Study of the Riemann Zeta function for odd integers

Jesús Sánchez

Independent Researcher, Bilbao, Spain
Email: jesus.sanchez.bilbao@gmail.com
https://www.researchgate.net/profile/Jesus-Sanchez-21
https://vixra.org/author/jesus_sanchez
ORCID 0000-0002-5631-8195

Copyright © 2024 by author
CC BY-NC-SA 4.0

Abstract

In this paper we will try to find a solution for the Riemann Zeta function for odd integers.

We will start with ζ(3) (the Riemann Zeta function with s=3) emulating the "Basel problem". But instead of using a sine or cosine function, using functions similar to these:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

$$f(x) = \frac{1}{3}\left(e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

We will discover that the process itself seems ok, but with a problem. The solutions of the above functions are not periodic, so we cannot emulate the "Basel problem" perfectly, obtaining the following value:

$$\zeta(3) = \frac{\frac{\pi^3}{3!\left(\sqrt{3}\right)^3}}{1 - \frac{1}{2^3}} = 1.1366020 \neq 1.202056903$$

With a small correction, we arrive to:

$$\zeta(3) = \frac{\frac{\pi^3}{3!\left(\sqrt{3}\right)^3}}{1 - \frac{1}{2^3}} \frac{e^{\frac{2}{\sqrt{3}}}}{3} = 1.202173775 \approx 1.202056903\ [7]$$

But not getting the correct value anyhow. The only way of obtaining the correct value would be to find a function of the form:

$$g(x) = 1 - r(3)\cdot\frac{x^3}{3!} + r(6)\cdot\frac{x^6}{6!} - r(9)\cdot\frac{x^9}{9!} + \cdots$$

That has periodic zeros. Where r(n) is an unknown function to be calculate/discovered.
We have also generalized this study to calculate a general $\zeta(k)$ where k can be higher odd numbers, or even numbers. Having $\zeta(k)$ for even numbers would lead to obtaining a closed equation for the Bernoulli numbers.
If a generalization for k as a general complex number was possible, we could even consider k=½+it, obtaining a closed function for the zeros of the Riemann Zeta function.

Keywords

Riemann Zeta function, Odd integers, Basel Problem, Non trivial zeros

1. Introduction

In this paper we will try to find a solution for the Riemann Zeta function for odd integers.

We will start with ζ(3) (the Riemann Zeta function with s=3) emulating the "Basel problem". But instead of using a sine or cosine function, using a different type of function that we will develop in chapter 5.

In chapters 2-4 we will make some previous calculations to understand the following chapters.

In chapters 5, 6 and 7 we will try to find the solution for ζ(3), ζ(5) and a general ζ(k) respectively. And in the chapter 8 we will comment the conclusions of the study.

2. Sum of the odd elements of the Riemann Zeta function

The first thing we will make is to calculate the sum of the odd integers of the Riemann zeta function [1]. This is:

$$\zeta_{odd}(s) = 1^{-s} + 3^{-s} + 5^{-s} + 7^{-s} \ldots$$

To do it, we start with the definition of the Riemann zeta function [1]:

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots$$

We can separate the even and the odd numbers this way:

$$\zeta(s) = 2^{-s} + 4^{-s} + 6^{-s} \ldots + 1^{-s} + 3^{-s} + 5^{-s} + \cdots$$

$$\zeta(s) = \zeta_{even}(s) + \zeta_{odd}(s)$$

We continue operating:

$$\zeta(s) = 2^{-s}(1^{-s} + 2^{-s} + 3^{-s} \ldots) + 1^{-s} + 3^{-s} + 5^{-s} + \cdots$$

$$\zeta(s) = 2^{-s}\zeta(s) + 1^{-s} + 3^{-s} + 5^{-s} + \cdots$$

$$\zeta(s) = 2^{-s}\zeta(s) + \zeta_{odd}(s)$$

$$\zeta(s) - 2^{-s}\zeta(s) = \zeta_{odd}(s)$$

So, we get:

$$\zeta_{odd}(s) = \zeta(s)(1 - 2^{-s})$$

And therefore:

$$\zeta(s) = \frac{\zeta_{odd}(s)}{1 - 2^{-s}}$$

3. Calculation of ζ(2) using the Cosine instead of the Sine

We will remake the Basel problem [2] by Euler, but using the cosine instead of the sine. We will see later why.

We know that the Taylor expansion series [3][6] of the cosine is:

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We also know that the polynomial of the series should correspond to the product of the solutions of the function [4] (in this case, the points where the cosine is zero). This is:

$$cos(x) = \cdots \left(1 - \frac{x}{\frac{5\pi}{2}}\right)\left(1 - \frac{x}{\frac{3\pi}{2}}\right)\left(1 - \frac{x}{\frac{\pi}{2}}\right)\left(1 - \frac{x}{\left(-\frac{\pi}{2}\right)}\right)\left(1 - \frac{x}{\left(-\frac{3\pi}{2}\right)}\right)\left(1 - \frac{x}{\left(-\frac{5\pi}{2}\right)}\right)\ldots$$

$$cos(x) = \cdots \left(1 - \frac{x}{\frac{5\pi}{2}}\right)\left(1 - \frac{x}{\frac{3\pi}{2}}\right)\left(1 - \frac{x}{\frac{\pi}{2}}\right)\left(1 + \frac{x}{\frac{\pi}{2}}\right)\left(1 + \frac{x}{\frac{3\pi}{2}}\right)\left(1 + \frac{x}{\frac{5\pi}{2}}\right)\ldots$$

