Contribution to Lonely Runner Conjecture

Radomir Majkic
E-mail: radomirm@hotmail.com

Abstract: The Lonely Runner conjecture finds its mathematical description in winding the runner's linear paths into the complete cycle, c-cycle, on the unit track circle. All runners, to finish the competition, must complete the c-cycle simultaneously. Any collection $\mathcal{R}_{\text{N}}$ of the N integer speeds runners and maximum speed $v_{\text{N}} = n$ is a subset of the enveloping collection $\mathcal{R}_n = \{1, 2, 3, \cdots n\}$ of the n, $n > \text{N}$, runners with the maximum speed n.

The time period $1_{\text{T}} = 1/n$ of the fastest runner, f-runner n, defines the set of n right half open time f-segments of the measure 1_{T}, which cover the c-cycle time domain of the measure 1. The winding mapping of the linear paths $X(t)$ associates with the c-cycle Graph G, the union of the n individual graphs $g = (t, X(t))$, reduced to the domain of the c-cycle. The time domain segmentation partitions the Graph G into n Subgraphs G_i, each one on the one of the n f-segments. The final Subgraph bundle sinks into the point $(1, 1)$.

At the end of the first f-segment, all the runners arranged into f-constellation at the n fixed, stationary points on the unit circle in the sequence of the increasing speeds. However, at the final f-segment, the runners, on the way to the starting point, are arranged at the decreasing speed order at the same stationary points. The speed order inversion inverts the slope order of the graphs on the final Subgraph bundle. Finally, the infimum graph g_{n-1} of the Subgraph bundle of the $n-1$ runner's mutual separation graphs, the graph of the largest slope connects the points $(0, n-1)1_{\text{L}}$ and $(1, 1)1_{\text{L}}$. Consequently, the Lonely Runner conjecture is true on the set $\mathcal{R}_n$, and must be true on any of its subset $\mathcal{R}_{\text{N}}$,

Key words: Lonely runner, linear functions, graph, winding.

Introduction

The problem, originally stated in 1967 by German mathematician Jorg M. Wills as a problem of the number theory, is today an unsolved problem, known in its popular version as the Lonely Runner conjecture.

Problem: *Consider* N *runners on a circular track of unit length. At* $t = 0$ *all runners start from the same point with the distinct integer speeds. A runner is lonely at a timet if it is far for at least* $1/\text{N}$ *of any other runner. The Lonely Runner conjecture states that each runner must be lonely at some time.*

The Lonely Runner project is in the study phase. The conjecture is true for $\text{N} < 7$, and for some specific number of the runners and the choice of the integer speeds, see [1]. The existing proofs are the distinct inventive searches of the researches to find an approach wich leads to solution of the problem

Description

The runners are designated by integers identical to their integer constant velocities $v_i = i$ and placed in the set $\mathcal{R}_{\rm N} = \{1_{\rm N}, 2_{\rm N}, 3_{\rm N}, \cdots, v_{\rm N} = n\}$ of the size N. Clearly, $\rm N < n$. The set $\mathcal{R}_{\rm N}$ is the part of the continuously populated enveloping set $\mathcal{R}_n = \{1, 2, 3, \cdots, n\}$ of the size n and maximum speed n. Further, the relevant is the enveloping set and the fastest, f-runner n.

