without explanations, and P/ω was his finish result for the torque. Taking into account the $\mathbf{P}\times\mathbf{E}$ term, he must obtain our result $2\mathsf{P}/\omega$.

2. Experimental verification

The work of Simpson at al.[14] rather confirms our result (1.5), (1.8) as well. The authors trapped ~2-μm-diameter Teflon particles by a $\mathrm{LG}_{p=0}^{l=1}$ beam of λ = 1064 nm and power P = 25 mW. If the $\mathrm{LG}_{p=0}^{l=1}$ beam is linearly polarized, it carries an orbital angular momentum flux of $\mathsf{P}/\omega = 1.3\cdot 10^{-17}$ J. In this case the trapped particles were rotated with the rotational rate ~1 Hz. This implies that the torque on the particles equaled $\tau = 8\pi\eta r^3\Omega = 1.6\cdot 10^{-19}$ J (formula (3) from,[14] here $\eta = 10^{-3}$ kg/m sec is the viscosity, $r = 10^{-6}$ m is the particle radius, and $\Omega = 2\pi/\mathrm{sec}$). Because $\tau = 0.012\mathsf{P}/\omega$, it was concluded that the particles absorbed approx 1.2% of the beam. However, this conclusion probably needs to be corrected. The point is a Laguerre-Gaussian beam can exert a torque on particles not only when absorbing, but also when being converted into Hermite-Gaussian beams.

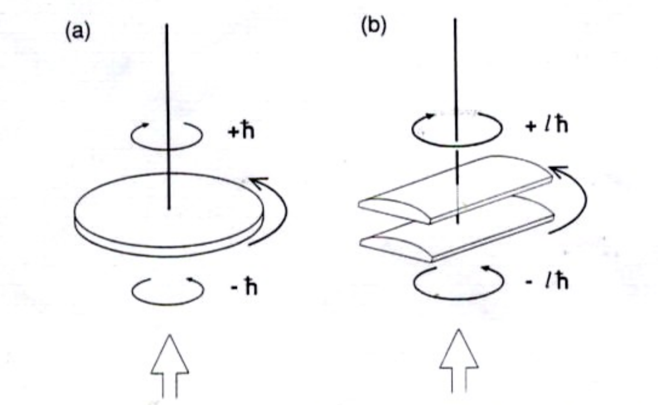


FIG. 1. (a) A suspended $\lambda/2$ birefringent plate undergoes torque in transforming right-handed into left-handed circularly polarized light. (b) Suspended cylindrical lenses undergo torque in transforming a Laguerre-Gaussian mode of orbital angular momentum $-l\hbar$ per photon, into one with $+l\hbar$ per photon.

Allen et al. show that a torque exerts on a converter of a Laguerre-Gaussian beam when converting (Fig. 1 from[15]), because the converter change the phase difference between the Hermite-Gaussian modes that constitute the Laguerre-Gaussian beam (Fig. 13

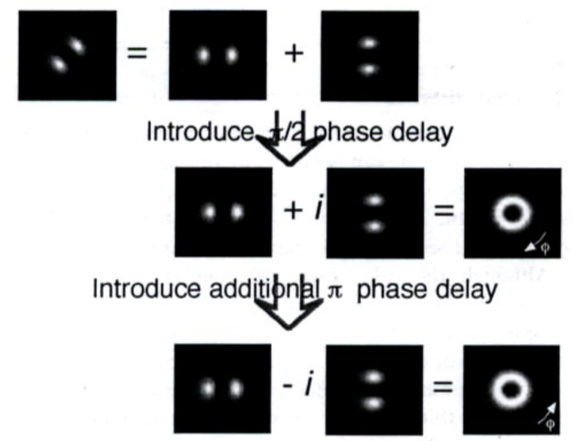
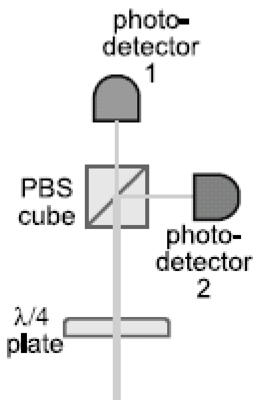