Now, we apply the law that the product of a sum and a difference is the difference of squares. We do this by pairs of elements:

$$cos(x) = \left(1 - \left(\frac{x}{\frac{\pi}{2}}\right)^2\right)\left(1 - \left(\frac{x}{\frac{3\pi}{2}}\right)^2\right)\left(1 - \left(\frac{x}{\frac{5\pi}{2}}\right)^2\right)\ldots$$

$$cos(x) = \left(1 - \frac{x^2}{\frac{\pi^2}{2^2}}\right)\left(1 - \frac{x^2}{\frac{3^2\pi^2}{2^2}}\right)\left(1 - \frac{x^2}{\frac{5^2\pi^2}{2^2}}\right)\ldots$$

$$cos(x) = \left(1 - \frac{2^2x^2}{\pi^2}\cdot\frac{1}{1^2}\right)\left(1 - \frac{2^2x^2}{\pi^2}\cdot\frac{1}{3^2}\right)\left(1 - \frac{2^2x^2}{\pi^2}\cdot\frac{1}{5^2}\right)\ldots$$

We can check that the product of elements that do not include x (the constant part of the polynomial is:

$$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \cdots = \mathbf{1}$$

And it is ok with the first element of the polynomial that is 1.

$$cos(x) = \mathbf{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

So, no escalation factor is needed to adjust the results.

Then we check the second element. The one that has $\mathbf{x^2}$.

$$cos(x) = 1 - \frac{\boldsymbol{x^2}}{\mathbf{2}!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

This should correspond with following sum:

$$-\frac{2^2x^2}{\pi^2}\cdot\frac{1}{1^2} - \frac{2^2x^2}{\pi^2}\cdot\frac{1}{3^2} - \frac{2^2x^2}{\pi^2}\cdot\frac{1}{5^2} - \cdots = -\frac{\boldsymbol{x^2}}{\mathbf{2}!}$$

The reason, is because the only way of obtaining a polynomial in x^2, is multiplying each element that contains x^2 in the product by an infinite product of 1s. The moment you multiply by another element you get at least an element with x^4.

So, operating:

$$-\frac{2^2x^2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)=-\frac{x^2}{2!}$$

$$-\frac{2^2x^2}{\pi^2}\zeta_{odd}(2)=-\frac{x^2}{2!}$$

$$\zeta_{odd}(2)=\frac{\pi^2}{2!\,2^2}=\frac{\pi^2}{8}$$

And now we apply the equation we obtained in the chapter 2 regarding the odd elements of the Riemann zeta function:

$$\zeta(s)=\frac{\zeta_{odd}(s)}{1-2^{-s}}$$

$$\zeta(2)=\frac{\zeta_{odd}(2)}{1-2^{-2}}=\frac{\frac{\pi^2}{8}}{1-\frac{1}{4}}=\frac{\frac{\pi^2}{8}}{\frac{4-1}{4}}=\frac{\frac{\pi^2}{8}}{\frac{3}{4}}=\frac{4\pi^2}{24}=\frac{\pi^2}{6}$$

If you are familiar with the Basel problem, you will check that the result is correct. The way to arrive to the result is similar but more complicated than using the sine function.

The reason I have used the cosine function is because it is easier to check that the element without x in the polynomial (in this case, 1) is coherent with the infinite product of solutions. You will understand why later.

4. Generalization of the product of a sum and a difference but using more than two elements

In the previous chapter, we have used the property of multiplying a sum by a difference to get the difference of squares. A specific example could have this form:

$$\left(1+\frac{A}{B}\right)\left(1-\frac{A}{B}\right)=1-\left(\frac{A}{B}\right)^2=1-\frac{A^2}{B^2}$$

If we put it using complex numbers, one of the possible ways will be the following:

$$\left(1+\frac{A}{B}\right)\left(1-\frac{A}{B}\right)=\left(1-\frac{A}{e^{\pi i}B}\right)\left(1-\frac{A}{e^{2\pi i}B}\right)=\left(1-\frac{A}{e^{\frac{1}{2}(2\pi i)}B}\right)\left(1-\frac{A}{e^{\frac{2}{2}(2\pi i)}B}\right)$$

So, we will have:

$$\left(1-\frac{A}{e^{\frac{1}{2}(2\pi i)}B}\right)\left(1-\frac{A}{e^{\frac{2}{2}(2\pi i)}B}\right)=1-\frac{A^2}{B^2}$$

And this leads, to the following question. Can we generalize this for more than two elements? For example, the next expression is true?

$$\left(1-\frac{A}{Be^{\frac{1}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{2}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{3}{3}(2\pi i)}}\right)=1-\frac{A^3}{B^3}$$

