Detecting Whether a Graph Has a Fixed-point-free Automorphisms Is in Polynomial Time

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Abstract

The problem that to determining whether a graph has a fixed-point-free automorphism is NP-complete. We show that it is solvable in polynomial time. First, we obtain the automorphisms of an input graph G by using a spectral method. Next, we prove the Theorem used to detect whether there is a fixed-point free automorphism in G. Next, we construct an algorithm to detect whether G has a fixed-point-free automorphism using this result. The computational complexity of this problem is $\mathcal{O}(n^5)$. Then, the complexity classes P and NP are the same.

Index Terms

fixed-point-free automorphism, graph spectrum, polynomial time computation, NP-complete problem.

I. INTRODUCTION

The P versus NP problem [1], [2] is one of the major problems in theoretical computer science. An answer to this problem would determine whether problems that can be verified in polynomial time can also be solved in polynomial time. Attempts have been made to prove that P is not equal to NP. However, it has been shown that this cannot be proven or is difficult to prove using the methods of relativizing proofs [3], natural proofs [4], and algebrizing proofs [5]. On the other hand, many attempts have been made to show the lower bound of NP problems [6], [7], mainly by pruning conditional searches [8], but it is still unclear whether P and NP are equal. In contrast, we will show a lower bound on the computational complexity of an NP-complete problem by introducing a spectral method to handle multiple states at once.

If a problem is NP and all other NP problems are polynomial-time reducible to it, the problem is NP-complete [9]. If one of the NP-complete problems can be solved in polynomial time, the complexity classes P and NP are the same. The problem of determining whether a given graph has a fixed-point-free automorphism is NP-complete [10]. In this paper, we show that it is solvable in polynomial time.

First, we define the following functions. Let S be a vertex-weighted graph. Let $V_{w0}(S)$ be the set of vertices with weight 0 of S. Let $Sg(S, v, w)$ be a vertex-weighted graph in which a weight $w \in \mathbb{N}$ is given to a vertex v of S. Let $Ev(S)$ be the eigenvalue set of the adjacency matrix of S. Next, we use Theorem II.2 [11] to obtain the automorphisms of an input graph G using eigenvalue sets.

Theorem II.2. Let $S_{v_i} = Sg(S, v_i, w)$ and $S_{v_j} = Sg(S, v_j, w)$ with $v_i, v_j \in V_{w0}(S)$, $v_i \neq v_j$ and $w > 0$. When $Ev(S_{v_i}) = Ev(S_{v_j})$, S_{v_i} and S_{v_j} are isomorphic.

Next, we prove Theorem II.5 to detect whether there is a fixed-point-free automorphism in G.

Theorem II.5. We obtain the vertex sets $V_\lambda \subset V$ with the same $\lambda_v = Ev(Sg(S, v, w))$, $v \in V$, $w > 0$. $V_{\lambda_{v_i}} > 1$ for all vertex sets of λ_{v_i} if, and only if, G has a fixed-point-free automorphism.

Next, we construct an algorithm to detect whether a graph has a fixed-point-free automorphism using this result. Since the elements of an adjacency matrix of a vertex-weighted graph are all integers, the coefficients of the eigenequation of this matrix are all integers. Then, we calculate the Frobenius normal form [12], [13] to obtain the coefficients of the eigenequation of this matrix without real number calculations. Then, we compare the coefficients to determine whether the sets of eigenvalue are the same. The computational complexity of detecting whether a graph has a fixed-point-free automorphism is $\mathcal{O}(n^5)$. Since one of the NP-complete problems is solvable in polynomial time, the complexity classes P and NP are the same.

This paper is organized as follows. Section II provides the proofs used to determine whether a given graph has a fixed-point-free automorphism. Section III presents an algorithm to solve this problem. Finally, Section IV presents a conclusion regarding the result of this paper.

II. PROOF

In this section, we provide the proofs for the results of detecting whether a given graph has a fixed-point-free automorphism.

A. Preparation

We define the following functions, which will be used in the proofs and the methods. Suppose S is a vertex-weighted graph. Let $V_{w0}(S)$ be the set of vertices of S with weight 0. Let $Sg(S, v, w)$ be the vertex-weighted graph in which the weight $w \in \mathbb{N}$ is given to vertex v of S. Denote the adjacency matrix of S by $A(S)$. Let $Ev(S)$ be the set (with multiplicities) of eigenvalues of $A(S)$.

B. Obtain the automorphisms

In this subsection, we provide the proof and use the Theorem and the Corollary for obtaining the automorphisms in G.