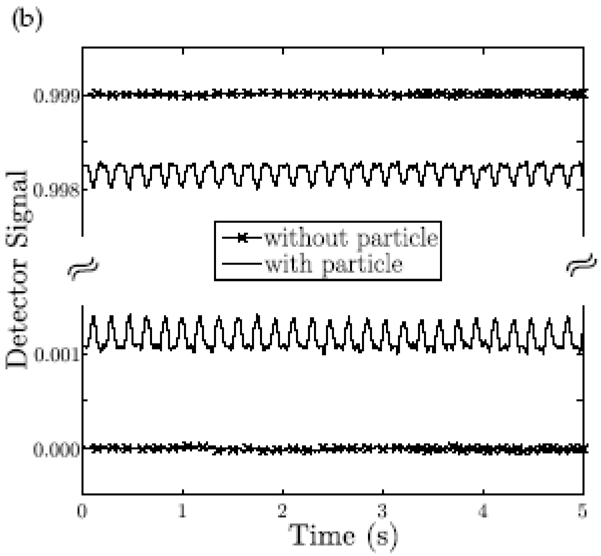
Figure 13. Two orthogonal Hermite–Gaussian modes can be added to give a Hermite–Gaussian mode at 45° or added with a phase delay to give a Laguerre–Gaussian mode.

from[16]). Because the particles had an irregular form, and because ~99% of $LG^{l=1}_{p=0}$ beam passed through the particles in the experiment, it was very possibly that at least 0.6% of the $LG^{l=1}_{p=0}$ beam were converted into HG modes. In this case, the absorption of 0.6% only, instead of 1.2%, could provide the torque $\tau = 1.6 \cdot 10^{-19}$ J.

The main point of the Simpson's experiment[14] was a cessation of rotating of the particles when the linearly polarized $LG^{l=1}_{p=0}$ beam became a circularly polarized one if the handedness was opposite to the rotation sense. Thus, we must conclude that the torque associated with the circular polarization equals $2P/\omega$ because $\tau = 0.006 \cdot 2P/\omega$. In any case, because of the possible $LG \to HG$ conversion, we must conclude that the circular polarization is related with an angular momentum flux which is larger than P/ω.

The recent work[17] confirms rather the formula (1.5) as well. In this work a linearly polarized $LG^{l=2}_{p=0}$ beam of λ = 1064 nm and power P = 20 mW rotates a trapped particle with the rotational rate 2.4 Hz, and, when circularly polarized, the beam rotates the particle with 2.9 Hz. This increase in the angular velocity, $\Delta\Omega$ = $2\pi\,0.5$/sec, causes the corresponding increase in the drag torque acting on the rotating particle (formula (3) from[17]): $\Delta\tau = 12\pi\eta a^3 \Delta\Omega = 1.2 \cdot 10^{-19}$ J (here $a = 10^{-6}$ m is the particle parameter). On the other hand, the increase in the drag torque is provided with change in the degree of circular polarization σ of the beam as the beam passes through the particle. This change is determined by signals of photo-detectors 1 and 2 (see the fragment of Fig. 1 from[17] here).