We can check that it is correct. Just operating:

$$\left(1-\frac{A}{Be^{\frac{1}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{2}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{3}{3}(2\pi i)}}\right)$$
$$=\left(1-\frac{A}{Be^{\frac{2}{3}(2\pi i)}}-\frac{A}{Be^{\frac{1}{3}(2\pi i)}}+\frac{A^2}{B^2e^{\frac{3}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{3}{3}(2\pi i)}}\right)$$
$$=1-\frac{A}{Be^{\frac{3}{3}(2\pi i)}}-\frac{A}{Be^{\frac{2}{3}(2\pi i)}}+\frac{A^2}{B^2e^{\frac{5}{3}(2\pi i)}}-\frac{A}{Be^{\frac{1}{3}(2\pi i)}}+\frac{A^2}{B^2e^{\frac{4}{3}(2\pi i)}}$$
$$+\frac{A^2}{B^2e^{\frac{3}{3}(2\pi i)}}-\frac{A^3}{B^3e^{\frac{6}{3}(2\pi i)}}$$
$$=1+\left(-\frac{A}{Be^{\frac{1}{3}(2\pi i)}}-\frac{A}{Be^{\frac{2}{3}(2\pi i)}}-\frac{A}{Be^{\frac{3}{3}(2\pi i)}}\right)$$
$$+\left(\frac{A^2}{B^2e^{\frac{3}{3}(2\pi i)}}+\frac{A^2}{B^2e^{\frac{4}{3}(2\pi i)}}+\frac{A^2}{B^2e^{\frac{5}{3}(2\pi i)}}\right)-\frac{A^3}{B^3e^{\frac{6}{3}(2\pi i)}}$$
$$=1+\left(-\frac{Ae^{-\frac{1}{3}(2\pi i)}}{B}-\frac{Ae^{-\frac{2}{3}(2\pi i)}}{B}-\frac{Ae^{-\frac{3}{3}(2\pi i)}}{B}\right)$$
$$+\left(\frac{A^2e^{-\frac{3}{3}(2\pi i)}}{B^2}+\frac{A^2e^{-\frac{4}{3}(2\pi i)}}{B^2}+\frac{A^2e^{-\frac{5}{3}(2\pi i)}}{B^2}\right)-\frac{A^3}{B^3e^{\frac{6}{3}(2\pi i)}}$$
$$=1+\left(-\frac{A\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}{B}-\frac{A\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}{B}-\frac{A\cdot 1}{B}\right)$$
$$+\left(\frac{A^2\cdot 1}{B^2}+\frac{A^2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}{B^2}+\frac{A^2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}{B^2}\right)-\frac{A^3}{B^3e^{\frac{6}{3}(2\pi i)}}$$
$$=1+(0)+(0)-\frac{A^3}{B^3e^{\frac{6}{3}(2\pi i)}}=1-\frac{A^3}{B^3}$$

So, yes, the expression is true:

$$\left(1-\frac{A}{Be^{\frac{1}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{2}{3}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{3}{3}(2\pi i)}}\right)=1-\frac{A^3}{B^3}$$

And it is valid for any number of elements. The following is for 5 elements:

$$\left(1-\frac{A}{Be^{\frac{1}{5}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{2}{5}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{3}{5}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{4}{5}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{5}{5}(2\pi i)}}\right)=1-\frac{A^5}{B^5}$$

And this is the general case:

$$\left(1-\frac{A}{Be^{\frac{1}{n}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{2}{n}(2\pi i)}}\right)\left(1-\frac{A}{Be^{\frac{3}{n}(2\pi i)}}\right)\ldots\left(1-\frac{A}{Be^{\frac{n}{n}(2\pi i)}}\right)=1-\frac{A^n}{B^n}$$

$$\prod_{k=1}^{n}\left(1-\frac{A}{Be^{\frac{k}{n}(2\pi i)}}\right)=1-\frac{A^n}{B^n}$$

Again, we will see later why we have done this.

5. Using a specific function to try to calculate the value of ζ(3)

In the chapter 3, we calculated ζ(2), the value of the Riemann zeta function for s=2 using the cosine function. The question is, can we use another function to calculate the value of ζ(3)?

To be able to obtain an exponent of 3 in the numbers, we will need a function of this form as we will see later:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

This function is the same as the following function, as we will demonstrate now:

$$f(x) = \frac{1}{3}\left(e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

To demonstrate it, we can obtain the Taylor series of above function at x=0 (McLaurin series). First we calculate the value of the series at the point x=0.

$$f(0) = \frac{1}{3}\left(e^{-0\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-0\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-0\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

$$f(0) = \frac{1}{3}(1+1+1) = \frac{1}{3}(3) = 1$$

Now, we take the first derivative of the series:

$$f'(x) = \frac{1}{3}\left(-e^{\frac{1}{3}(2\pi i)}\cdot e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} - e^{\frac{2}{3}(2\pi i)}\cdot e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} - e^{\frac{3}{3}(2\pi i)}e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

And, again, we calculate its value for x=0.

$$f'(0) = \frac{1}{3}\left(-e^{\frac{1}{3}(2\pi i)} - e^{\frac{2}{3}(2\pi i)} - e^{\frac{3}{3}(2\pi i)}\right) = \frac{1}{3}\left(-\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 1\right)$$
$$= \frac{1}{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2} - 1\right) = 0$$

Now, we take the second derivative and its value at x=0.

$$f''(x) = \frac{1}{3}\left(-e^{\frac{2}{3}(2\pi i)}\cdot e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} - e^{\frac{4}{3}(2\pi i)}\cdot e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} - e^{\frac{6}{3}(2\pi i)}e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

$$f''(0) = \frac{1}{3}\left(-e^{\frac{2}{3}(2\pi i)} - e^{\frac{4}{3}(2\pi i)} - e^{\frac{6}{3}(2\pi i)}\right) = \frac{1}{3}\left(-\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - 1\right)$$
$$= \frac{1}{3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} - i\frac{\sqrt{3}}{2} - 1\right) = 0$$