Let $Aut(G)$ be the automorphism group of G. We can obtain $\psi \in Aut(G)$ by composition of automorphisms of order two.

Corollary II.1. *There are certain automorphisms $\psi_1, \psi_2, \ldots, \psi_m \in Aut(G)$ of order two, and we can explain $\psi = \psi_m \psi_{m-1} \cdots \psi_1$.*

Proof. Permuting vertices of ψ consists of transposition and cycling. Suppose that a composition of automorphisms has a cycle $\sigma_i = (\sigma_1 \to \cdots \to \sigma_j \to \sigma_{j+1} \to \cdots \to \sigma_r \to \sigma_1)$ of order $r > 3$. At this time, there exists the transposition such that $\sigma_{j,j+1} = (\sigma_j \leftrightarrow \sigma_{j+1})$. Then, $\sigma_i = \sigma_i \sigma_{j,j+1}^{-1} \sigma_{j,j+1} = \sigma_{i+1} \sigma_{j,j+1}$. Thus, the order of σ_{i+1} is $r - 1$. Therefore, Corollary II.1 holds. $\square$

An automorphism of order two contains transposing two vertices. So, we remove fixed points by compositions of automorphisms that contain transposing two vertices. Thus, we use Theorem II.2 and Corollary II.3 [11] to obtain the automorphisms that contain transposing two vertices of an input graph G using eigenvalue sets.

Theorem II.2. *Let $S_{v_i} = Sg(S, v_i, w)$ and $S_{v_j} = Sg(S, v_j, w)$ with $v_i, v_j \in V_{w0}(S)$, $v_i \neq v_j$ and $w > 0$. If $Ev(S_{v_i}) = Ev(S_{v_j})$, then S_{v_i} and S_{v_j} are isomorphic.*

Corollary II.3. *Let $S_{v_i} = Sg(S, v_i, w)$ and $S_{v_j} = Sg(S, v_j, w)$ with $v_i, v_j \in V_{w0}(S)$, $v_i \neq v_j$ and $w > 0$. If $Ev(S_{v_i}) \neq Ev(Sv_j)$, then S_{v_i} and S_{v_j} are not isomorphic.*

So, when $Ev(S_{v_i}) = Ev(S_{v_j})$ if, and only if, there is an automorphism in G that contain the transposition of v_i and v_j.

C. Detect whether there is a fixed-point-free automorphism

In this subsection, we provide the proofs for the results of detecting whether a given graph has a fixed-point-free automorphism from obtained the automorphisms.

Lemma II.4 and Theorem II.5 prove that it is possible to detect whether there is a fixed-point-free automorphism in G.

Lemma II.4. *Suppose that a graph $G = (E, V)$ has nontrivial automorphisms $\psi_a, \psi_b \in Aut(G)$, where $\psi_a \neq \psi_b$. Let ψ_a have fixed points $V_{fixed,\psi_a} = \{v | \psi_a(v) = v, v \in V\}$. Now, ψ_b has the vertex transposition*

Algorithm 1 A function that determines whether a graph G has a fixed-point-free automorphism.

1: **function** HAS_FIXED_POINT_FREE_AUTOMORPHISM($G = (V, E)$)
2:     $S \leftarrow G$ with all vertex weights are 0
3:     $w \leftarrow 2|V|$
4:     clear hash h
5:     **for each** $v \in V$ **do**
6:         $\lambda \leftarrow Ev(Sg(S, v, w))$
7:         **if** $h(\lambda) = \emptyset$ **then** $h(\lambda) \leftarrow \{v\}$
8:         **else** $h(\lambda) \leftarrow h(\lambda) \cup \{v\}$
9:         **end if**
10:     **end for**
11:     **for each** $T \in h$ **do**
12:         **if** $|T| = 1$ **then**
13:             **return** FALSE
14:         **end if**
15:     **end for**
16:     **return** TRUE
17: **end function**

$\psi_b(v_a) = v_b$ *and* $\psi_b(v_b) = v_a$, $v_a \in V_{fixed,\psi_a}$. *When we apply* ψ_b *following* ψ_a, *the set of fixed points become* $V_{fixed,\psi_a} \cap V_{fixed,\psi_b}$.

