On the Elliptic Integrals and the Arc Length of an Ellipse

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Abstract

Elliptic integral is an integral equation that appears in the process of calculating the arc length of an ellipse. It does not provide an exact solution, and the approximation equation for the solution is complicated. The arc length of an ellipse is simply given as $l = a\theta$.

If the coordinates of an ellipse are expressed in polar coordinates, it can be written as

$$x = a\cos\theta, \;\; y = b\sin\theta. \tag{1}$$

And from this, the ellipse is given as follows,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2}$$

The arc length of an ellipse is given as follows,

$$\begin{aligned} l &= \int_0^{\phi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\; d\theta \\ &= \int_0^{\phi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\; d\theta . \end{aligned} \tag{3}$$

If we apply the eccentricity $\left(k = \sqrt{1 - \frac{b^2}{a^2}}\right)$ we get the following equation, which

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is called the elliptic integral of the second kind,

$$l = aE_k = a\int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\, d\theta. \tag{4}$$

Here "k" is the eccentricity.

The solution of this integral cannot be obtained by direct integration, and the approximation equation is very complicated, too.

However, from equations (3) and (1), we can simply obtain the solution of the above integral as follows,

$$\begin{aligned} l &= \int_0^{\phi} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\ d\theta \\ &= \int_0^{\phi} \sqrt{x^2 + y^2}\, d\theta \\ &= r\phi, \end{aligned} \tag{5}$$

because $x^2 + y^2 = r^2$ represents a circle with the radius of r.

It can be seen that the arc length of an ellipse is the same as the arc length of a circle.

The length of an elliptical arc can be obtained by using the semi-major axis "a" of an ellipse instead of r in the equation (5) and substituting $\phi = \pi$, we get

$$l = a\pi. \tag{6}$$

The area of an ellipse is calculated in the same way as the area of a circle,

$$S = \int_0^{b} a\pi\, dx = ab\pi, \tag{7}$$

where "b" represents the semi-minor axis of the ellipse.

The result is already known as the area of an ellipse.