The elementary solution of the Navier-Stokes existence and smoothness with uniqueness

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This article is written as mathematical conjecture. It is a challenge to build a elementary theory without semi-group theory or apriori estimates of the Navier-Stokes equation. If you have any ideas or questions, please contact to MasatoshiOhrui1993@gmail.com . I'm also looking for people to study together.

I thought about the uniqueness and smoothness of the weak solution, which was unsolved in the Leray-Hopf's weak solution. I thought of a elementary argment in the sense that there are no long or complicated calculations, and the theory of evolution equations is not used at all. The existence of the solution is actually known, and the proof that already exists is very wonderful. For example, Fujita-Kato Theory, Shibata Theory: Takayoshi Ogawa [26], Yoshihiro Shibata [22], Shibata-Kubo [24], Kakita-Shibata [3], Okamoto [20]. But I don't think these are elementary. Also, I'm not good at complex calculations, so I want to say the existence of solutions without calculating too much, specifically, "Fundamental theorem of distributions with compact support":

"The fundamental solution of any linear partial differential operator with constant coefficients L on $\mathbb{R}^N$, that is, $E \in \mathcal{D}'$ that satisfies $LE = \delta$, the distribution with compact support $f \in \mathcal{E}'$ or the C^∞ -function $f \in C_0^\infty$, one of the solutions of the equation $Lu = f$ is $u = E * f \in \mathcal{D}'$ or $u = E * f \in C^\infty$.

Here if $f \in \mathcal{E}'$ then $\langle E * f, \varphi \rangle = \langle E(x), \langle f(y), \varphi(x+y) \rangle \rangle,$

if $f \in C_0^\infty$ then $(E * f)(x) = \langle E(y), f(x-y) \rangle$ ".

I thought about it as an application of real analysis and "fundamental theorem of distributions with compact support".

The policy is, let L be the heat operator $\partial_t - \Delta$ in the Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u = f - \nabla \mathfrak{p} - (u \cdot \nabla) u \\ \operatorname{div} u = 0 \end{cases},$$

erase the pressure $\mathfrak{p}$ and to approximate the nonlinear term $(u\cdot\nabla)u$ by a sequence of smooth functions, use the fundamental theorem for the difference between the external force f and the approximation term, and show that the limit in Sobolev space is the solution.

[definition of symbols]
For convenience , write the index of the component of the vector in the upper right corner. "Function space" and "space" are abbreviations for "linear topological space" (of functions or distributions), other than pressure $\mathfrak{p}$ are $\mathbb{R}^3$ -values. The absolute value of the function in the norm of normal function space is interpreted as the length of the number vector (the absolute value of $\mathbb{R}^3$) in the norm of the space of the $\mathbb{R}^3$ -value function. We write the space of the real numeric function and the space of the $\mathbb{R}^3$ -value function in the same symbol to make the symbol easy. For any positive number δ , let $B_\delta(0,y)$ be the δ -neighborhood of point $(0,y)$. Let Ω be a bounded open set contained in $\mathbb{R}\times\mathbb{R}^3$ whose for any $y\in\mathbb{R}^3$, there exists δ such that $B_\delta(0,y)\cap\Omega=\emptyset$ and have smooth boundary. Let $t_0=\inf\{s\in\mathbb{R}:\exists y\in\mathbb{R}^3,(s,y)\in\overline{\Omega}\}$. Let $|\Omega|$ be its Lebesgue measure. Let χ_Ω be the characteristic function on Ω , the support compact and the divergence for special valuables 0 . For any natural number $m>\max\{0+4/1,0+4/2\}=4$, $p=1,2$,
let $V^{m,p}_\sigma(\Omega)=\{u\in C^\infty(\Omega):\|u\|_{W^{m,p}(\Omega)}<\infty,\operatorname{div}u=0\}$,
$W^{m,p}_\sigma(\Omega)$ be the Sobolev space defined by
$V^{m,p}_\sigma$'s completion by norm of $W^{m,p}_\sigma(\Omega)=\overline{V^{m,p}_\sigma(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}$. Let $\mathcal{D}(\Omega)$ be the space of the test functions ($C^\infty_0(\Omega)$ as a set), $\mathcal{D}_\sigma(\Omega)$ is the space of the test functions where the divergence is 0 for spatial variables (see [Supplement 1]). Let $P:L^2(\Omega)\to L^2_\sigma(\Omega)$ be the projection. Let $C^{k,\varepsilon}(\overline{\Omega})$ be the Hölder space.