All the runners $\mathcal{R}_n$ start from the initial point O on the circular track of a unit length. The competition lasts until all runners complete the single cycle, called the c-cycle. The further competition is the repetition of the c-cycle. Clearly, the c-cycle duration is the time $\mathrm{T}_1 = 1/v_1 = 1$ which the slowest runner needs to complete its cycle. A runner i completes its cycle for a time equal to its period $\mathrm{T}_i = 1/i$. The collection of all periods is the set $\mathcal{T} = \{1, 1/2, 1/3, \cdots, 1/n$. The number of the cycles, i-cycles i runner completes during a single c-cycle is the frequency $\nu_i = \mathrm{T}_1/\mathrm{T}_i = 1/\mathrm{T}_i = i$, and the set of the frequencies of all runners is $\mathcal{N} = \{1, 2, 3, \cdots n\}$. We call the period $\mathrm{T}_1 = 1$ of the slowest runner natural period, which turns out to be also the c-cycle period, and the smallest period, the period of the f-runner we choose to be the f-time unit $1_{\rm T} = 1/n$. The f-length unit is the n^{th} part $1_{\rm L} = 1/n$, of the unit circle track. Notice that $1_{\rm L} = v_1 1_{\rm T}$, so that the time domain of the c-cycle contains n f-length units and the cycle domain $D = n \cdot 1_{\rm T} = 1 = n \cdot 1_{\rm L}$. Further $\mathrm{T}_1 \cdot i = 1 \cdot i = n \cdot 1_{\rm T} i = i(n \cdot 1_{\rm T}) = i \cdot 1$, so that all runners complete the c-cykle simultaneously.

The runner's frequencies $\nu(i)$ do partition the c-cycle domain into segments, and all segments of a runner i are the right half-open intervals $\sigma_i^a = \{t : (a-1)\mathrm{T}_i \leq t < a\mathrm{T}_i,\}$ $a = 1, 2, 3, \cdots, \nu_i$. The collection Σ_i of all segments of a runner i covers continuously and disconnectedly the c-cycle domain. We will use the most fine cover $\Sigma_n = \{\sigma_n^a : a = 1, 2, 3, \cdots, n\}$, runners f-constellations by the segments of the fastest runner, f-cover.

The f-time unit translation operator $\hat{\mathbf{1}}_{\rm T}$ displaces a runner i to the point P_i at the distance $1_{\rm L} = \hat{\mathbf{1}}_{\rm T}(i) = 1_{\rm T} i$, and creates the pair (i, P_i) on the circular track. If $|O, |0123 \cdots n\rangle$ represents all runners at the start point the operator

$$\hat{\mathbf{1}}_{\rm T} : \quad (0, \mathcal{R}_n\rangle \;\rightarrow\; |(1, \mathrm{P}_1), (\mathrm{P}_2), (3, \mathrm{P}_3), \cdots, (n, \mathrm{P}_n) \equiv (n, O)\rangle = \Pi, \tag{1}$$

distributes equidistantly runners in the increasing order of the speeds on the circular track in the first f-runner cycle. The fixed points $P = |\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \cdots, \mathrm{P}_n\rangle$ on the circular track are the stationary points and the any runner's distribution sitting at the points is the f-runners constellation or the f-cycle. Successive application of the displacement operator $\hat{\mathbf{1}}_{\rm T}$ on a runner i

$$\hat{\mathbf{1}}_{\rm T}(i)(i, \mathrm{P}_i), \;\; \hat{\mathbf{1}}^2_{\rm T}(i) = (i, \mathrm{P}_{i+i}) = (i, \mathrm{P}_{2i}), \cdots \tag{2}$$

moves the runner throughout the set of stationary points.

Corollary 1. *The runner's competition is the motion between the stationary points. The runners distribute on the finishing segment σ_n is the order reversed to the one on the segment σ_1. The initial Π_1 and final Π_2 runner's f-constellations are symmetric in their positions as well as in their speeds.*

■ The c-cycle completes when the slowest runner $i = 1$ reaches the initial point. It requires the $n-$ times repeated use of the displacement operator $\hat{\mathbf{1}}_{\mathrm{T}}$, so that the competition is the motion between the stationary points.
At the competition start the runners $|123\cdots n\rangle$ distribute according to the increasing speed sequence $|1, 2, 3, \cdots, n\rangle$ to reach the f-cycle $\Pi_1 = (1, \mathrm{P}_1), (2, \mathrm{P}_2), (3, \mathrm{P}_3), \cdots, (n, \mathrm{P}_n)\rangle$. The runners position sequence on the final segment σ_n must be the the position inverted sequence of the cycle Π_1. For, if originally $i < j$ and the position $j \prec i$ at the end, the runners would not reach the initial position simultaneously, a contradiction. Thus

$$\hat{\mathbf{1}}_{\mathrm{T}}: \quad \big|(1, \mathrm{P}_1), (2, \mathrm{P}_2), (3, \mathrm{P}_3), \cdots, (n, \mathrm{P}_n)\big\rangle \quad \rightarrow \quad \big|(n, \mathrm{P}_n), \cdots, (3, \mathrm{P}_3), (2, \mathrm{P}_2), (1, \mathrm{P}_1)\big\rangle,$$

and the initial and final f-constellations are the position and the speed symmetric. ■