Now, the third derivative and its value for x=0.

$$f'''(x) = \frac{1}{3}\left(-e^{\frac{3}{3}(2\pi i)}\cdot e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} - e^{\frac{6}{3}(2\pi i)}\cdot e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} - e^{\frac{9}{3}(2\pi i)}e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

$$f'''(0) = \frac{1}{3}\left(-e^{\frac{3}{3}(2\pi i)} - e^{\frac{6}{3}(2\pi i)} - e^{\frac{9}{3}(2\pi i)}\right) = \frac{1}{3}(-1-1-1) = \frac{1}{3}(-3) = -1$$

If we continue with the next derivatives and we use the Taylor series definition we obtain, the expected:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

We can again obtain the derivatives and their value at x=0 to check that everything hold. You can skip, this part if you want as it is just a verification that both functions commente din the beginning of the chapter are the same:

$$f(0) = 1$$

$$f'(x) = -\frac{3x^2}{3!} + \frac{6x^5}{6!} - \frac{9x^8}{9!} + \cdots$$

$$f'(x) = -\frac{3x^2}{3 \cdot 2!} + \frac{6x^5}{5 \cdot 5!} - \frac{9x^8}{9 \cdot 8!} + \cdots$$

$$f'(x) = -\frac{x^2}{2!} + \frac{x^5}{5!} - \frac{x^8}{8!} + \cdots$$

$$f'(0) = 0$$

$$f''(x) = -\frac{2x}{2!} + \frac{5x^4}{5!} - \frac{8x^7}{8!} + \cdots$$

$$f''(x) = -\frac{2x}{2 \cdot 1!} + \frac{5x^4}{5 \cdot 4!} - \frac{8x^7}{8 \cdot 7!} + \cdots$$

$$f''(x) = -\frac{x}{1!} + \frac{x^4}{4!} - \frac{x^7}{7!} + \cdots$$

$$f''(0) = 0$$

$$f'''(x) = -\frac{1}{1!} + \frac{4x^3}{4!} - \frac{7x^6}{7!} + \cdots$$

$$f'''(x) = -\frac{1}{1!} + \frac{4x^3}{4 \cdot 3!} - \frac{7x^6}{7 \cdot 6!} + \cdots$$

$$f'''(x) = -\frac{1}{1!} + \frac{x^3}{3!} - \frac{x^6}{6!} + \cdots$$

$$f'''(x) = -f(x)$$

$$f'''(x) = -1$$

So, we have this function expressed in these following forms. And the question is, what do we do now?

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

$$f(x) = \frac{1}{3}\left(e^{-x \cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x \cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x \cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

The theory tells us that if we calculate the zeros of above function and we make an infinite product of the solutions, as we made in chapter 3 with the cosine, we will be able to obtain ζ(3).

The problem is that above statement is only true if the solutions to the function are periodic. And as we will check now, they are "almost" periodic but not "perfectly" periodic.

To find the solutions, we will use the programs in Mathematica that you have in Annexes 1 and 2. The program in Annex 1 obtains the solutions for:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

And the program in Annex 2, obtains the solutions for:

$$f(x) = \frac{1}{3}\left(e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

And, as expected, the solutions are the same, as the functions are the same (expressed in different forms).

The solutions, expressed in polar way (modulus, argument) are:

```
Zeros in polar format(modulus, argument):
Out[•]= {{56.22778, -2.0943951}, {56.22778, 2.0943951}, {52.600182, -2.0943951}, {52.600182, 2.0943951}, {48.972583, -2.0943951}, {48.972583, 2.0943951},
 {45.344984, -2.0943951}, {45.344984, 2.0943951}, {41.717385, -2.0943951}, {41.717385, 2.0943951}, {38.089787, -2.0943951}, {38.089787, 2.0943951},
 {34.462188, -2.0943951}, {34.462188, 2.0943951}, {30.834589, -2.0943951}, {30.834589, 2.0943951}, {27.20699, -2.0943951}, {27.20699, 2.0943951},
 {23.579392, -2.0943951}, {23.579392, 2.0943951}, {19.951793, -2.0943951}, {19.951793, 2.0943951}, {16.324194, -2.0943951}, {16.324194, 2.0943951},
 {12.696596, -2.0943951}, {12.696596, 2.0943951}, {9.0689975, -2.0943951}, {9.0689975, 2.0943951}, {5.4412334, -2.0943951}, {5.4412334, 2.0943951},
 {1.8498128, -2.0943951}, {1.8498128, 2.0943951}, {1.8498128, 0.}, {5.4412334, 0.}, {9.0689975, 0.}, {12.696596, 0.}, {16.324194, 0.}, {19.951793, 0.},
 {23.579392, 0.}, {27.20699, 0.}, {30.834589, 0.}, {34.462188, 0.}, {38.089787, 0.}, {41.717385, 0.}, {45.344984, 0.}, {48.972583, 0.}}
```