[Existence of elementary weak solutions]
For any $f\in\mathcal{D}(\Omega)$, there are weak solutions $(u,\mathfrak{p})$ of the Navie-Stokes equation in the following sense: $u(t_0,x)$ is bounded on $\Omega_0=\{(t,x)\in\overline{\Omega}:t=t_0\}$ and for any natural number $m>4$, $u\in W^{m,1}_\sigma(\Omega)\cap W^{m,2}_\sigma(\Omega),$
$\mathfrak{p}\in L^2_{\mathrm{loc}}(\Omega)$.
Let the fundamental solution of $\partial_t-\Delta$ be E . That is, $(\partial_t-\Delta)E(t,x)=\delta(t,x)=\delta(t)\otimes\delta(x)$.
$u(t,x)=\int_{\mathbb{R}\times\mathbb{R}^3}E(s,y)\chi_\Omega(t-s,x-y\ )(Pf(t-s,x-y)-P((u\cdot\nabla)u)(t-s,x-y))dsdy,$

For any $\varphi\in\mathcal{D}_\sigma(\Omega)$,
$\langle\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla\mathfrak{p}-f,\varphi\rangle=0,$
for any $\varphi\in\mathcal{D}(\Omega)$,
$\langle\operatorname{div}u,\varphi\rangle=0.$
Here $(u\cdot\nabla)u^i=\sum_{j=1}^3 u^j\partial_{x^j}u^i,$
$\langle w,\varphi\rangle=(w,\varphi)_{L^2(\Omega)}$
$=\int_\Omega\sum_{i=1}^3 w^i(t,x)\varphi^i(t,x)dtdx$

$= \int_\Omega w(t,x)\cdot\varphi(t,x)dtdx$
$(w=(w^1,w^2,w^3),\varphi=(\varphi^1,\varphi^2,\varphi^3))$.

In general, if for two Banach spaces X, Y , there exists linear Hausdorff space Z such that $X, Y \subset Z$, then $X \cap Y$ is a Banach space with the norm given by $\|u\|_X + \|u\|_Y$ or $\max\{\|u\|_X, \|u\|_Y\}$. $\max\{\|u\|_X, \|u\|_Y\} \leq \|u\|_X + \|u\|_Y \leq 2\max\{\|u\|_X, \|u\|_Y\}$ so these are equivalent.

If $f \neq 0$ then $u \neq 0$. Ω can be arbitrary large, so $u, \mathfrak{p}$ are time global.

[Intuitive proof]
For any $\varphi \in \mathcal{D}_\sigma(\Omega)$,
$\mathrm{div}(\varphi)=0$, so by integration by parts
$\langle\nabla\mathfrak{p},\varphi\rangle$
$= \int_\Omega \sum_{i=1}^3 (\nabla\mathfrak{p})^i(t,x)\varphi^i(t,x)dtdx$
$= -\int_\Omega \mathfrak{p}(t,x)\mathrm{div}(\varphi)(t,x)dtdx = 0.$

Therefore, boundness of $u, \partial_{x^j}u$ by the Sobolev's embedding theorem and $|\Omega| < \infty$, we have $(u\cdot\nabla)u \in L^2(\Omega)$, so by the Helmholtz decomposition,
if we let $f = Pf + \nabla\mathfrak{f}, (u\cdot\nabla)u = P((u\cdot\nabla)u) + \nabla\mathfrak{u}$
then
$\langle f,\varphi\rangle = \langle Pf,\varphi\rangle, \langle\ (u\cdot\nabla)u,\varphi\rangle = \langle P((u\cdot\nabla)u),\varphi\rangle$, hence we solve