Motion Graphs

The competition of the $\mathcal{R}_n$ runners is the motion of n points, defined by the n linear functions $X_i(t) = it, \quad t > 0$. The linear graph $\gamma_i = (t, X_i(t))$ associates to a runner i, and the bundle of all straight lines, linear trajectories, on the time axes t makes the linear Graph $\Gamma = \{\gamma_i : i = 1.2, 3, \cdots, n\}$, the Graph Γ on the Figure 1. However, the runners follow the circular trajectory, and every linear graph γ_i is wound ν_i times on the circular track to complete the c-cycle. Consequently, each linear graph $\gamma_i \in \Gamma$ must undergo the winding transformation

$$\hat{\varpi}: \quad \gamma_i \rightarrow g_i = \{x_i^a \bmod T_i, \;\; a = 1, 2, 3, \cdots, \nu_i\} \tag{3}$$

to the unit circle, see Graph G_i on Figure 1. Further, the union $\cup_{i=1}^{n} g$ of all n winding graphs on the time domain $D = \{t : 0 \leq t \leq 1\}$ of the c-cycle is the winding Graph G.
Each graph $g_i \in \mathrm{G}$ rests on a segmental cover $\Sigma_i = \cup_{a=1}^{\nu_i} \sigma_{ai}$ of the c-cycle time domain D. The graphs in Figure 1 rest on the segmental f-cover $\Sigma_n = \cup_{a=1}^{n} \Sigma_a$. Such segmental cover of the time domain necessarily partitions the winding Graph G into a sequence of n connected f-Sub-graphs G_a, and their union $\mathrm{G} = \cup_{a=1}^{n} \mathrm{G}_a$ is the winding Graph itself.
The Corollary 2 points out the properties of the winding Graph G essential for the development of the presentation. The proof of the Corollary 2 relies on the Figure 1.

Corollary 2. *Subgraphs Γ_1 and Γ_n in the winding Graph G are central symmetric, and the individual graphs of the Γ_n are ordered from the $\inf_g \Gamma_n = g_n$ to $\sup_g \Gamma_n = g_1$ in the their slope descending order .*

■ **1.** The order of the individual graphs in the linear bundle Γ is the slopes increasing sequence between the $\inf_\gamma \Gamma = \gamma_1$ and $\sup_\gamma \Gamma = \gamma_n$ graphs. The winding Subgraph G_1 on the f-segment Σ_1 is the part of the bundle Γ, so we will make the identification $\Gamma \sim G_1 \equiv \Gamma_1$. Thus,the graphs of the Γ_1 preserves the slope order $|123\cdots n\rangle$ of Γ, and

$$\inf_g \Gamma_1 = g_1 \prec \gamma_2 \prec \gamma_3 \cdots \prec \sup_g \Gamma_n = g_n.$$

The frequencies of the c-cycle and the cycle of the runner $i = 1$ are identical, so that the graph $g_1 = (t, X_1(t))$ is the diagonal of the graph G, and connects its points $(0, 0)$ and $(1, 1)$ on the Figure

1. Consequently, all individual graphs in the Γ_1 are above or identical to the diagonal graph g_1.

2. The f-graph $g_n = (t, X_n(t))$ is the linear function on the segment σ_1, starts at zero at the beginning of the segment rises to one to suffer the break to zero at the end of the segment, starts again from zero at the beginning of the next segment, rises to reach one and so on, until it covers all domain D. Since the f-segment is the smallest among the segments, the graph γ_n is the first one to undergo the diagonal inversion and the recreation at the beginning of the f-segment σ_2. Diagonal inversions are followed by the inversion of slope order of the individual graphs at the Graph G on Figure 1, and the first single graph inversion is $|n; 123 \cdots n-1\rangle$.

