Proof of the Generalized Continuum Hypothesis

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continuum | hypothesis | set | ZFC | independence

Abstract

The Continuum Hypothesis has recently been proven in a form that might have been accepted had it appeared before ZFC but after Hilbert's challenge in 1900. This work will develop the same technique to prove the Generalized Continuum Hypothesis by induction on $\aleph$ and $\beth$ subscripts.

1 Significance Statement

Set theory exists to render rigorous the study of infinite numbers. Two series of such transfinite numbers (or cardinals) are the aleph ($\aleph$) and beth ($\beth$) numbers, where $\aleph_{n+1}$ is the very next cardinal after $\aleph_n$, and $\beth_{n+1}$ is $2^{\beth_n}$. The generalized continuum hypothesis (GCH) states that these two sequences are the same. Because earlier work has shown GCH neither provable nor disprovable in the most popular set theory axiomatization (ZFC), having a plausible proof of GCH outside of ZFC may point to a new or strengthened axiomatization that will be able to prove the new result. Once such a strengthened set theory is constructed, many might wish to study its other theorems.

2 Introduction

We [1] showed that the Continuum Hypothesis (CH) may be proved by a relatively simple argument involving pairing some paths in the binary tree with all of the countable ordinals.

In summary, $\aleph_1$ many "ordinal paths" cannot crowd into one final tree path-segment because of the definition of an injection (as being from and to distinct elements), implying that [2], sooner or later, each node with $\aleph_1$ ordinal paths passing through it either has two children also with $\aleph_1$ ordinal paths passing through them or has a descendant that has two such children.

This endless division of the paths that contain only such "$\aleph_1$-nodes" makes these "$\aleph_1$-node-paths" continuum many, while the mapping of only countable ordinals makes these same paths $\aleph_1$ many.

The referenced paper contains an explicit bijection between all binary strings of length ω and the "$\aleph_1$-node-paths," completing the proof. Some objections gathered from MathOverflow were also answered.

3 Induction Hypothesis (Concrete Example) $\aleph_2 = \beth_2$

We will show how to carry on from the Continuum Hypothesis (CH) as the base case to the next step in the induction. The full induction hypothesis covers all successor ordinals, but to aid the understanding we will first prove that $\aleph_2 = \beth_2$ given $\aleph_1 = \beth_1$.

Consider a transfinite binary tree having paths of length ω_1. Such a tree has $2^{\aleph_1} = 2^{\beth_1} = \beth_2$ paths, where the foregoing identity makes use of the earlier step, which in this instance is the base case of the induction, $\aleph_1 = \beth_1$. There is an injection from ω_2 to the paths in this tree, because $\aleph_2 \leq \beth_2$.

We call a path a "tree path" regardless of whether any ordinal is mapped to it by this injection. We will call a path an "ordinal path" if this injection assigns a member of ω_2 to that path. By an $\aleph_n$-node we will mean that $\aleph_n$ ordinal paths pass through that node. Finally, an $\aleph_n$-node-path is a path whose every node is an $\aleph_n$-node.

Figure 1 shows a situation in which an $\aleph_2$-node's descendants have only one $\aleph_2$-node at every lower level of the tree. This may be thought of as "removing" from the left (red) path, $\aleph_1$ many paths, $\aleph_1$ many times. This still leaves $\aleph_2$ ordinal paths mapped to one tree path, which contradicts the definition of an injection. After eliminating all subtrees like this one, we see the opposite condition is required [2] for the mapping to be an injection: that beneath every $\aleph_2$-node there are either two $\aleph_2$-node children or a descendant with two $\aleph_2$-node children.

Now label each pair of nodes in which two $\aleph_2$-nodes share a parent as follows. The first such pair are labelled "0" (for the left child) and "1" (for the right child). For the rest, append "0" to the nearest ancestor's label for the left child and "1" to the nearest ancestor's label for the right child. Because the tree is transfinite, these labels include not only finite (for nodes in the first part of the tree) and ω-length labels (for ω-length paths), but labels of all lengths up to ω_1 for paths in the whole tree. Figure 2 shows how the labels are built incrementally and Figure 3 shows some examples of labels. Figure 4 gives an explicit bijection between a set of size $\beth_2$ and a set of size $\aleph_2$.

4 Induction Hypothesis (Successor Ordinals)

In the foregoing section, replace $\aleph_2$ with $\aleph_n$, ω_1 with ω_{n-1}, and $\beth_2$ with $\beth_n$. The result is a proof of $\aleph_n = \beth_n$ for all successor ordinals n.

