# MINIMAL DIVIDING ODD SUBSETS ARE RELATED TO THE GOLDEN RATIO

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Abstract. In this note, we give a proof for a lower bound (though not a satisfying one) of the density of minimal dividing odd subsets where the golden ratio surprisingly appears. We also provide some other properties of such integer subsets and some new insight on the relationship between minimal dividing odd subsets and the Goldbach conjecture. We argue that the study of minimal dividing odd subsets is an interesting starting point to prove the Goldbach conjecture.

## 1. Introduction

We define as in [Antonetti(2024)]

$$m(E) = \max\{m \in \mathbb{N}+1 \mid 2[\![1,m]\!] \subset E+E\}, \tag{1}$$

and for any $n \in \mathbb{N}+1$,

$$E_n = \underset{E\subset 2\mathbb{N}+1, \mathrm{Card}(E)=n}{\mathrm{argmax}} m(E). \tag{2}$$

Then by definition, $E_n$ contains all the subsets $E$ of at most $n$ elements such that $E+E$ contains $[\![1,m]\!]$ with $m$ as large as possible. In the sequel, we are interested in $m(E_n) = \max_{E\in E_n} m(E) = \min_{E\in E_n} m(E)$ and more precisely in $d(n) = \frac{n}{m(E_n)}$. In fact, $d(n)$ is the density of odd numbers necessary to retrieve the even numbers up to $2m(E_n)$. That is why $d$ is an interesting function to study.

In order to explore further the realm of dividing odd subsets, we try to rigorously define this concept.

**Definition 1.1.** $(A_n)_{n\in\mathbb{N}}$ *is an odd dividing subsets sequence if and only if*

$$\forall m \in \mathbb{N}+1, \exists n \in \mathbb{N}, \quad 2[\![1,m]\!] \subset A_n + A_n.$$

**Remark 1.2.** *When we say "odd dividing subset", it is actually an abuse of language. This means that it induces an odd dividing subsets sequence (i.e. its form gives the elements of the sequence).*

Once again, under the Goldbach conjecture, $\mathcal{P} = (\mathcal{P}_n)_n$ is an odd dividing subsets sequence (where $\mathcal{P}_n = \{1, p_2, ..., p_n\}$ with $p_n$ being the $n$-th prime). However, we have $\frac{p_n}{2} \leq m(\mathcal{P}_n) \leq p_n$ (using the properties 2.1) and therefore obtain with the prime number theorem

$$1 \leq \liminf_{n\to+\infty} \left(\frac{n}{m(\mathcal{P}_n)}\right) \ln(n) \leq \limsup_{n\to+\infty} \left(\frac{n}{m(\mathcal{P}_n)}\right) \ln(n) \leq 2.$$

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Thus the Goldbach conjecture actually gives a very bad estimation of $d(n)$. This means we need something stronger than the Goldbach conjecture and so maybe we could also prove back the conjecture by using those stronger results.

We recently proved the following results in [Antonetti(2024)] :

**Theorem 1.3.** *Let $n \in \mathbb{N}+1$, we have*

$$d(n) \leq \frac{n}{2n(p(n)+1) - 2p(n)(2p(n)+1) - 1} = U(n), \tag{3}$$

*where* $p(n) = \begin{cases} \frac{n}{4} \text{ if } 4 \mid n \\ \left\lfloor \frac{n-1}{4} \right\rfloor \text{ otherwise} \end{cases}.$

**Corollary 1.4.** *We have* $\limsup\limits_{n\to+\infty} nd(n) = 4$.

This note shows that we can find a lower bound as well and this lower bound heavily depends on the golden ratio $\varphi$, showing its entanglement with the minimal odd dividing subsets.

## 2. Elementary properties

We can derive the following elementary properties on odd dividing subsets and the function $m$.

**Properties 2.1.** *Let $A \subset 2\mathbb{N}+1$, we have*

$$m(A) \leq \max(A).$$

*If $(A_n)_n$ is an odd dividing subsets sequence, then*

$$\exists N \in \mathbb{N}, \forall n \geq N, \quad \{1,3\} \subset A_n. \quad (-)$$

*Moreover, if $(A_n)_n$ is increasing (i.e. such that $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$) and $A = A_n$ for some $n \in \mathbb{N}$, then*

$$m(A) \geq \frac{\max(A)+1}{2}.$$

*Proof.* We have by the definition (1) of $m$ that $2m(A) = a+b \leq 2\max(A)$ for some $a, b \in A$. Therefore, $m(A) \leq \max(A)$.