The point is an elliptically polarized beam consists of right and left circularly polarized constituents. The electrical field may have the form

$$\mathbf{E} = \exp(ikz - i\omega t)[r(\mathbf{x} + i\mathbf{y}) + l(\mathbf{x} - i\mathbf{y})]E_0/\sqrt{2}, \qquad (2.1)$$

where $rE_0/\sqrt{2}$ and $lE_0/\sqrt{2}$ are the amplitudes of the circularly polarized constituents. The degree of circular polarization of the beam is defined as

$$\sigma = \frac{r^2 - l^2}{r^2 + l^2}. \qquad (2.2)$$

To determine *r* and *l*, the authors send the beam to a circular polarization detection system consisting of the $\lambda/4$ plate, the polarizing beam splitter cube, and the photo-detectors. The $\lambda/4$ plate converts a circularly polarized constituent to a linearly polarized one by introducing $\pi/2$ phase shift of *y*-components, i.e. by multiplying the *y*-components in (2.1) by *i*.

$$\mathbf{E} = \exp(ikz - i\omega t)[r(\mathbf{x} + i\mathbf{y}) + l(\mathbf{x} - i\mathbf{y})]E_0/\sqrt{2} \to \mathbf{E} = \exp(ikz - i\omega t)[r(\mathbf{x} - \mathbf{y}) + l(\mathbf{x} + \mathbf{y})]E_0/\sqrt{2}. \qquad (2.3)$$

According to Figure (b) from,[17] the input polarization is 0.999, and the output polarization is 0.9982 – 0.0012 = 0.997. I.e. $\Delta\sigma$ = 0.002. These results mean that $\Delta\sigma P/\omega \cong 0.2 \cdot 10^{-19}$ J (here P = 20 mW and $\omega = 2\pi c/\lambda = 1.9 \cdot 10^{15}$/sec). So, we have, according to,[17] $\Delta\tau \cong 6\Delta\sigma P/\omega$ instead of $\Delta\tau = 2\Delta\sigma P/\omega$, according to eqn. (1.8), and instead of $\Delta\tau = \Delta\sigma P/\omega$, according to eqn. (1.7). This sizeable polarization contribution to the total torque confirms our statement (1.8).

We tried to confirm our formula (1.8) by the work,[8] but we could not find data in the paper. For example, FIG. 2 of the paper shows the angular velocity of a trapped birefringent particle was $\Omega = 94 \cdot 2\pi = 590/\mathrm{sec}$ when the output polarization of the beam was $\sigma = 0.9 - 0.1 = 0.8$, i.e. $\Delta\sigma = 0.2$, but the radius of the particle and the power of the beam were not given. However, as one can understand from the text and from FIG. 3 of the paper, the radius was $r = 1.2\,\mu\mathrm{m}$ and the power was P = 100 mW. From this assumption we get $(1-\sigma)\mathsf{P}/\omega = 10^{-17}$ J and $\tau = 8\pi\eta r^3\Omega = 2.7 \cdot 10^{-17}$ J. So, $\tau = 2.7\Delta\sigma\mathsf{P}/\omega$, which is rather in accordance with (1.8).

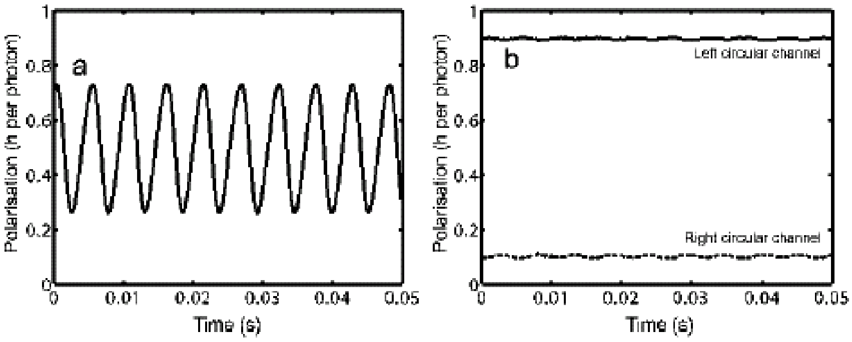


FIG. 2: a. Signal recorded by the linear polarization measurement apparatus during rotation of a vaterite crystal. The frequency of rotation is (94.0 ± 0.7) Hz. b. Signal traces from the circular polarization measurement apparatus during rotation of a vaterite crystal.