(N-S)' $\partial_t u - \Delta u = f - (u\cdot\nabla)u \,\mathrm{in}\, \mathcal{D}'_\sigma(\Omega)$. By the way,
$E^i(t,x) = \begin{cases} \frac{1}{\sqrt{4\pi t}^3} e^{-\frac{|x|^2}{4t}} & (t>0) \\ 0 & (t \leq 0) \end{cases}$
is locally integrable, $W^{m,p}_\sigma(\Omega)$ is completion, so with any $\{u_n\} \subset V^{m,p}_\sigma(\Omega)$,
$\lim_{n,n'\to\infty}\|u_n - u_{n'}\|_{W^{m,p}(\Omega)} = 0, \lim_{n,n'\to\infty}\|E*\chi_\Omega P((u_n\cdot\nabla)u_n) - E*\chi_\Omega P((u_{n'}\cdot\nabla)u_{n'})\|_{W^{m-1,p}(\Omega)} = 0$, some $u \in W^{m,p}_\sigma(\Omega)$ exists such that $\lim_{n\to\infty}\|u_n - u\|_{W^{m,p}(\Omega)} = 0, \lim_{n\to\infty}\|E*\chi_\Omega P((u_n\cdot\nabla)u_n) - E*\chi_\Omega P((u\cdot\nabla)u)\|_{W^{m-1,p}(\Omega)} = 0$.

u satisfies $\mathrm{div}, u = 0$ in the sense of a distribution belonging to $\mathcal{D}'(\Omega)$ (See [28]). That is, for any $\varphi \in \mathcal{D}(\Omega)$, $\langle\mathrm{div}\,u,\varphi\rangle = -\sum_{j=1}^3\langle u^j, \partial x^j\varphi\rangle = 0$.

$\chi_\Omega f \in C_0^\infty(\Omega)$, the support of $\chi_\Omega(u_n\cdot\nabla)u_n$ is also compact:
$\mathrm{supp}(\chi_\Omega(u_n\cdot\nabla)u_n^i) \subseteq \bigcup_{j=1}^3 (\mathrm{supp}\,u_n^j) \cap (\mathrm{supp}\,\partial_{x^j}u_n^i) \cap \overline{\Omega} \subseteq \overline{\Omega}$ therefore $\chi_\Omega(f - (u_n\cdot\nabla)u_n) \in C_0^\infty(\Omega)$. So the solution of the approximate equation

(N-S)'' $\partial_t v_n - \Delta v_n = \chi_\Omega(Pf - P((u_n \cdot \nabla)u_n))$
is
$v_n = E * \chi_\Omega(Pf - P((u_n \cdot \nabla)u_n)) \in V_\sigma^{m-1,p}(\Omega)$.

Therefore, the solution of (N-S)''
$v_n(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u_n \cdot \nabla)u_n)(t-s,x-y))dsdy.$
The way to take $\{u_n\}$ is arbitrary, so we can take the Cauchy sequence $\{u_n\}$ such that the limit of $\{v_n\}$ and the limit of $\{u_n\}$ coincide. Later we can justify it. We show that $u = v$ is the solution of (N-S)' :

$$v_n(t,x)$$
$$= \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u_n \cdot \nabla)u_n)(t-s,x-y))dsdy,$$
$$u_n \to u = v \leftarrow v_n.$$

$$\partial_t v_n(t,x) - \Delta v_n(t,x)$$
$$= \langle(\partial_t E(t-s,x-y) - \Delta E(t-s,x-y)), \chi_\Omega(s,y)(Pf(s,y) - P((u_n \cdot \nabla)u_n)(s,y))\rangle$$
$$= \langle\delta(\tau)\otimes\delta(z), \chi_\Omega(t-\tau,x-z)(Pf(t-\tau,x-z) - P((u_n \cdot \nabla)u_n)(t-\tau,x-z))\rangle$$
$$= Pf(t,x) - P((u_n \cdot \nabla)u_n)(t,x).$$

Therefore, the above calculation and the continuity of the heat operator on $\mathcal{D}'_\sigma(\Omega)$:
$|\langle\partial_t v_n - \Delta v_n, \varphi\rangle - \langle\partial_t u - \Delta u, \varphi\rangle| \to 0$, and from the Hölder's inequality, $\|P\| = 1$, and product of the function $L^2(\Omega)\times L^2(\Omega) \ni (u,v) \mapsto uv \in L^1(\Omega)$ is continuous (See [Supplement 2]), so

$$|\int_\Omega (P((u_n \cdot \nabla)u_n)(t,x)$$
$$-P((u \cdot \nabla)u)(t,x))) \cdot \varphi(t,x)dtdx|$$
$$\leq \|((u_n \cdot \nabla)u_n)(t,x) - ((u \cdot \nabla)u)(t,x)\|_{L^1(\Omega)}\|\varphi(t,x)\|_{L^\infty(\Omega)} \to 0, (n \to \infty)$$,

hence
$\partial_t u - \Delta u = Pf - P((u \cdot \nabla)u)$ holds, so we have
$u(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u \cdot \nabla)u)(t-s,x-y))dsdy$.

