A UNIFORM LOWER BOUND FOR THE PROBABILITY OF k PLAYERS TIED FOR FIRST PLACE USING SUPERTELESCOPING SERIES

MATHIS ANTONETTI

Abstract. In this note, we establish a uniform lower bound (w.r.t. the number of players) for the probability of k players tied for first place in the geometric case. To derive this bound, we introduce the concept of supertelescoping series as a generalization of telescoping series. We also provide an insight on the relationship between supertelescopic series and supermartingales.

1. Introduction

The concept of supertelescoping series is a generalization of telescoping series sequences.

Definition 1.1. *A triple sequence $a = (a_{k,j,n})_{k,j,n\in\mathbb{N}}$ is a supertelescoping sequence if and only if there exists two real sequences $(c_j)_{j\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}^*}$ such that*

$$\forall k, n \geq 0, \forall j \geq 1, \quad d_{n+1}a_{k+1,j,n} \geq a_{k+1,j-1,n+1} - (1 - c_{j-1})a_{k,j-1,n+1}. \tag{1}$$

A supertelescoping series is a series $\sum_k a_{k,j,n}$ such that $(a_{k,j,n})_{k,j,n}$ is a supertelescoping sequence. Such series have the following fondamental property.

Property 1.2. *If $\sum_k a_{k,j,n}$ converges and $a_{0,j-1,n+1} = 0$ for some $j \geq 1, n \geq 0$, then*

$$d_{n+1}\sum_{k=1}^{+\infty} a_{k,j,n} \geq c_{j-1}\sum_{k=1}^{+\infty} a_{k,j-1,n+1}$$

Proof. Since $\sum_k a_{k,j,n}$ converges, we know that $a_{k,j,n} \underset{k\to+\infty}{\to} 0$ thus by summing over k the equation (1), we get the result. $\square$

2. Probability of a tie for first place

A non trivial example of such telescoping series would be the probability of j players (among n players) tied for first place in the geometric case which is (see [Bennett et al.(1993)])

$$\mathbb{P}(N_n = j) = (1-q)^j\binom{n}{j}\sum_{k=0}^{+\infty} a_{k,j,n-j},$$

where $0 < q < 1$, $n \in \mathbb{N}^*$ and

$$\forall k, j, n \in \mathbb{N}, \quad a_{k,j,n} = q^{kj}(1-q^k)^n. \tag{2}$$

Date: April 2024.

The asymptotic results on $\mathbb{P}(N_n = j)$ are well-known since 1993 but no exact lower bound was provided in [Bennett et al.(1993)]. We derive here a uniform lower bound with respect to n. Surprisingly, the uniformity naturally appears when we do the calculus.

Theorem 2.1. *We have*

$$\mathbb{P}(N_n = j) \geq \frac{1-q}{j!} \prod_{i=1}^{j-1} \left(q(1-q^i)\right).$$

Proof. Indeed, we have for all $j \geq 1$,

$$\begin{aligned}\frac{1-q}{q} a_{k+1,j,n} = q^{(j-1)(k+1)} \left(q^k(1-q)(1-q^{k+1})^n\right) &\geq q^{(j-1)(k+1)} \left(q^k(1-q^{k+1})^n\right) \\ &\geq q^{(j-1)(k+1)} \int_{1-q^k}^{1-q^{k+1}} x^n dx\end{aligned}$$

using $\int_a^b f \leq \max(f)(b-a)$. Thus we obtain

$$\begin{aligned}(n+1)\frac{1-q}{q} a_{k,j,n} &\geq q^{(j-1)(k+1)} \left((1-q^{k+1})^{n+1} - (1-q^k)^{n+1}\right) \\ &\geq a_{k+1,j-1,n+1} - q^{j-1} a_{k,j-1,n+1}.\end{aligned}$$

It is clear that for this particular case, $\sum_k a_{k,j,n}$ converges and $a_{0,j,n} = 0$ for all $j \geq 1, n \geq 0$. Hence using Property 1.2, we get

$$\sum_{k=0}^{+\infty} a_{k,j,n} \geq \prod_{i=1}^{j-1} \frac{c_i}{d_{n+i}} \sum_{k=0}^{+\infty} a_{k,1,n+j-1} \geq \frac{(n!)q^{j-1} \prod_{i=1}^{j-1} \frac{1-q^i}{1-q}}{(n+j)!} \sum_{k=0}^{+\infty} (a_{k+1,0,n+j} - a_{k,0,n+j}) = \frac{(n!)q^{j-1}}{(n+j)!} \prod_{i=1}^{j-1} \frac{1-q^i}{1-q}.$$

Thus

$$\mathbb{P}(N_n = j) \geq (1-q)^j \binom{n}{j} \frac{(n-j)!}{n!} \prod_{i=1}^{j-1} \frac{\left(q(1-q^i)\right)}{1-q}.$$

The result follows. □

3. Supertelescoping is related to supermartingales

Let $E = \{y_k \mid k \in \mathbb{N}^*\}$ a discrete state space, and $M = (M_n)$ a discrete stochastic process on E (see [Doob(1962)]). Take $A_{k,j,n} = \mathbb{P}(M_{j+1} = y_k \mid M_{n+j}, M_{n+j-1}, ..., M_1)$. We have the following fundamental property :

Property 3.1. *If $(A_{k,j,n})_{k,j,n}$ is a.s. a supertelescoping sequence with $d_n = 1 = c_j$ then M is a supermartingale.*

Proof. If $A_{k,j,n}$ is a.s. a supertelescoping sequence with $d_n = 1 = c_j$ then we have
$\forall k \in \mathbb{N}^*, \mathbb{P}(M_{n+1} = y_k \mid M_n, ..., M_1) = a_{k,n,0} \leq a_{k,n-1,1} = \mathbb{P}(M_n = y_k \mid M_n, ..., M_1) = \mathbb{1}_{M_n = y_k} a.s.$,
thus $\mathbb{E}[M_{n+1} \mid M_n, ..., M_1] \leq M_n$. The result follows. □

4. Conclusion

We derived a new uniform lower bound using a new powerful concept called supertelescoping series. We also saw that the supertelescoping series seem to be deeply related to other concepts in mathematics such as supermartingales. This will be the entire subject of a future paper that will investigate more deeply the applications of supertelescoping series.

References

[Bennett et al.(1993)] Eisenberg Bennett, Stengle Gilbert, and Strang Gilbert. 1993. The asymptotic probability of a tie for first place. *The Annals of Applied Probability* 3 (1993), 731–745.
[Doob(1962)] Joseph L. Doob. 1962. *Stochastic Processes*.

Email address: mathis.antonetti@gmail.com