Proof. Suppose $\psi_a : V_{a,s} \mapsto V_{a,d}$, $V_{a,s} \cup V_{a,d} \cup V_{fixed,\psi_a} = V$. When we apply ψ_b following ψ_a, we obtain $\psi_b : V_{a,s} \cup V_{fixed,\psi_a} \mapsto V_{a,s} \cup V_{fixed,\psi_a}$, $V_{a,d} \cup V_{fixed,\psi_a} \mapsto V_{a,d} \cup V_{fixed,\psi_a}$, so the vertices belonging to $V_{a,s}$ and $V_{a,d}$ are not returned to the original point by ψ_b. The automorphic transformation ψ_b maps at least one vertex v to another. Thus, Lemma II.4 holds. $\square$

Theorem II.5. *Consider a graph* $G = (E, V)$. *Let the vertex weighted graph* $S = G$. *We obtain the vertex sets* $V_\lambda \subset V$ *with the same* $\lambda_v = Ev(Sg(S, v, w))$, $v \in V$, $w > 0$. $V_{\lambda_{v_i}} > 1$ *for all vertex sets of* λ_{v_i} *if, and only if,* G *has a fixed-point-free automorphism.*

Proof. From Lemma II.4, applying multiple automorphic transformations to G does not increase the size of the set of fixed points. When $V_{\lambda_{v_i}} > 1$ for all vertex sets, there exists ψ such that $\psi(v) \neq v$ at every vertex v. Suppose there is a set of vertices such that $|V_{\lambda_{v_j}}| = 1$. There is no ψ such that $\psi(v) \neq v$ at $v \in V_{\lambda_{v_j}}$. Then, v becomes a fixed point. Therefore, Theorem II.5 holds. $\square$

III. ALGORITHM

In this section, we present a polynomial-time algorithm to determine whether a graph has a fixed-point-free automorphism. We assume that the number of vertices of the graph is n.

Since the elements of an adjacency matrix of a vertex-weighted graph are all integers, the coefficients of the eigenequation of this matrix are all integers. We use the set of coefficients of the eigenequation of the adjacency matrix of a vertex-weighted graph instead of its set of eigenvalues. We calculate the Frobenius normal form to obtain the set of coefficients without real number calculations. The amount of computation required to convert an adjacency matrix into the Frobenius normal form is $\mathcal{O}(n^4)$.

Function 1 determines whether a graph G has a fixed-point-free automorphism. First, by adding a weight $w > 0$ to a vertex, we obtain a set of vertices V_λ with the same eigenvalue set. Thus, we obtain automorphisms of G from Theorem II.2 and Corollary II.3. Next, we check if the size of the vertex set V_λ is 1 or above to determine whether there is a fixed-point-free automorphism based on Theorem II.5. The computational complexity of this function is $\mathcal{O}(n^5)$.
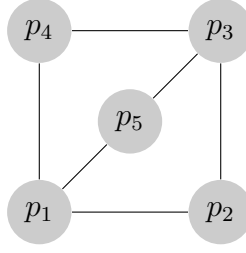
Fig. 1. An example of detecting a fixed point.

Figure 1 shows an example of detecting a fixed point for the graph $G = (V, E)$. Let the vertex weighted graph $S = G$. We obtain the vertex sets $V_\lambda \subset V$ with the same $\lambda_v = Ev(Sg(S, v, w))$, $v \in V$, $w > 0$. Then, we obtain $V_{\lambda_1} = \{v_1, v_3\}$, $V_{\lambda_2} = \{v_2, v_4\}$ and $V_{\lambda_3} = \{v_5\}$. Thus, if $|V_{\lambda_3}| = 1$, then G has no fixed-point-free automorphism.

IV. Conclusion

In this paper, we have presented an algorithm to detect whether a given graph G has a fixed-point-free automorphism. It has polynomial time complexity. Since one of the NP-complete problems is solvable in polynomial time, the complexity classes P and NP are the same.

Note that this algorithm has a limitation in that it can only detect whether G has a fixed-point-free automorphism.

Appendix A
Definition

In this section, we give the definitions used in this paper.

Definition A.1. A graph $G = (V, E)$ is a pair consisting of a non-empty finite vertex set $V \neq \emptyset$ and an edge set E that is a subset of V^2. The graph's size is the number of its vertices $1 < n = |V|$. The number of vertices in a graph is assumed to be finite. In addition, we align the set V with $\{v_1, \ldots, v_n\}$. There is an edge between vertices v_a and v_b when (v_a, v_b) is an element of the set E. Also, edges have no direction. Moreover, the graph has no multiple edges between a pair of vertices, and there are no loops (i.e., (v_a, v_a) is never an edge).

Definition A.2. A vertex-weighted graph $S = (V, E, w)$ is a graph with a function $w : V \to \mathbb{N}$ that gives the weights of the vertices. Then, a graph is a vertex-weighted graph in which the weights of all its vertices are 0.