It has been shown that it is a solution in the sence of a distribution in $\mathcal{D}'_\sigma(\Omega)$ of (N-S)' (See [Supplement 3]).

" $\varphi \in \mathcal{D}_\sigma(\Omega) \Rightarrow \langle U, \varphi\rangle = 0$ "
$\Longleftrightarrow$
" there exist $\mathfrak{p}$ such that $U = \nabla\mathfrak{p}$ "

(See [14]), therefore there exist $\mathfrak{p}$ such that $\partial_t u + (u \cdot \nabla)u - \Delta u - f = -\nabla \mathfrak{p}$ holds.

$u(t, x) \in W^{m,p}(\Omega) \subset C^{(m-4/p)-1,\varepsilon}(\overline{\Omega})$, and if the function is bounded as variables (t, x) then it is also bounded as variable x , therefore $u(t_0, x)$ is bounded.
(END)

[Smoothness and boundness of elementary weak solutions]
Solution $(u, \mathfrak{p})$ are C^∞ -functions and bounded.

[Proof]
m can be arbitrarily large, so the embedding theorem to Hölder space (See [18] theorem 6.12) "if $\mathbb{N} \ni m - 4/p > 0$ then $W^{m,p}(\Omega) \subset C^{(m-4/p)-1,\varepsilon}(\overline{\Omega})$ for $\varepsilon \in (0, 1)$ ", in the sence of existence of suitable representative elements, u is bounded on $\overline{\Omega}$ and C^∞ -function.

f is smooth and $\partial_t u + (u \cdot \nabla)u - \Delta u - f = -\nabla \mathfrak{p}$ because $-\nabla \mathfrak{p}$ is smooth, so $\mathfrak{p}$ is also smooth.
(END)

[The uniqueness of elementary weak solutions]
Let the solutions are u, v .
If $\partial_t u + (u \cdot \nabla)u - \Delta u - f = \partial_t v + (v \cdot \nabla)v - \Delta v - f$ then $u = v$.

[Proof]
u, v are smooth, so if i $u \neq v$,
$\partial_t u + (u \cdot \nabla)u - \Delta u - f \neq \partial_t v + (v \cdot \nabla)v - \Delta v - f$. This is a contradiction.
Therefore $u = v$.
(END)

[Supplement 1]
As functions φ that diverges for spatial variables $\mathrm{div}\varphi = \nabla \cdot \varphi = 0$, it is sufficient to take any $\psi \in \mathcal{D}(\Omega)$ and set to $\varphi = \mathrm{curl}\psi$. (See [10])

[Supplement 2]
Let $\|u_n - u\|_{L^2(\Omega)} \to 0, \|v_n - v\|_{L^2(\Omega)} \to 0$. By the triangle inequality, we have $|\|u_n\|_{L^2(\Omega)} - \|u\|_{L^2(\Omega)}| \leq \|u_n - u\|_{L^2(\Omega)}$ for any sufficientaly large n . On the other hand, $\|u_n\|_{L^2(\Omega)} < \|u\|_{L^2(\Omega)} + 1$. Therefore
$\|u_n v_n - uv\|_{L^1(\Omega)} \leq \|u_n\|_{L^2(\Omega)}\|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|u_n - u\|_{L^2(\Omega)} <$
$(\|u\|_{L^2(\Omega)} + 1)\|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|u_n - u\|_{L^2(\Omega)} \to 0.$

[Supplement 3]
Let $|\alpha| \leq m-1$.

$\int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)(Pf(t-s,x-y)-P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx$
$= \int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3-B_\delta(0,0)} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)(Pf(t-s,x-y)-P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx$
$+ \int_\Omega |\int_{B_\delta(0,0)} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)(Pf(t-s,x-y)-P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx.$

$E^i(t,x) = \begin{cases} \frac{1}{\sqrt{4\pi t}^3} e^{-\frac{|x|^2}{4t}} & (t>0) \\ 0 & (t\leq 0) \end{cases}$, so $E^i(s,y)$ is a locally integrable function, therefore
$\int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)Pf(t-s,x-y))dsdy|^p dtdx$
is a finite value.

