The problem of the «negative frequencies» of the solutions of the D'Alembert equation

Marcello Colozzo

Abstract

The appearance of solutions with negative frequency in the D'Alembert wave equation can be removed with a change of variable. The corresponding positive frequencies describe waves propagating from the "future" towards the "past". This argument was developed in the 1940s by the Italian mathematician Luigi Fantappiè [1] in the analysis of the solutions of the D'Alembert equation, but also of the Klein-Gordon equation (quantum particles of spin 0) and the Dirac equation (spin 1/2 particles).

1 The D'Alembert equation

As is known, the D'Alembert wave equation

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{1}$$

is a linear, second-order partial differential equation (PDE) in $\psi(x, y, z, t)$. It is often written as:

$$\Box^2 \psi = 0,$$

where

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

is the *Delambertian*. The solutions of (1) si classificano in:are classified into: A) plane waves; B) spherical waves; C) standing waves. We are interested in case A. For the remaining cases, please refer to [2].

Rammentiamo che a differenza delle equazioni differenziali ordinarie (ODE), nelle PDE non interessa l'integrale generale, ma soluzioni soddisfacenti particolari condizioni al contorno o iniziali.

Given this, plane waves (described by a wave function $\psi(x, y, z, t)$) are characterized by a constant propagation direction verifying the following property: on every plane normal to this direction, the d'function wave ψ depends only on the variable t. It follows that by orienting the x axis in the direction of propagation, the (1) is rewritten:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{2}$$

Definition 1 *We say* **solution** *of the (2) any $\psi \in C^2(\mathbb{R}^2)$ which verifies (2).*

Notation 2 *The definition (1) can be weakened by incorporating any finite discontinuities of the derivatives of ψ and of ψ itself.*

Theorem 3 *A necessary and sufficient condition for $\psi \in C^2(\mathbb{R}^2)$ to be a solution of (2), is that it admits a decomposition of the type:*

$$\psi(x, t) = f(x - ct) + g(x + ct), \quad f, g \in C^2(\mathbb{R}) \tag{3}$$

Proof. The sufficiency of the condition is immediate, since $f(x-ct)$ and $g(x-ct)$ are manifestly solutions of (2). To demonstrate the need, we perform the coordinate transformation in the xt plane:

$$(x,t) \to (\xi,\eta)\,, \tag{4}$$

whose transformation equations are:

$$\xi = x - ct,\ \ \eta = x + ct, \tag{5}$$

so that (4) is manifestly invertible:

$$x = \frac{1}{2}(\xi+\eta)\,,\ \ t = \frac{1}{2c}(\eta-\xi) \tag{6}$$

The (5) imply $\psi(x,t) \equiv \psi[\xi(x,t),\eta(x,t)]$. Applying the derivation rule of composite functions:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x}$$

From (5) $\frac{\partial\xi}{\partial x} = 1$, $\frac{\partial\eta}{\partial x} = 1$, so

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi} + \frac{\partial\psi}{\partial\eta}, \tag{7}$$

which can be rewritten as:

$$\frac{\partial\psi}{\partial x} = \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\psi \tag{8}$$

This relation is valid for every differentiable function ψ. This circumstance suggests writing the formal expression

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}, \tag{9}$$

which links the partial differentiation operator with respect to x, to the differentiation operators with respect to the variables ξ and η. To determine the second partial derivative $\frac{\partial^2\psi}{\partial x^2}$, we can then write:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right) \\ &= \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta\partial\xi} + \frac{\partial^2}{\partial\eta^2}\end{aligned} \tag{10}$$

It is clear that we can write

$$\frac{\partial^2}{\partial\xi\partial\eta} = \frac{\partial^2}{\partial\eta\partial\xi}$$

if and only if this operator acts on a function that verifies the hypotheses of Schwarz's theorem on the invertibility of partial differentiation, i.e. of class C^2 on an assigned field A of $\mathbb{R}^2$. Since we are looking for solutions $\psi \in C^2(\mathbb{R}^2)$, this condition is satisfied, so the (10) is written:

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial\xi^2} + 2\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2}, \tag{11}$$

so

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial\xi^2} + 2\frac{\partial^2\psi}{\partial\xi\partial\eta} + \frac{\partial^2\psi}{\partial\eta^2} \tag{12}$$

Proceeding in the same way for the second derivative $\frac{\partial^2 \psi}{\partial t^2}$

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \left(\frac{\partial^2 \psi}{\partial \eta^2} - 2\frac{\partial^2 \psi}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \xi^2} \right) \tag{13}$$

At this point we impose that ψ is a solution of the D'Alembert equation:

$$0 = \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 4\frac{\partial^2 \psi}{\partial \xi \partial \eta}$$

i.e.

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0, \tag{14}$$

which is the D'Alembert equation written in coordinates (ξ, η), and integrates immediately. Indeed:

$$\frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial \xi}\right) = 0 \Longrightarrow \frac{\partial \psi}{\partial \xi} = \theta\left(\xi\right),$$

being $\theta\left(\xi\right) \in C^2\left(\mathbb{R}\right)$ an arbitrary function. Integrating again:

$$\psi\left(\xi, \eta\right) = \int \theta\left(\xi\right) d\xi + g\left(\eta\right), \tag{15}$$

where the arbitrary function $g\left(\eta\right) \in C^2\left(\mathbb{R}\right)$ plays the role of "constant" of integration (with respect to the variable ξ). We therefore set:

$$f\left(\xi\right) \stackrel{def}{=} \int \theta\left(\xi\right) d\xi,$$

so

$$\psi\left(\xi, \eta\right) = f\left(\xi\right) + g\left(\eta\right)$$

By restoring the variables (x, t) the statement follows. ■

Definition 4 *The solutions $f(x - ct)$ and $g(x + ct)$ are called* **progressing wave** *and* **regressive wave**.