{{56.22778, -2.0943951}, {56.22778,
2.0943951}, {52.600182, -2.0943951}, {52.600182,
2.0943951}, {48.972583, -2.0943951}, {48.972583,
2.0943951}, {45.344984, -2.0943951}, {45.344984,
2.0943951}, {41.717385, -2.0943951}, {41.717385,
2.0943951}, {38.089787, -2.0943951}, {38.089787,
2.0943951}, {34.462188, -2.0943951}, {34.462188,
2.0943951}, {30.834589, -2.0943951}, {30.834589,
2.0943951}, {27.20699, -2.0943951}, {27.20699,
2.0943951}, {23.579392, -2.0943951}, {23.579392,
2.0943951}, {19.951793, -2.0943951}, {19.951793,
2.0943951}, {16.324194, -2.0943951}, {16.324194,
2.0943951}, {12.696596, -2.0943951}, {12.696596,
2.0943951}, {9.0689975, -2.0943951}, {9.0689975,
2.0943951}, {5.4412334, -2.0943951}, {5.4412334,
2.0943951}, {1.8498128, -2.0943951}, {1.8498128,
2.0943951}, {1.8498128, 0.}, {5.4412334, 0.}, {9.0689975,
0.}, {12.696596, 0.}, {16.324194, 0.}, {19.951793, 0.}, {23.579392,
0.}, {27.20699, 0.}, {30.834589, 0.}, {34.462188, 0.}, {38.089787,
0.}, {41.717385, 0.}, {45.344984, 0.}, {48.972583, 0.}}

When the argument is 2.0943951 it corresponds to: $e^{\frac{1}{3}(2\pi i)}$ and the argument -2.0943951 corresponds to $e^{-\frac{1}{3}(2\pi i)}$ which is the same as $e^{\frac{2}{3}(2\pi i)}$.

When the argument is 0, it corresponds to a real number. Which is the same as saying that it corresponds to $e^{\frac{3}{3}(2\pi i)}$.

We can check that for each modulus, we have three solutions, with these three arguments. As we can see below for the first five modulus as an example:

{1.8498128, 2.0943951}, {1.8498128, -2.0943951}, {1.8498128, 0.}

{5.4412334, 2.0943951}, {5.4412334, -2.0943951}, {5.4412334, 0.}

{9.0689975, 2.0943951}, {9.0689975, -2.0943951}, {9.0689975, 0.}

{12.696596, 2.0943951}, {12.696596, -2.0943951}, {12.696596, 0.}

{16.324194, 2.0943951}, {16.324194, -2.0943951}, {16.324194, 0.}

This is the same as:

$$1.8498128 \cdot e^{\frac{1}{3}(2\pi i)}, 1.8498128 \cdot e^{\frac{2}{3}(2\pi i)}, 1.8498128 \cdot e^{\frac{3}{3}(2\pi i)}$$

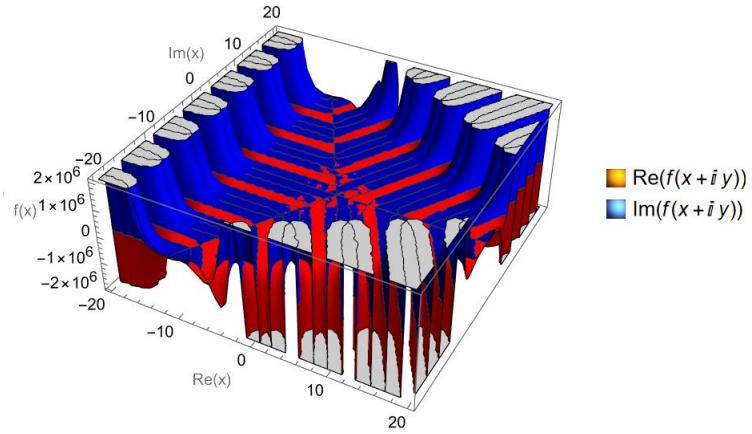
$$5.4412334 \cdot e^{\frac{1}{3}(2\pi i)}, 5.4412334 \cdot e^{\frac{2}{3}(2\pi i)}, 5.4412334 \cdot e^{\frac{3}{3}(2\pi i)}$$

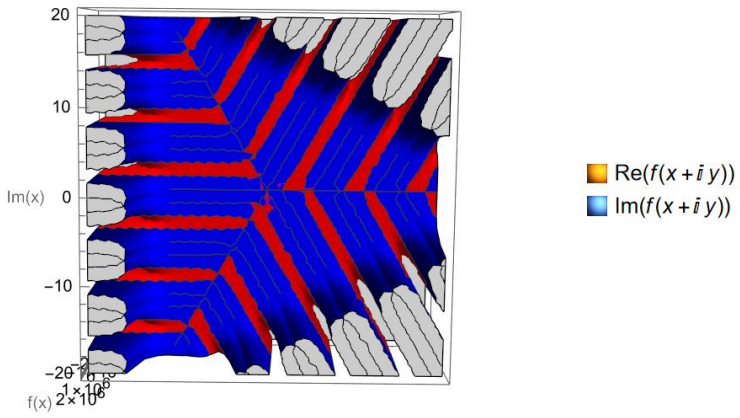
$$9.0689975 \cdot e^{\frac{1}{3}(2\pi i)}, 9.0689975 \cdot e^{\frac{2}{3}(2\pi i)}, 9.0689975 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$12.696596 \cdot e^{\frac{1}{3}(2\pi i)}, 12.696596 \cdot e^{\frac{2}{3}(2\pi i)}, 12.696596 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$16.324194 \cdot e^{\frac{1}{3}(2\pi i)}, 16.324194 \cdot e^{\frac{2}{3}(2\pi i)}, 16.324194 \cdot e^{\frac{3}{3}(2\pi i)}$$

We can see that the solutions divide the complex plane in three equal parts, as we can see in the graph of the function. See Annex A1 and A2 for more details.