The property $(-)$ is obvious. Now, consider an increasing odd dividing subsets sequence $(A_n)_n$ and take $A = A_n$ for some $n \in \mathbb{N}$. Then we have

$$\forall k \geq n, \quad \max(A_n) < \min(A_k \setminus A_n), \ (*)$$

thus

$$\forall (a,b) \in (A_k \setminus A_n) \times A_k, \quad \max(A_n) + 1 < a + b$$

. Assume that $\max(A) + 1 - 2p \notin A_n + A_n$ for some $p \in \mathbb{N}$. Then clearly,

$$\forall k \leq n, \quad \max(A) + 1 - 2p \notin A_k + A_k., \ (**)$$

Moreover, if $k \geq n+1$ and $\max(A) + 1 - 2p \in A_k + A_k$, then we have $a, b \in A_k$ such that

$$\max(a,b) \leq a + b = \max(A) + 1 - 2p,$$

so $a, b \notin A_k \setminus A_n$, therefore $a, b \in A_n$ and thus we have a contradiction with $(*)$. Thus we obtain

$$\forall k \geq n+1, \quad \max(A) + 1 - 2p \notin A_k + A_k. \ (***)$$

Using $(**)$ and $(***)$, we get $\max(A)+1-2p \notin A_k + A_k$ for all $k \in \mathbb{N}$ which contradicts the definition (1.1) of an odd dividing subsets sequence. The result follows by contradiction. $\square$

Another remarkable property is that $(m(E_n))_n$ is strictly increasing, i.e.

$$\forall n \in \mathbb{N}, \quad m(E_n) + 1 \leq m(E_{n+1}). \tag{4}$$

Indeed, we can build $F = \{f_1, ...f_n, k\}$ such that $\{f_1, ...f_n\} \in E_n$, $f_1 < ... < f_n$ and $k = 2m(F) - f_1$ so that $m(E_{n+1}) \geq m(F) = m(E_n) + 1$.

## 3. A conjecture and the main result

We have seen that $\limsup\limits_{n\to+\infty} nd(n) \leq 4$ in [Antonetti(2024)] but no result was given for $\liminf\limits_{n\to+\infty} nd(n)$. Surprisingly, it is more difficult to prove a decent lower bound of $d$ than to prove the previously given upper bound. Based on the experiments introduced later in this paper, it seems like $d(n) \underset{n\to+\infty}{\sim} \frac{C}{n}$ for some $C \geq \frac{5}{2}$ but to the best of the authors' knowledge, no proof has been written in the litterature. However, we give here the following partial result related to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

**Theorem 3.1.** *Let $n \in \mathbb{N}+1$, we have*

$$d(n) \geq \frac{n(3\varphi+1)}{(4\varphi+2)\varphi^n + 2\varphi(-\varphi^{-1})^n - (3\varphi+1)} = L(n).$$

As before, we deduce the following result.

**Corollary 3.2.** *We have $\liminf\limits_{n\to+\infty} \frac{d(n)}{n\varphi^{-n}} \geq \frac{3\varphi+1}{4\varphi+2}$.*

*Proof.* Define by induction $v_k$ such that $v_{k+2} = v_{k+1} + v_k + 1$, $v_1 = 1$, $v_2 = 2$ and $\tilde{E}_n = \{\{x_1, ..., x_n\} \mid x_1 < ... < x_n \text{ and } \forall k \in [\![1, n]\!],\ x_k \in 2[\![k, v_k]\!] - 1\}$.

<u>Step 1 :</u> We claim that $E_n \subset \tilde{E}_n$. We prove this by strong induction. The initial cases $E_1 \subset \tilde{E}_1$, $E_2 \subset \tilde{E}_2$, $E_3 \subset \tilde{E}_3$ are easy to check. Now, assume that $\forall k \leq n, E_k \subset \tilde{E}_k$ for some $n \in \mathbb{N}+3$. Let $F = \{f_1, ..., f_{n+1}\} \in E_{n+1}$ with $f_1 < ... < f_{n+1}$. Under such assumption, we have for all $k \in [\![4, n+1]\!]$, $2v_{k-1} > v_{k-1} + v_{k-2} + 2$ and

$$\forall F \in E_n \subset \tilde{E}_n, \quad \{v_{k-2}+2, v_{k-1}\} \not\subset F,$$