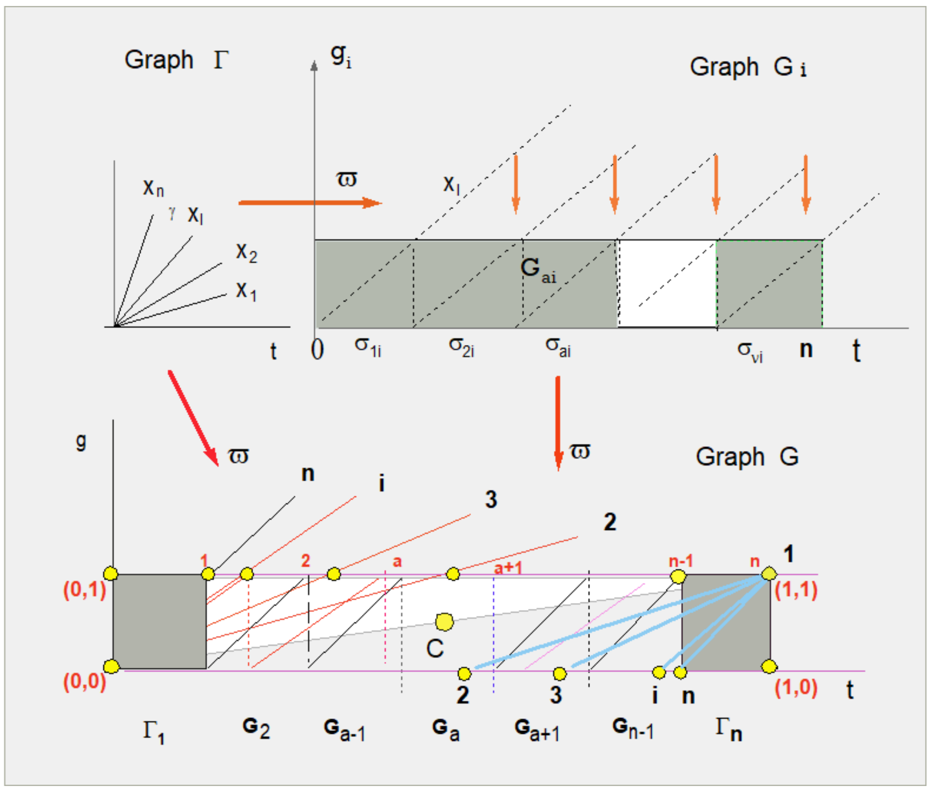

Figure 1: Lonely Runner Winding Graphs

Further, as time progresses throughout the f-segments, more and more graphs from the ascending order invert to the descending order $n, (n-1), (n-2) \cdots$, which is, more and more graphs which once all had been over diagonal recreate below the diagonal. Notice that each graph g_i intersect the graph diagonal g_1 at a point $S_{ia} = g_1 \cap g_i$ dependent on the segment σ_a.
At this point, we must recall that the runner's motion is the transition between the stationary points $\mathcal{P}$, and that all runners must end at the same moment at the starting point. It means that between the initial and last segments the graphs are mixed by the slope order, and finally reach the inverted

f-constellation distribution $\Pi_n = |(n, \mathrm{P}_n), \cdots, (3, \mathrm{P}_3), (2, \mathrm{P}_2), (1, \mathrm{P}_1)\rangle$ at the last segment σ_n.