But are these solutions periodic? Let's check it. If we subtract:

$$16.324194 - 12.696596 = 3.627598 \approx \frac{2\pi}{\sqrt{3}}$$

$$12.696596 - 9.0689975 = 3.6275985 \approx \frac{2\pi}{\sqrt{3}}$$

$$9.0689975 - 5.4412334 = 3.6277641 \approx \frac{2\pi}{\sqrt{3}}$$

$$5.4412334 - 1.8498128 = 3.5914206 \neq \frac{2\pi}{\sqrt{3}}$$

Considering that $\frac{2\pi}{\sqrt{3}} = 3.6275987$ we can see that the solutions tend to have that period the larger they are. It will be periodic with that exact period of $\frac{2\pi}{\sqrt{3}}$ asymptotically at infinity.

Considering that:

$$\frac{\pi}{\sqrt{3}} = 1.8137993$$

$$\frac{3\pi}{\sqrt{3}} = 5.4413980$$

$$\frac{5\pi}{\sqrt{3}} = 9.0689968$$

$$\frac{7\pi}{\sqrt{3}} = 12.696596$$

$$\frac{9\pi}{\sqrt{3}} = 16.324194$$

A perfect periodic function will have these solutions:

$$1.8137993 \cdot e^{\frac{1}{3}(2\pi i)}, 1.8137993 \cdot e^{\frac{2}{3}(2\pi i)}, 1.8137993 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$5.4413980 \cdot e^{\frac{1}{3}(2\pi i)}, 5.4413980 \cdot e^{\frac{2}{3}(2\pi i)}, 5.4413980 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$9.0689968 \cdot e^{\frac{1}{3}(2\pi i)}, 9.0689968 \cdot e^{\frac{2}{3}(2\pi i)}, 9.0689968 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$12.696596 \cdot e^{\frac{1}{3}(2\pi i)}, 12.696596 \cdot e^{\frac{2}{3}(2\pi i)}, 12.696596 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$16.324194 \cdot e^{\frac{1}{3}(2\pi i)}, 16.324194 \cdot e^{\frac{2}{3}(2\pi i)}, 16.324194 \cdot e^{\frac{3}{3}(2\pi i)}$$

And the separation between all the solutions would be $\frac{2\pi}{\sqrt{3}} = 3.6275987$ as expected. In fact, we can check that the last two solutions correspond to the solutions of our function (with the number of decimals considered of course, in reality the complete numbers would not be the same). Putting the solutions in another way, it will be:

$$\frac{\pi}{\sqrt{3}} \cdot e^{\frac{1}{3}(2\pi i)}, \frac{\pi}{\sqrt{3}} \cdot e^{\frac{2}{3}(2\pi i)}, \frac{\pi}{\sqrt{3}} \cdot e^{\frac{3}{3}(2\pi i)}$$

$$\frac{3\pi}{\sqrt{3}} \cdot e^{\frac{1}{3}(2\pi i)}, \frac{3\pi}{\sqrt{3}} \cdot e^{\frac{2}{3}(2\pi i)}, \frac{3\pi}{\sqrt{3}} \cdot e^{\frac{3}{3}(2\pi i)}$$

$$\frac{5\pi}{\sqrt{3}} \cdot e^{\frac{1}{3}(2\pi i)}, \frac{5\pi}{\sqrt{3}} \cdot e^{\frac{2}{3}(2\pi i)}, \frac{5\pi}{\sqrt{3}} \cdot e^{\frac{3}{3}(2\pi i)}$$

$$\frac{7\pi}{\sqrt{3}} \cdot e^{\frac{1}{3}(2\pi i)}, \frac{7\pi}{\sqrt{3}} \cdot e^{\frac{2}{3}(2\pi i)}, \frac{7\pi}{\sqrt{3}} 6 \cdot e^{\frac{3}{3}(2\pi i)}$$

$$\frac{9\pi}{\sqrt{3}} \cdot e^{\frac{1}{3}(2\pi i)}, \frac{9\pi}{\sqrt{3}} \cdot e^{\frac{2}{3}(2\pi i)}, \frac{9\pi}{\sqrt{3}} \cdot e^{\frac{3}{3}(2\pi i)}$$

If we had this perfect function (let's call it g(x)) with these solutions, we could do the same as in chapter 3 due to [4]:

$$g(x) = \left(1 - \frac{x}{\frac{\pi}{\sqrt{3}} e^{\frac{1}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{\pi}{\sqrt{3}} e^{\frac{2}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{\pi}{\sqrt{3}} e^{\frac{3}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{3\pi}{\sqrt{3}} e^{\frac{1}{3}(2\pi i)},}\right) \cdot$$

$$\cdot \left(1 - \frac{x}{\frac{3\pi}{\sqrt{3}} e^{\frac{2}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{3\pi}{\sqrt{3}} e^{\frac{3}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{5\pi}{\sqrt{3}} e^{\frac{1}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{5\pi}{\sqrt{3}} e^{\frac{2}{3}(2\pi i)},}\right)\left(1 - \frac{x}{\frac{5\pi}{\sqrt{3}} e^{\frac{3}{3}(2\pi i)},}\right) \ldots$$