$\int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx$
is also finite.

$\int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)P((u\cdot\nabla)u)(t-s,x-y)dsdy|^p dtdx$
$= \int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3-B_\delta(0,0)} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx$
$+ \int_\Omega |\int_{B_\delta(0,0)} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx.$

This first term is a finite value:

$\int_\Omega |\int_{\mathbb{R}\times\mathbb{R}^3-B_\delta(0,0)} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx$
$\leq \sup\{E^i(s,y) : (s,y)\in\mathbb{R}\times\mathbb{R}^3-B_\delta(0,0)\}^p \int_\Omega |\int_{\{(s,y):(t-s,x-y)\in\Omega\}} \partial^\alpha(P((u\cdot\nabla)u)(t-s,x-y))dsdy|^p dtdx$
$\leq \sup\{E^i(s,y) : (s,y)\in\mathbb{R}\times\mathbb{R}^3-B_\delta(0,0)\}^p \sup\{|\partial^\alpha(P((u\cdot\nabla)u))(s,y)| : (s,y)\in\Omega\}^p |\Omega|^{1+p}$
$< \infty.$

Also, the second term is also a finite value:by
Hölder's inequality,
$\int_\Omega |\int_{B_\varepsilon(0,0)} E(s,y)\partial^\alpha(\chi_\Omega(t-s,x-y)P((u\cdot\nabla)u)(t-s,x-y)dsdy|^p dtdx$
$\leq \|E\|^p_{L^1(B_\varepsilon(0,0))} \|\partial^\alpha(P((u\cdot\nabla)u))\|^p_{L^\infty(B_\varepsilon(0,0))} |\Omega|$
$< \infty.$
(END)

We think the formula of the above solution is the existence of a fixed point in the function space $X=\bigcap_{m=5}^{\infty} W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)$. X is a Banach space with the norm given by $\|u\|_X=\sum_{m=5}^{\infty}\frac{1}{m!^5}\|u\|_{W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)}$.

[Proof]
Let $\{u_n\}$ be the Caucy sequence in X . Then, $\{u_n\}$ is the Caucy sequence of $W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)$. $W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)$ is a Banach space, so $\{u_n\}$ converges. Let the limit be u . If $u\notin X$, for any positive number R , there exists natural number m' such that $\sum_{m=5}^{m'}\frac{1}{m!^5}\|u\|_{W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)}>R$. Then $\|u\|_{W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)}>CR$. This is a contradiction, so $u\in X$. If $\lim_{n\to\infty}\|u_n-u\|_X=0$ does not hold, there exists positive number R' such that for any natural number $M'\geq 5$, $\sum_{m=5}^{M'}\frac{1}{m!^5}\|u_n-u\|_{W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)}>R'$. Then $\|u_n-u\|_{W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)}>C'R'$. This is a contradiction, too. So $\lim_{n\to\infty}\|u_n-u\|_X=0.$
(END)

$\chi_\Omega\in X$ so $X\neq\{0\}$.

A constant $C>0$ exists such that
$\left\|u^iv^i\right\|_X\leq C\|u^i\|_X\|v^i\|_X$
(Separation of the product)
and
$\|\partial_{x^j}u\|_X\leq C\|u\|_X$
(absorption of differential)
holds for $u\in X$.

[Proof]
For binomial coefficients $c_{\alpha,\beta}$, let
$c_\alpha=\sum_{\beta\leq\alpha}c_{\alpha,\beta}$.
There is a continuous embedding $X\subset C^{k,\varepsilon}(\overline{\Omega})$ for any natural number k , because $\|u_n-u\|_X\to 0$
$\Rightarrow\|u_n-u\|_{W_{\sigma}^{m,1}(\Omega)\cap W_{\sigma}^{m,2}(\Omega)}\to 0$
$\Rightarrow\|u_n-u\|_{C^{k,\varepsilon}(\overline{\Omega})}\to 0$, so there exists constant $c'>0$ such that if $|\alpha|\leq k$, by Leibniz' formula,
$\|\partial^\alpha(u^iv^i)\|_{L^p(\Omega)}$
$\leq c_\alpha\|u^i\|_{C^{k,\varepsilon}(\overline{\Omega})}\|v^i\|_{C^{k,\varepsilon}(\overline{\Omega})}|\Omega|^{1/p}$
$\leq c_\alpha c'|\Omega|^{1/p}\|u^i\|_X c'\|v^i\|_X$
$\leq c_\alpha c'^2|\Omega|^{1/p}\|u^i\|_X\|v^i\|_X$. Therefore,
$\|\partial^\alpha(u^iv^i)\|_{L^p(\Omega)}\leq c_\alpha c'^2|\Omega|^{1/p}\|u^i\|_X\|v^i\|_X$, so there exists a constant $C>0$ such that $\|u^iv^i\|_X\leq C\|u^i\|_X\|v^i\|_X$.