Definition A.3. The adjacency matrix A of a vertex-weighted graph $S = (V, E, w)$ with $n = |V|$ is an $n \times n$ symmetric matrix that is given as follows. The entries $a_{i,j}$, $v_i, v_j \in V$, $0 < i, j \leq n$ of A satisfy:

$$\begin{cases} (v_i, v_j) \in E & \text{if} \quad a_{i,j} = a_{j,i} = 1, \\ (v_i, v_j) \notin E & \text{if} \quad a_{i,j} = a_{j,i} = 0, \\ a_{i,i} = w(v_i). \end{cases}$$

Definition A.4. Suppose that a graph $G = (E, V)$ has an automorphism. Let ψ be the automorphic transformation. A fixed-point-free automorphism is an automorphism such that $\psi(v) \neq v$ at all vertices $v \in V$.

APPENDIX B
PROOF

This chapter reproduces the proofs from the reference [11].

The following Theorem II.2 and Corollary II.3 [11] prove that it is possible to obtain the automorphisms of S using eigenvalue sets.

Theorem II.2. Let $S_{v_i} = Sg(S, v_i, w)$ and $S_{v_j} = Sg(S, v_j, w)$ with $v_i, v_j \in V_{w0}(S)$, $v_i \neq v_j$ and $w > 0$. If $Ev(S_{v_i}) = Ev(S_{v_j})$, then S_{v_i} and S_{v_j} are isomorphic.

Proof. We show that if $Ev(S_{v_i}) = Ev(S_{v_j})$, then S_{v_i} and S_{v_j} are not cospectral but isomorphic.

Let $A(S_{v_i})$ and $A(S_{v_j})$ be A_{v_i} and A_{v_j}, respectively. When there exists a permutation matrix P such that $A_{v_i} = P^t A_{v_j} P$, S_{v_i} and S_{v_j} are isomorphic. Denote the eigenfunctions of A_{v_i} and A_{v_j} by f_{v_i} and f_{v_j}, respectively. When f_{v_i} and f_{v_j} are the same, the eigenvalue sets of A_{v_i} and A_{v_j} are the same. Therefore, we will prove that such a nontrivial permutation matrix exists when $f_{v_i} - f_{v_j} = 0$.

Without loss of generality, we may assume $i = 1$ and $j = 2$. We show the characteristic polynomials f_{v_1} and f_{v_2} as below.

$$
f_{v_1} = |A_{v_1} - \lambda I|
$$
$$
= \begin{vmatrix}
w - \lambda & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\
a_{2,1} & -\lambda & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\
a_{3,1} & a_{3,2} & w_3 - \lambda & a_{3,4} & \cdots & a_{3,n} \\
a_{4,1} & a_{4,2} & a_{4,3} & \ddots & & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \vdots \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & \cdots & w_n - \lambda
\end{vmatrix},
$$

$$
f_{v_2} = |A_{v_2} - \lambda I|
$$
$$
= \begin{vmatrix}
-\lambda & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\
a_{2,1} & w - \lambda & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\
a_{3,1} & a_{3,2} & w_3 - \lambda & a_{3,4} & \cdots & a_{3,n} \\
a_{4,1} & a_{4,2} & a_{4,3} & \ddots & & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \vdots \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & \cdots & w_n - \lambda
\end{vmatrix}.
$$

The weights of the vertices are w, $w_3, and \ldots w_n$, all of which are integers. Then,

$$
f_{v_1} - f_{v_2} = w \begin{vmatrix}
0 & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\
a_{3,2} & w_3 - \lambda & a_{3,4} & \cdots & a_{3,n} \\
a_{4,2} & a_{4,3} & \ddots & & a_{3,n} \\
\vdots & \vdots & & \ddots & \vdots \\
a_{n,2} & a_{n,3} & \cdots & \cdots & w_n - \lambda
\end{vmatrix} - w \begin{vmatrix}
0 & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\
a_{3,1} & w_3 - \lambda & a_{3,4} & \cdots & a_{3,n} \\
a_{4,1} & a_{4,3} & \ddots & & a_{3,n} \\
\vdots & \vdots & & \ddots & \vdots \\
a_{n,1} & a_{n,3} & \cdots & \cdots & w_n - \lambda
\end{vmatrix} \tag{1}
$$
$$
= 0.
$$

If $n = 2$, f_{v_1} and f_{v_2} are the same. Hence, in this case, S_{v_1} and S_{v_2} are isomorphic.