Now, applying the property commented in chapter 4 that said:

$$\left(1 - \frac{A}{Be^{\frac{1}{3}(2\pi i)}}\right)\left(1 - \frac{A}{Be^{\frac{2}{3}(2\pi i)}}\right)\left(1 - \frac{A}{Be^{\frac{3}{3}(2\pi i)}}\right) = 1 - \frac{A^3}{B^3}$$

We can apply it, leading to:

$$g(x) = \left(1 - \frac{x^3}{\left(\frac{\pi}{\sqrt{3}}\right)^3,}\right)\left(1 - \frac{x^3}{\left(\frac{3\pi}{\sqrt{3}}\right)^3,}\right)\left(1 - \frac{x^3}{\left(\frac{5\pi}{\sqrt{3}}\right)^3,}\right) \ldots$$

$$= \left(1 - \frac{(\sqrt{3})^3 x^3}{\pi^3} \frac{1}{1^3}\right)\left(1 - \frac{(\sqrt{3})^3 x^3}{\pi^3} \frac{1}{3^3}\right)\left(1 - \frac{(\sqrt{3})^3 x^3}{\pi^3} \frac{1}{5^3}\right) \ldots$$

Now, as we made in chapter 4, if the function g(x) had the same Taylor polynomial expansion [3] than f(x) (we know it hasn't but let's try):

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

We can check that the product of the 1s in g(x) results in 1, as in the polynomial, so the scalation is ok.

Now if we compare the elements with x^3 similarly as we made in chapter 4 with the elements in x^2:

$$-\frac{(\sqrt{3})^3 x^3}{\pi^3}\frac{1}{1^3} - \frac{(\sqrt{3})^3 x^3}{\pi^3}\frac{1}{3^3}1 - \frac{(\sqrt{3})^3 x^3}{\pi^3}\frac{1}{5^3} - \cdots = -\frac{x^3}{3!}$$

$$-\frac{(\sqrt{3})^3 x^3}{\pi^3}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\right) = -\frac{x^3}{3!}$$

$$-\frac{(\sqrt{3})^3 x^3}{\pi^3}(\zeta_{odd}(3)) = -\frac{x^3}{3!}$$

$$\zeta_{odd}(3) = \frac{\pi^3}{3!(\sqrt{3})^3} = \frac{\pi^3}{18\sqrt{3}} = 0.994526788$$

Now, applying the equation in chapter 2:

$$\zeta(s) = \frac{\zeta_{odd}(s)}{1 - 2^{-s}}$$

$$\zeta(3) = \frac{\zeta_{odd}(3)}{1 - 2^{-3}} = \frac{\frac{\pi^3}{3!(\sqrt{3})^3}}{1 - \frac{1}{2^3}} = \frac{\frac{\pi^3}{18\sqrt{3}}}{\frac{7}{8}} = \frac{4\pi^3}{63\sqrt{3}} = 1.1366020 \neq 1.202056903\ [7]$$

We can see that the result does not correspond to the value of ζ(3). The reason for this is because we have cheated clearly. We have used the "perfect periodic" solutions of a hypothetical function g(x) and we have compared it with the polynomial of the function f(x) that we know does not have those "perfect periodic" solutions. The function f(x) has "similar" solutions than g(x) at infinity but not the same solutions (and even more difference in the first solutions). They are not the same function; we are mixing apples with pears.

The only way to obtain the correct result is to find the polynomial of the function g(x). This polynomial should be very similar than the polynomial of f(x) but with some changes in the coefficients.

This is, we know f(x) is:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

$$f(x) = \frac{1}{3}\left(e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

And g(x) probably has a form similar like one of the following, where h(x) or t(x,n) are unknown functions to be discovered/calculated:

$$g(x) = \frac{1}{3}\left(e^{-h(x)x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-h(x)x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-h(x)x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

$$g(x) = \frac{1}{3}\left(t(x,1)\cdot e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + t(x,2)e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + t(x,3)e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

Once, we have discovered how g(x) is, we can use the Taylor series expansion [3] to calculate its polynomial and use the real value of the element in x^3. The form would be something like this (being r(n) an unknown function to be discovered/calculated):

$$g(x) = 1 - r(3)\cdot\frac{x^3}{3!} + r(6)\cdot\frac{x^6}{6!} - r(9)\cdot\frac{x^9}{9!} + \cdots$$

And now, yes, we could calculate the real value of $\zeta_{odd}(3)$:

$$-\frac{\left(\sqrt{3}\right)^3 x^3}{\pi^3}\left(\zeta_{odd}(3)\right) = -r(3)\cdot\frac{x^3}{3!}$$

And with:

$$\zeta(3) = \frac{\zeta_{odd}(3)}{1 - 2^{-3}}$$

We could calculate the desired value of ζ(3).

The problem, we do not know which this g(x) is, or even if it exists.