Let $\{u_n\} \subset X$ satisfies $u_n \to u, \partial_{x^j} u_n \to v$. From the Hölder's inequality, we have $|\langle \partial_{x^j} u_n - v, \varphi \rangle| \le \|\partial_{x^j} u_n - v\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)} \to 0$ and the weak differentiation is continuous in $\mathcal{D}'_\sigma(\Omega)$, so $\partial_{x^j} u_n \to \partial_{x^j} u \,\mathrm{in}\, \mathcal{D}'_\sigma(\Omega)$. From $v = \partial_{x^j} u \in X$, $\{u \in X : \partial_{x^j} u \in X\} = X$, therefore the absorption of differentiation is true by the closed graph theorem.

(END)

$X \ni u \mapsto E * (\chi_\Omega u) \in X$ is a bounded operator and a constant $C > 0$ exists such that for any $u \in X$,

$$\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)u(t-s,x-y)dsdy\|_X$$
$$\le C\|u\|_X$$

holds.

[Proof]

As a function of (s,y) , for any $(t,x) \in \Omega$,

$$\mathrm{supp}(E^i(s,y)\chi_\Omega(t-s,x-y)u^i(t-s,x-y))$$
$$\subseteq -\overline{\Omega} + (t,x)$$
$$= \overline{\{(s,y) \in \mathbb{R}\times\mathbb{R}^3 : (t-s,x-y) \in \Omega\}}$$

is the translation of reverse of $\overline{\Omega}$, so it is compact, and $|\partial^\alpha_{t,x}(E^i(s,y)\chi_\Omega(t-s,x-y)u^i(t-s,x-y))| \le E^i(s,y)\sup\{|\partial^\alpha_{t,x}u^i(t-s,x-y)| : (t-s,x-y) \in \Omega\} \in L^1_{s,y}(\Omega)$, so combine the theorem of differentiation under the integral sign, the Hölder's inequality and assumption of Ω , we have

$$\|\partial^\alpha(E*(\chi_\Omega u))\|_{L^p(\Omega)}$$
$$\le \|E*(\partial^\alpha(\chi_\Omega u))\|_{L^p(\Omega)}$$
$$\le \|\|E(s,y)\|_{L^2_{s,y}(-\Omega+(t,x))}\|\partial^\alpha u(t-s,x-y)\|_{L^2_{s,y}(-\Omega+(t,x))}\|_{L^p_{t,x}(\Omega)}$$
$$\le \sup_{(t,x)\in\Omega}\|E\|_{L^2(-\Omega+(t,x))}\|\partial^\alpha u\|_{L^2(\Omega)}|\Omega|^{1/p}$$
$$\le c\|\partial^\alpha u\|_{L^1(\Omega)\cap L^2(\Omega)}$$
$$< \infty .$$

So we have

$$\|E*(\chi_\Omega u)\|_X \le C\|u\|_X.$$

(END)

For a constant M , let S be a subspace of X : $S = \{u \in X : \|u\|_X \le M\}$. We take M the smaller one while satisfying $2C^3M < 1, C(1+3C^2)M \le 1$. Let the external force $f \in S$ and $\|f\|_X \le M^2$.

As similar to the intuitive argument, we solve

(N-S)' $\partial_t u - \Delta u = f - (u\cdot\nabla)u$, that is, $u(t_0,x) \in L^\infty(\Omega_0)$, $u \in W^{m,1}_\sigma(\Omega) \cap W^{m,2}_\sigma(\Omega),$

$\mathfrak{p} \in L^2_{\mathrm{loc}}(\Omega)$, for any $\varphi \in \mathcal{D}_\sigma(\Omega)$,

$$\langle \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\mathfrak{p} - f, \varphi\rangle = 0,$$

for any $\varphi \in \mathcal{D}(\Omega)$,
$\langle \operatorname{div} u, \varphi \rangle = -\sum_{j=1}^{3} \langle u^j, \partial_{x^j} \varphi \rangle = 0$.