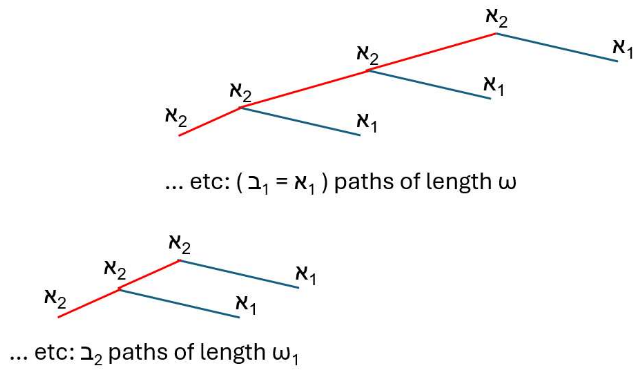

Figure 1: A subtree like this violates the definition of an injection as mapping to distinct elements. In this case each $\aleph_2$-node has an $\aleph_1$-node child so that the red path has $\aleph_1$ ordinal paths removed, $\aleph_1$ times, so that $\aleph_2$ ordinal paths are left over to map to the red path.

5 Limit Step (Concrete Example) ($\aleph_\omega = \beth_\omega$)

Recall the definitions of $\aleph_\omega$: $\lim\{\aleph_0, \aleph_1, \aleph_2, ...\}$ and $\beth_\omega$: $\lim\{\beth_0, \beth_1, \beth_2, ...\}$. By the base case and the induction hypothesis, we have shown that $\aleph_n = \beth_n$, that is, the two limits are limits of sequences that are term-by-term equivalent. Since the two limits are equal, so are the values $\aleph_\omega$ and $\beth_\omega$.

6 Limit Step ($\aleph_\alpha = \beth_\alpha$)

In a similar way, higher limit $\aleph$ and $\beth$ numbers are defined by limits that are term-by-term equivalent as shown at earlier stages of the induction, including the base case, induction hypothesis, and $\aleph_\omega$ case. $\aleph_\alpha = \beth_\alpha$ for all limit ordinals α, and thus (combined with earlier results) for all ordinals α.

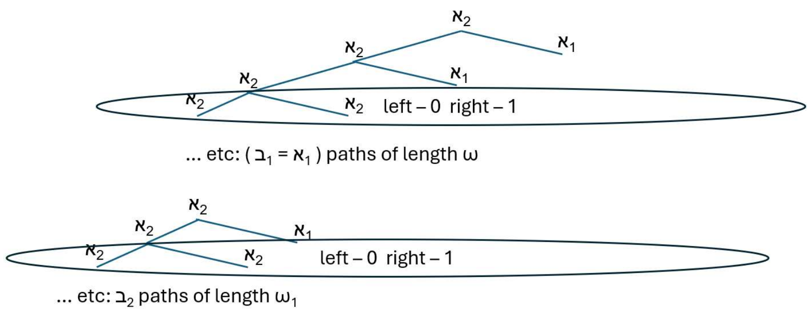

Figure 2: Append 0 to left children's labels and 1 to right children's labels

References

[1] Selke, David (2024) *The continuum theorem.* Preprint online at `https://vixra.org/abs/2402.0031`

[2] Gustason, William and Ulrich, Dolph (1989) *Elementary Symbolic Logic, 2nd Edition.* Waveland Press, Inc. Prospect Heights, Illinois. pp. 119–124

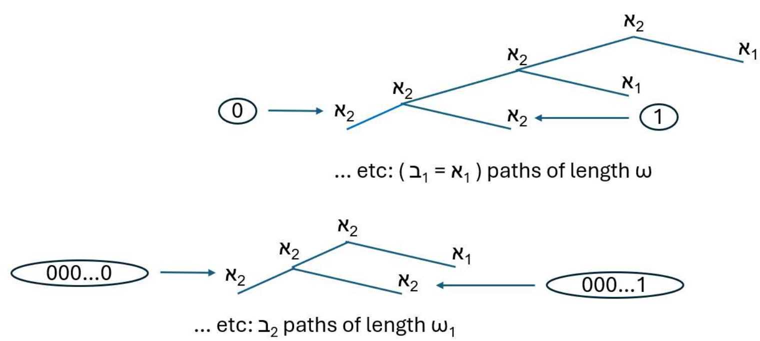

Figure 3: Some examples of labels including after the first ω-length segment

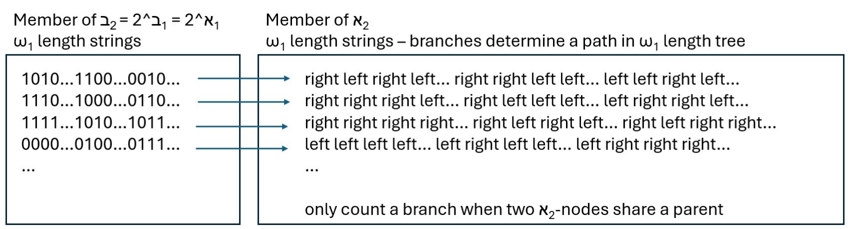

Figure 4: Explicit bijection from a set of size $\beth_2$ to a set of size $\aleph_2$