At the same time we were puzzled by the fact that “the rotation may be stopped by aligning the $\lambda/4$ plate located before the objective so that the polarization is made linear.” It seems that if a birefringent particle converts a circularly polarized beam to partially linear polarized one and is rotated, the particle must convert a linearly polarized beam to partially circularly polarized one and be rotated as well. Unfortunately, the authors did not measure the degree of output circular polarization when input polarization was linear.

We are interested in works that show how a particle rotates simultaneously around its own axis (due to spin) and around the beam’s axis (due to orbital angular momentum). So we consider the paper.[18] As is shown in Fig. 1 of the paper (see a reproduction here), a particle of a radius approx $r = 1\ \mu\mathrm{m}$ rotates around its own axis with rotational rate $\Omega_{\mathrm{spin}} = 18/\mathrm{sec}$ and around the beam’s axis with rotational rate $\Omega_{\mathrm{orbit}} = 2.4/\mathrm{sec}$ along a circle of radius $R = 2.9\ \mu\mathrm{m}$. The beam is a high-order J_2 Bessel beam ($l = 2$). The azimuthal component of the linear momentum density, $\omega l u^2/R$, yields the azimuthal force on the particle $F_\phi = \omega l u^2 \pi r^2/R$. If we use the Stokes’s law, $D = 6\pi\eta r v$, for the particle, we obtain $\Omega_{\mathrm{orbit}} = \omega l u^2 r/6\eta R^2$.

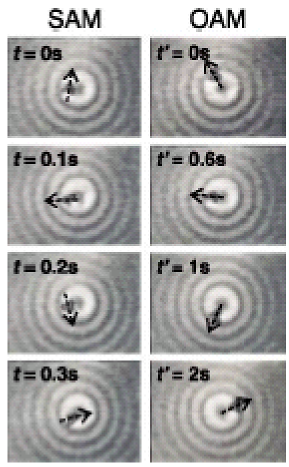


FIG. 1 (color online). A birefringent particle trapped in the first ring of a HOBB rotates simultaneously (i) around its own axis (due to SAM) and (ii) around the beam’s axis (due to OAM). The frames were taken from a video at the time indicated in each box.

At the same time, the quantity (1.7) for the spin torque is $\tau = \mathsf{P}/\omega = \omega u^2\sigma\pi r^2$. If we use formula (3) from [14], $\tau = 8\pi\eta r^3\Omega_{\mathrm{spin}}$, we obtain $\Omega_{\mathrm{spin}} = \omega u^2\sigma/8\eta r$. So we obtain the ratio $\Omega_{\mathrm{spin}}/\Omega_{\mathrm{orbit}} = 3R^2/8r^2 = 3.2$, but in reality the ratio is $\Omega_{\mathrm{spin}}/\Omega_{\mathrm{orbit}} = 18/2.4 = 7.5$. However, if we use our formula (1.8), $\tau = 2\mathsf{P}/\omega$, instead of (1.7), we obtain $\Omega_{\mathrm{spin}}/\Omega_{\mathrm{orbit}} = 6.4$, instead of 3.2, what confirms our theory.

Authors of the interesting work[19] also deal with probe particles, which rotates around their own axises and around the beam’s axis. Unfortunately, this work is not quantitative one. Nevertheless, this work confirms an extremely sizeable contribution from the circular polarization of a beam. The authors watched a rotation of a calcite fragment around its own axis due to σ and could not observe this fragment orbiting though they used a Laguerre-Gaussian beam of $l = 8$ $(LG_{p=o}^{l=8})$.