$\Phi : S \to S$ can be defined as
$\Phi[u](t, x)$
$= \int_{\mathbb{R}\times\mathbb{R}^3} E(s, y)\chi_\Omega(t-s, x-y)(Pf(t-s, x-y) - P((u\cdot\nabla)u)(t-s, x-y))dsdy$. we take the function sequence $\{u_n\} \subset S$ as $u_0 \in S$, if $n \geq 0$ then $u_{n+1}(t, x) = \Phi[u_n](t, x)$
$= \int_{\mathbb{R}\times\mathbb{R}^3} E(s, y)\chi_\Omega(t-s, x-y)(Pf(t-s, x-y) - P((u_n\cdot\nabla)u_n)(t-s, x-y))dsdy$. If X is a complete metric space, then S is complete because it is a closed subspace that is not empty, and if it can be said that Φ is a contraction mapping, according to the Banach's fixed point theorem, the uniqueness and the existence of a fixed point of Φ :

Some $u \in S$ exists uniquely and $\Phi[u] = u$.

Then, due to the uniqueness of the fixed point in Banach's fixed point theorem and the same argment as the intuitive argment, it can be said that u is a unique weak solution. If $f \neq 0$ then $u \neq 0$. Ω can be arbitrary large, so $u, \mathfrak{p}$ are time global.

[Proof of the possibility that Φ can be defined as a contraction mapping]
$u \in S \Rightarrow \|E * (\chi_\Omega(Pf - P((u\cdot\nabla)u)))\|_X < \infty$
holds. Therefore
$\|\Phi[u]\|_X \leq M.$

$\|P\| = 1$, so
$\|\chi_\Omega(Pf - P((u\cdot\nabla)u))\|_X$
$\leq \|f\|_X + \|u^1\partial_{x^1}u + u^2\partial_{x^2}u + u^3\partial_{x^3}u\|_X$
$\leq M^2 + 3C^2M^2 < \infty$.

If
$\|\Phi[u]\|_X$
$\leq CM^2 + 3C^3M^2$
$\leq M$, M must be $C(1+3C^2)M \leq 1$.
(END)

$\Phi : S \to S$ is Lipschitz continuous: there is a constant $L > 0$ such that
$\|\int_{\mathbb{R}\times\mathbb{R}^3} E(s, y)\chi_\Omega(t-s, x-y)(P((v\cdot\nabla)v)(t-s, x-y) - P((u\cdot\nabla)u)(t-s, x-y))dsdy\|_X$
$\leq L\|u - v\|_X.$

may be possible. If the Lipschitz continuity established,

$$
\begin{aligned}
&\| \varPhi[u] - \varPhi[v] \|_X \\
&\leq \| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v)(t-s,x-y) - P((u\cdot \\
&\nabla)u)(t-s,x-y))dsdy\|_X \\
&\leq L\|u-v\|_X
\end{aligned}
$$

follows. Here, if

[$\varPhi$ may be a contraction mapping]

$$L < 1$$

holds, the argument is justified.

[Proof of Lipschitz continuity]

$$
\begin{aligned}
&(v\cdot\nabla)v(t-s,x-y) - (u\cdot\nabla)u(t-s,x-y) \\
&= \sum_{j=1}^3 v^j(\partial_{x^j}v(t-s,x-y) - \partial_{x^j}u(t-s,x-y)) + (v^j\partial_{x^j}u(t-s,x-y)) - \\
&(u^j\partial_{x^j}u(t-s,x-y))
\end{aligned}
$$

, so we have

$$
\begin{aligned}
&\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v)(t-s,x-y) - P((u\cdot\nabla)u)(t- \\
&s,x-y))dsdy\|_X \\
&\leq C^2\|v\|_X \max_j(\|\partial_{x^j}v - \partial_{x^j}u\|_X) + C^2\|v-u\|_X \max_j(\|\partial_{x^j}u\|_X) \\
&\leq C^3M\|v-u\|_X + C^3M\|v-u\|_X \\
&= 2C^3M\|u-v\|_X.
\end{aligned}
$$

Therefore, we can make it $L = 2C^3M$.