3. After the runners achieve the inverted f-constellation, all individual graphs are in the linear bundle Γ_n at the point $(1,1)$, all placed below the diagonal in the slope-reversed arrangement on all segment σ_n. All intersection points S are on the diagonal graph g_1, approaching the accumulation point $(1,1)$. The infimum graph is $\inf_g \Gamma_n = g_n$. Accordingly, the Subgraphs Γ_1 and Γ_n are the center symmetric and

$$\inf_g \Gamma_n = g_n, \quad \sup_g \Gamma_n = g_1. \tag{4}$$

Corollary 3. *The winding graph infimum on the final f-segment is the graph γ_n of the fastest runner, and all other graphs are in descending slope order placed above the graph γ_n. The graphs of any other collection $\mathcal{R}_{\mathrm{N}} : \mathcal{R}_{\mathrm{N}} \subset \mathcal{R}_n$ of the runners are the part of the Subgraph Γ_n, respect the existing slope order and have the same infimum graph.*

Conclusion

The linear function $\hat{D}_i : \Gamma \to \Gamma - X_i(t)$ defines the distance of a runner i from all other runners. When the test runner pas trough the $i = 1, 2, 3, \cdots, n-1$ set the difference graph functions take the positive, negative, and zero values in the set $\Gamma_{\pm(n-1)}$. Thus, the individual graph distances run through the enveloping Graph $(-\Gamma_{(n-1)}, +\Gamma_{(n-1)})$ of the size $2n$. The zero graph represents the distance of the test runner from itself, while all other graphs represent its distances from the other runners. However, the signs of distances are irrelevant, so it is proper to take the absolute values graphs differences. With all n test runners, it makes $n-1$ positive difference graphs which are exactly the first $n-1$ graphs from the Graph Γ. Thus $\Gamma_{\pm(n-1)} \sim \Gamma_{(n-1)} \subset \Gamma$, valid on all time domain $D = [0, n1_{\mathrm{T}}] = [0, 1]$. Consequently, each graph $g_j : j = 1, 2, 3, \cdots, n-1$ separation of any test runner from others ends at the point $(1,1)$ in Figure 1.

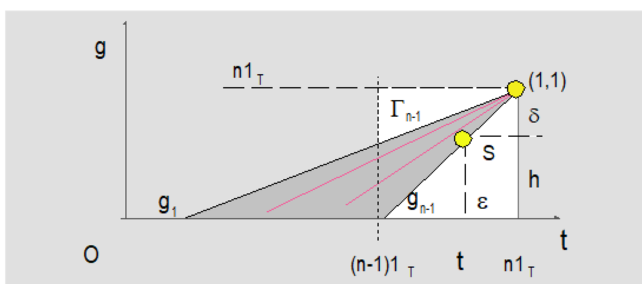

Figure 2: Lonely Runner Separations

Corollary 4. *The Lonely Runner conjecture holds for any number of the runners.*

■ The infimum graph $g_{n-1} = \gamma_{n-1}$ of the Subgraph $G_{n-1} = \Gamma_{n-1}$ connects the $(n-1, 0)1_{\mathrm{T}}$ and $(n, n)1_{\mathrm{T}}$ points on the f-segment σ_n, and

$$\forall h > 0\ \exists \varepsilon > 0\ \therefore\ \forall t > 1 - \varepsilon\ \Rightarrow\ \gamma_{n-1}(t) > h, \tag{5}$$
$$\Rightarrow\ \forall i = 1, 2, 3, \cdots, n,\ \gamma_i(t) > \gamma_n(t) > h \in (0, 1). \tag{6}$$

For an arbitrary $h \in (0, 1)$ there is a moment t' such that for a time $t : t' < t \leq n1_{ct} = 1$ all graphs from the linear bundle Γ_{n-1} at the point $(1, 1)$ are above the graph $\gamma = h(t) = const$. However, all the graphs of the mutual separation of the runners from the set $\mathcal{R}_{\mathrm{N}}$ are among the graphs of the Γ_{n-1} runners, and according to Corollary 3, the two sets of runners have the same infimum graph. Thus, the Lonely Runner conjecture holds for any number N of the runners. ■

References

[1] Ram Krishna Padey *A Note on Lonely Runner Conjecture* Journal of integer Sequences, Vol. 12 (2009) Article 09.4.6

[2] W. E. Deskins, *Abstract Algebra*, The MacMilan Company, New York,

[3] George E. Andrews, *Number Theory*, Dower Publications, Inc. New York.