so $2m(E_{k-1}) \leq v_{k-1} + v_{k-2}$. Thus if we assume by contradiction that $f_k \geq 2v_k + 1$, then $f_k \geq 2(v_{k-1} + v_{k-2} + 1) + 1 \geq 2(2m(E_{k-1}) + 1) + 1 \geq 2m(\{f_1, ..., f_{k-1}\}) + 3$. Thus $f_1 + f_k > 2m(\{f_1, ...f_{k-1}\}) + 2$ and so $2m(\{f_1, ...f_{k-1}\}) + 2 \notin F + F$. Thus $m(E_{n+1}) = m(F) \leq m(\{f_1, ...f_{k-1}\}) \leq m(E_{k-1})$ which is in contradiction with (4). This means that $f_k \leq 2v_k - 1$.

Moreover, we have $m(E_{n+1}) \geq n+1$ thus $f_1 = 1$ (it is necessary to have $2 \in F + F$), $f_2 = 3$ (necessary to have $4 \in F + F$) and $f_3 \in \{5, 7\}$ (to have $8 \in F + F$). We finally obtain $F \in \tilde{E}_{n+1}$ thus $E_{n+1} \subset \tilde{E}_{n+1}$.

<u>Step 2 :</u> With a classical computing method, we get

$$v_n = \frac{2\varphi+1}{3\varphi+1}\varphi^n + \frac{\varphi}{3\varphi+1}(-\varphi^{-1})^n.$$

Furthermore, we have $m(E_n) \leq \max\{\max(F) \mid F \in E_n\} \leq \max\{\max(F) \mid F \in \tilde{E}_n\} = 2v_n - 1$. The Theorem 3.1 follows. $\square$

## 4. Experiments

The lower bound proven previously seems really bad according to the following figure. We also observe that $nd(n)$ seems to converge (since it is increasing in average and upper-bounded (c.f. Corollary 1.4)). This justifies our conjecture that $d(n) \sim \frac{C}{n}$.

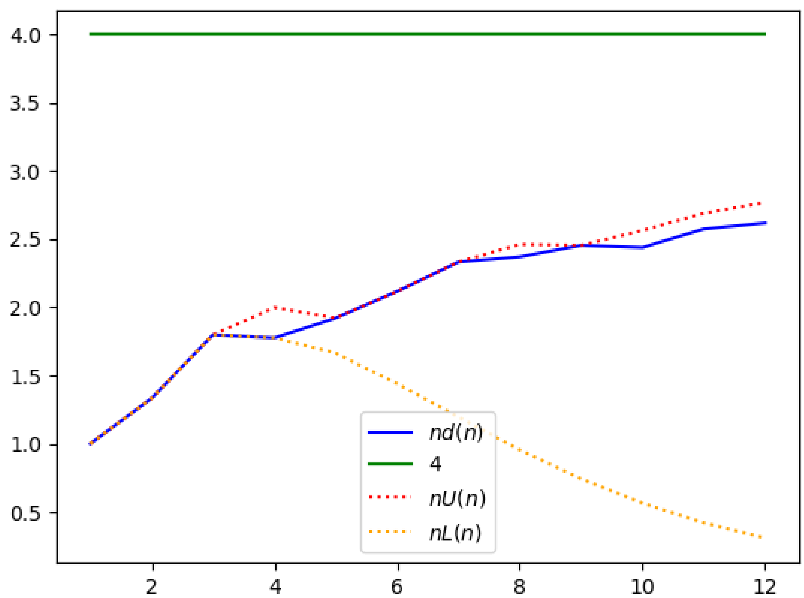


Figure 1. Comparison of $nd(n)$, $nU(n)$, $nL(n)$ and 4 for $n = 1, ..., 12$.

## 5. Conclusion

We have shown more properties of the odd dividing subsets and we showed the deep relationship between the golden ratio and the minimal odd dividing subsets. In particular, we have seen that the lower bound naturally derived from this relationship is not that good, hinting that we do not yet understand those subsets enough. We also showed that the minimal dividing odd subsets is deeply related to the Goldbach conjecture. We believe that it may be an interesting starting point to prove the Goldbach conjecture.

## References

[Antonetti(2024)] Mathis Antonetti. 2024. The Density of Minimal Dividing Odd Subsets for the Even Numbers is Asymptotically Normal. *viXra* (2024).

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