These names are suggested by the fact that taking time t as the real parameter, the graph of the function $f(x - ct)$ $[g(x + ct)]$ translates uniformly in the direction of the x axis and in the direction of the increasing [decreasing] abscissae. If t is the time, the translation occurs in both cases at speed c, as shown in the Figures 1-2.

2 Fundamental solutions

Fundamental solutions are those for which ψ depends sinusoidally on $x \pm ct$. They are called fundamental because from them we can reconstruct a more general solution by linear superposition (thanks to the linearity of (2)). For example:

$$\psi\left(x, t\right) = A\cos\left[\frac{2\pi}{\lambda}\left(x - ct\right)\right] \tag{16}$$

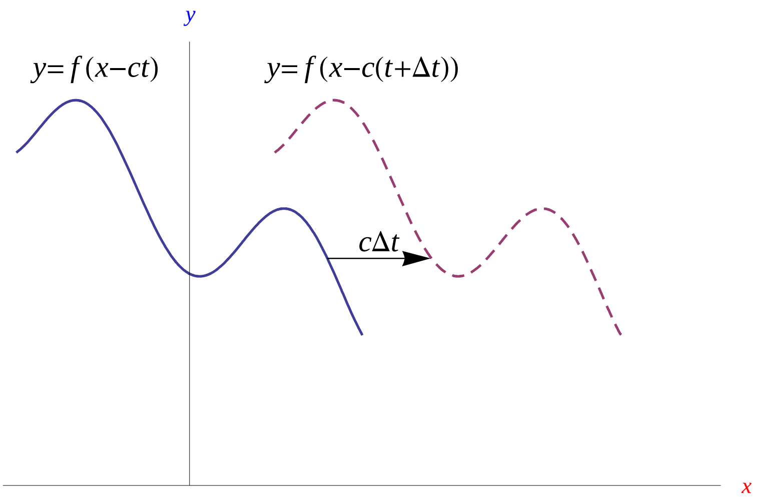

Figure 1: Progressive plane wave.

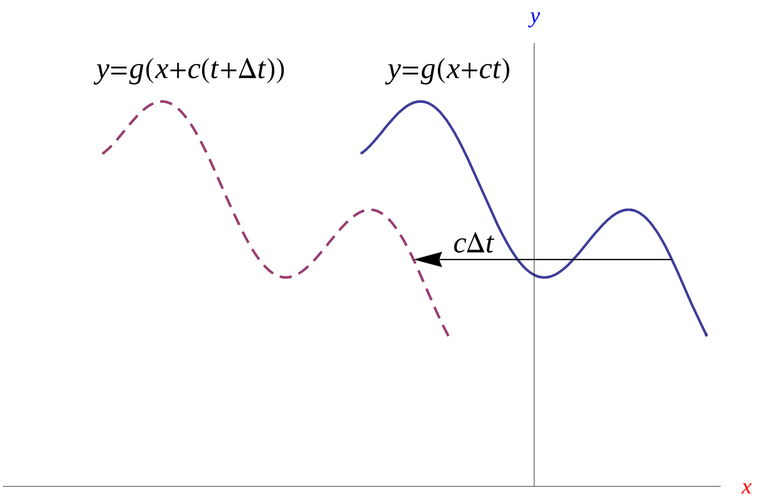

Figure 2: Regressive plane wave.

where $A > 0$ is the amplitude, while $\lambda > 0$ is the period of ψ with respect to x for a given instant. This quantity is called wavelength. Let's define

$$k \in \mathbb{R} \setminus \{0\} \mid |k| = \frac{2\pi}{\lambda}$$

We call the positive real number $|k|$ *wavenumber*. Continued

$$\psi(x,t) = A\cos\left[\left(|k|\,x - \omega t\right)\right] \tag{17}$$

having defined the angular frequency $\omega = c\,|k| = \frac{2\pi}{T}$ where T is the period of the function ψ with respect to t (for an assigned x). If in ((17) we free ourselves from $|k|$:

$$\psi(x,t) = A\cos\left[\left(kx - \omega t\right)\right] \tag{18}$$

which for $k < 0$ describes a regressive plane wave. Complex notation is preferable:

$$\psi(x,t) = Ae^{i(kx-\omega t)} \tag{19}$$

3 Solutions with negative frequency

The totality of (19) does not exhaust the set of solutions of (2) relative to the fundamental solutions. In fact, by imposing that (19) is a solution, we have

$$\omega^2 = c^2k^2$$

therefore negative frequencies are also allowed $\omega = -ck < 0$. In this case, the (19) is rewritten

$$\psi(x,t) = Ae^{i(kx+|\omega|t)} \tag{20}$$

Performing the change of variable $t' = -t$

$$\psi(x,t') = Ae^{i(kx-|\omega|t')} \tag{21}$$

having

$$-\infty < t = -t' < +\infty \Longrightarrow +\infty > t' > -\infty \tag{22}$$

It follows that while $\psi(x,t) = Ae^{i(kx+|\omega|t)}$ describes the propagation of a plane wave with initial instant $t_0 = -\infty$ («past») and with negative frequency, the function $\psi(x,t') = Ae^{i(kx-|\omega|t')}$ describes the propagation of a plane wave with initial instant $t'_0 = +\infty$ («future»), with positive frequency. This wave propagates backwards in time.

References

[1] Arcidiacono G., Fantappié e gli universi. Nuove vie della scienza.

[2] Fasano A., Marmi S., Analytical Mechanics.