Conclusions, Notes and Acknowledgements

This paper conveys new physics. We review existing works concerning electrodynamics spin and indicate that existing theory is insufficient to solve spin problems because spin tensor of the modern electrodynamics is zero. Then we show how to resolve the difficulty by introducing a true electrodynamics spin tensor. Our spin tensor, in particular, doubles a predicted angular momentum of a circularly polarized light beam without an azimuth phase structure

The idea of the classical spin and the concrete expression were rejected more than 350 times by scientific journals since the rejection by "JETP Letters" on May 21, 1998." For example (I show an approximate number of the rejections in parentheses): JETP Lett. (8), JETP (13), TMP (10), UFN (9), RPJ (75), AJP (16), EJP (4), EPL (5), IJTP (1), JOSAA (2), JOSAB (4), PRA (6), PRD (4), PRE (2), PRL (2), APP (5), FP (6), PLA (9), OC (5), JPA (4), JPB (1), JMP (6), JOPA (4), JMO (2), CJP (1), OL (5), NJP (5), MPEJ (3), arXiv (75). My submission to the 2007 CLEO/QELS Conference was rejected on February 28, 2007.

I am deeply grateful to Professor Robert H. Romer for publishing my question[20] (was submitted on Oct. 7, 1999) and to Professor Timo Nieminen for valuable discussions (Newsgroups: sci.physics.electromag).

REFERENCES

1. W. Heitler, *The Quantum Theory of Radiation*, (Clarendon, Oxford, 1954), p. 401.
2. F. J. Belinfante, Physica 6, 887 (1939).
3. L. Rosenfeld, “Sur le Tenseur D’Impulsion-Energie.” Memoires de l'Academie Royale des Sciences de Belgiques **8** No 6 (1940).
4. J. D. Jackson, *Classical Electrodynamics*, (John Wiley, 1999), p. 350.
5. D. E. Soper, *Classical Field Theory* (N.Y.: John Wiley, 1976).
6. F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Mass. 1965)
7. R. Loudon, “Theory of the forces by Laguerre-Gaussian light beams on dielectrics.” Phys. Rev. A68, 013806 (2003)
8. A. I. Bishop, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, “Optical Microrheology Using Rotating Laser-Trapped Particles.” Phys. Rev. Lett. **92**, 198104 (2004).
9. R.I. Khrapko. “True energy-momentum tensors are unique. Electrodynamics spin tensor is not zero,” physics/0102084
10. R. I. Khrapko, “Experimental verification of Maxwellian electrodynamics,” Measurement Techniques, **46**, No. 4, 317-321 (2003)
11. R. I. Khrapko, “Classical spin in space with and without torsion,” Gravitation & Cosmology, **10**, No. 1-2, 91-98 (2004)
12. R. I. Khrapko, “Absorption of a circularly polarized beam in a dielectric, *etc*.” www.sciprint.org (2006)
13. R. A. Beth, “Mechanical detection and measurement of the angular momentum of light” Phys. Rev. **50**, 115 (1936).
14. N. B. Simpson, K. Dholakia, L. Allen, and M. J. Padgett, “Mechanical equivalence of spin and orbital angular momentum of light: an optical spanner,” Opt. Lett. **22**, 52-54 (1997)
15. L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, “Orbital angular momentum of light and the transformation of Laguerre-Gaussian modes,” Phys. Rev. **A45**, 8185 (1992)
16. M. Padgett, L. Allen, “Light with a twist in its tail,” Contemp. Phys. **41**, 275-285 (2000)
17. S Parkin, G Knoner, T. A. Nieminen, N. R. Heckenberg and H. Rubinsztein-Dunlop, “Measurement of the total optical angular momentum transfer in optical tweezers,” Optics Express **14** 6963 (2006).
18. V. Garces-Chavez, D. McGloin, M. J. Padgett, et al., Phys. Rev. Lett., **91**, 093602 (2003).
19. A. T. O’Neil et al., “Intrinsic and Extrinsic Nature of the Orbital Angular Momentum of a light beam,” Phys. Rev. Lett. **88**, 053601 (2002).
20. R. I. Khrapko, “Does plane wave not carry a spin?” Amer. J. Phys., 69, 405 (2001).