We treat the case of $n = 3$ as follows. Equation 1 becomes

$$
f_{v_1} - f_{v_2} = w \begin{vmatrix} 0 & a_{2,3} \\ a_{3,2} & w_3 - \lambda \end{vmatrix} - w \begin{vmatrix} 0 & a_{1,3} \\ a_{3,1} & w_3 - \lambda \end{vmatrix}
$$
$$
= w(a_{2,3} a_{3,2} - a_{1,3} a_{3,1})
$$
$$
= 0.
$$

So, when $a_{2,3} = a_{1,3}$, f_{v_1} and f_{v_2} are the same. For this case, then, S_{v_1} and S_{v_2} are isomorphic.

Let $n > 3$. Suppose the matrix A' is as follows.

$$A' = \begin{pmatrix} w_3 & a_{3,4} & \cdots & a_{3,n} \\ a_{4,3} & \ddots & & a_{3,n} \\ \vdots & & \ddots & \vdots \\ a_{n,3} & \cdots & \cdots & w_n \end{pmatrix}.$$

Let vertex $u_1 = (a_{1,3}, a_{1,4}, \ldots, a_{1,n})^t$ and $u_2 = (a_{2,3}, a_{2,4}, \ldots, a_{2,n})^t$. Then, Equation 1 becomes as follows.

$$f_{v_1} - f_{v_2} = w \begin{vmatrix} 0 & u_2^t \\ u_2 & A' - \lambda I \end{vmatrix} - w \begin{vmatrix} 0 & u_1^t \\ u_1 & A' - \lambda I \end{vmatrix}$$
$$= 0.$$

In order for f_{v_1} and f_{v_2} to be the same, it is necessary that $f_{v_1} - f_{v_2} = 0$ for all λ. So, we assume $|A' - \lambda I| \neq 0$. Then,

$$\begin{aligned} f_{v_1} - f_{v_2} &= & w|A' - \lambda I||0 - u_2^t(A' - \lambda I)^{-1}u_2| \\ & - & w|A' - \lambda I||0 - u_1^t(A' - \lambda I)^{-1}u_1| \\ &= & w|A' - \lambda I|(u_2 - u_1)^t(A' - \lambda I)^{-1}(u_2 - u1) \\ &= & 0. \end{aligned}$$

When $u_1 = u_2$, f_{v_1} and f_{v_2} are the same. In this case, then, S_{v_1} and S_{v_2} are isomorphic.

Let $u_2 \neq u_1$. When $(u_2 - u_1)^t(A' - \lambda I)^{-1}(u_2 - u_1) = 0$, $u_2 - u_1$ and $(A' - \lambda I)^{-1}(u_2 - u_1)$ are orthogonal. So,

$$(u_2 - u_1)^t(A' - \lambda I)(u_2 - u_1) = u_2^t A' u_2 - u_1^t A' u_1 - u_2^t \lambda I u_2 + u_1^t \lambda I u_1$$
$$= 0.$$

In order for f_{v_1} and f_{v_2} to be the same, it is necessary that $f_{v_1} - f_{v_2} = 0$ for all λ. So, the number of elements with value 1 in u_2 and u_1 is the same.

Since $u_2 - u_1$ and $(A' - \lambda I)(u_2 - u_1)$ are orthogonal,

$$\begin{aligned} (u_2 - u_1)^t A'(u_2 - u_1) &= (u_2 - u_1)^t P'^t A' P'(u_2 - u_1) \\ &= (u_1 - u_2)^t P'^t A' P'(u_1 - u_2) \\ &= 0 \end{aligned}$$

with P' a liner operator. When A_1 and A_2 have the same eigenvalue set, there exists a set of nontrivial permutation matrices $\{P'|P'^t A' P' = A' \wedge (u_2 - u_1) = P'(u_1 - u_2))\}$. So, S_{v_1} and S_{v_2} are isomorphic. $\square$

Corollary II.3. Let $S_{v_i} = Sg(S, v_i, w)$ and $S_{v_j} = Sg(S, v_j, w)$ with $v_i, v_j \in V_{w0}(S)$, $v_i \neq v_j$ and $w > 0$. If $Ev(S_{v_i}) \neq Ev(Sv_j)$, then S_{v_i} and S_{v_j} are not isomorphic.

Proof. Using a permutation matrix P, $A(S_{v_i}) \neq P^t A(S_{v_j})P$. So, there is no bijection between S_{v_i} and S_{v_j}. Therefore, S_{v_i} and S_{v_j} are not isomorphic. $\square$

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