(END)

[Proof of the possibility that $\varPhi$ is a contraction mappig]

From the above argment $\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(P((v\cdot\nabla)v(t-s,x- y)) - P((u\cdot\nabla)u)(t-s,x-y))dsdy\|_X$

$$\leq 2C^3M\|u-v\|_X$$

and

$$2C^3M < 1.$$

(END)

[Solvability of the Navier-Stokes equations]

When taking $f \in S$ to $\|f\|_X \leq M^2$, the fixed point of

$\varPhi : S \to S$ will be the solution of (N-S)'.

[Proof]

We take the function sequence $\{u_n\} \subset S$ as $u_0 \in S$, if $n \geq 0$ then

$u_{n+1}(t,x) = \Phi[u_n](t,x)$

$= \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)(Pf(t-s,x-y) - P((u_n\cdot\nabla)u_n)(t-s,x-y))dsdy$.

$\operatorname{div}\Phi[u] = 0$ in the sense of distribution.

As similar to the intuitive argument, if we take the limit, u is a weak solution of (N-S)'.

(END)

[References]

[1]俣野博-神保道夫, 熱・波動と微分方程式, 岩波オンデマンドブックス, 岩波書店, 2018

[2]金子晃, 偏微分方程式入門, 基礎数学12, 東京大学出版会, 2013

[3]垣田高夫-柴田良弘, ベクトル解析から流体へ, 日本評論社, 2007

[4]谷島賢二, 数理物理入門 改訂改題, 基礎数学11, 東京大学出版会, 2018

[5]柴田良弘, ルベーグ積分論, 内田老鶴圃, 2006

[6]谷島賢二, 新版 ルベーグ積分と関数解析, 講座〈数学の考え方〉13, 朝倉書店, 2015

[7]コルモゴロフ-フォミーン, 函数解析の基礎 上, 岩波書店, 2012

[8]北田均, 新訂版 数理解析学概論, 現代数学社, 2016

[9]猪狩惺, 実解析入門, 岩波書店, 2013

[10]岡本久-中村周, 関数解析, 岩波オンデマンドブックス, 岩波書店, 2016

[11]黒田成俊, 関数解析, 共立数学講座15, 共立出版, 2011

[12]藤田宏-黒田成俊-伊藤清三, 関数解析, 岩波書店, 2009

[13]吉田耕作, Functional Analysis, Springer-Verlag, 1980

[14]増田久弥, 応用解析ハンドブック, 丸善出版, 2012

[15]溝畑茂, 偏微分方程式論, 岩波書店, 2010

[16]小薗英雄-小川卓克-三沢正史, これからの非線型偏微分方程式, 日本評論社, 2007

[17]垣田高夫, シュワルツ超関数入門, 日本評論社, 2015

[18]宮島静雄, ソボレフ空間の基礎と応用, 共立出版, 2020

[19]澤野嘉宏, ベゾフ空間論, 日本評論社, 2011

[20]岡本久, ナヴィエ-ストークス方程式の数理 新装版, 東京大学出版会, 2023

[21]柴田良弘, 流体数学の基礎 上, 岩波数学叢書, 岩波書店, 2022

[22]柴田良弘, 流体数学の基礎 下, 岩波書店, 2022

[23]L.ヘルマンダー, The Analysis of Linear Partial Differential Operators I: Distribution Theory And Fourier Analysis, Springer, 1990

[24]柴田良弘-久保隆徹, 非線形偏微分方程式, 現代基礎数学21, 朝倉書店, 2013

[25]八木厚志, 放物型発展方程式とその応用(上) 可解性の理論, 岩波数学叢書, 岩波書店, 2011

[26]小川卓克, 非線型発展方程式の実解析的方法, シュプリンガー現代数学シリーズ 第18巻, 丸善出版, 2013

[27]新井仁之, ルベーグ積分講義 ルベーグ積分と面積0の不思議な図形たち, 日本評論社, 2003

[28]Wasao SIBAGAKI, Hisako RIKIMARU『ON THE E. HOPF'S WEAK SOLUSION OF INITIAL VALUE PROBLEM FOR THE NAVIER-STOKES EQUATIONS』, 1967

[29]Adams-Fournier, Sobolev Spaces, Academic Press, 2003