Just as help, I have discovered the following via trial-error. If we start from the incorrect value obtained before:

$$\zeta(3) = \frac{\frac{\pi^3}{3!\left(\sqrt{3}\right)^3}}{1 - \frac{1}{2^3}} = \frac{4\pi^3}{63\sqrt{3}} = 1.1366020 \neq 1.202056903\ [7]$$

If we multiply by:

$$\zeta(3) = \frac{\frac{\pi^3}{3!\left(\sqrt{3}\right)^3}}{1 - \frac{1}{2^3}}\frac{e^{\frac{2}{\sqrt{3}}}}{3} = \frac{4\pi^3}{63\sqrt{3}}\frac{e^{\frac{2}{\sqrt{3}}}}{3} = 1.202173775 \approx 1.202056903\ [7]$$

Where the 3 in the denominator can refer to the number of elements of the function or to s directly and the $\frac{2}{\sqrt{3}}$ it could refer to $\frac{1}{sin\left(\frac{1}{3}2\pi\right)}$ or $\frac{2}{tan\left(\frac{1}{3}2\pi\right)}$ or related to the period we have obtained before $\frac{\frac{2\pi}{\sqrt{3}}}{\pi} = \frac{2}{\sqrt{3}}$. There are different possibilities but the number 2 and the square root of three are clearly related with the complex plane divided in three parts by the function.

This could be a hint of how g(x) should look like.

6. Similar study regarding the value of ζ(5)

In this chapter we will follow the same steps as in chapter 5, but with this function:

$$f(x) = 1 - \frac{x^5}{5!} + \frac{x^{10}}{10!} - \frac{x^{15}}{15!} + \cdots$$

$$f(x) = \frac{1}{5}\left(e^{-x\cdot e^{\frac{1}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{4}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{5}{5}(2\pi i)}}\right)$$

Both expressions above correspond to the same function (similarly as the f(x) we defined in chapter 5). In Annex A3 and A4 you can find the Mathematica programs for these functions. You can calculate their zeros and plot them. The problem is that the program in Annex 4 overflows (at least in my computer/software) so you do not get reliable graphs or results. In my case I have been obliged to use only the program in Annex A3.

The zeros (in polar (modulus, argument) format) obtained with the program in Annex A3 for the function f(x) above are:

```
Zeros in polar format(modulus, argument):
Out[•]= {{18.706788, -2.5132741}, {18.706788, 2.5132741}, {13.361992, -2.5132741}, {13.361992, 2.5132741}, {8.0171451, -2.5132741}, {8.0171451, 2.5132741},
  {2.6072512, -2.5132741}, {2.6072512, 2.5132741}, {2.6072512, -1.2566371}, {2.6072512, 1.2566371}, {8.0171451, -1.2566371}, {8.0171451, 1.2566371}, {2.6072512, 0.},
  {13.361992, -1.2566371}, {13.361992, 1.2566371}, {18.706788, -1.2566371}, {18.706788, 1.2566371}, {8.0171451, 0.}, {13.361992, 0.}, {18.706788, 0.}}
```

{{18.706788, -2.5132741}, {18.706788,
2.5132741}, {13.361992, -2.5132741}, {13.361992,
2.5132741}, {8.0171451, -2.5132741}, {8.0171451,
2.5132741}, {2.6072512, -2.5132741}, {2.6072512,
2.5132741}, {2.6072512, -1.2566371}, {2.6072512,
1.2566371}, {8.0171451, -1.2566371}, {8.0171451,
1.2566371}, {2.6072512, 0.}, {13.361992, -1.2566371}, {13.361992,
1.2566371}, {18.706788, -1.2566371}, {18.706788,
1.2566371}, {8.0171451, 0.}, {13.361992, 0.}, {18.706788, 0.}}

If we take the solutions with the lower modulus, we can see that they are:

{2.6072512, -2.5132741}, {2.6072512, 2.5132741}, {2.6072512, -1.2566371}, {2.6072512, 1.2566371}, {2.6072512, 0.},

Where the arguments are distributed in five parts around the complex plane:

$$1.2566371 = \frac{1}{5}(2\pi)$$

$$2.5132741 = \frac{2}{5}(2\pi)$$

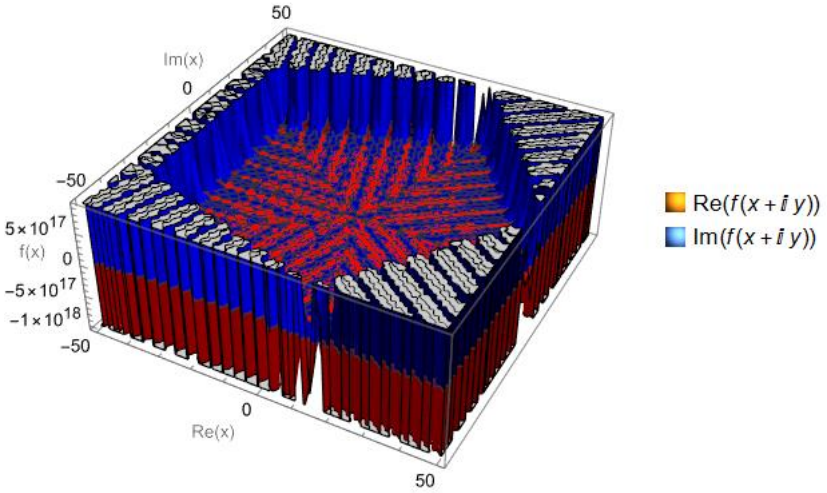
$$-2.5132741 = -\frac{2}{5}(2\pi) \rightarrow \frac{3}{5}(2\pi)$$

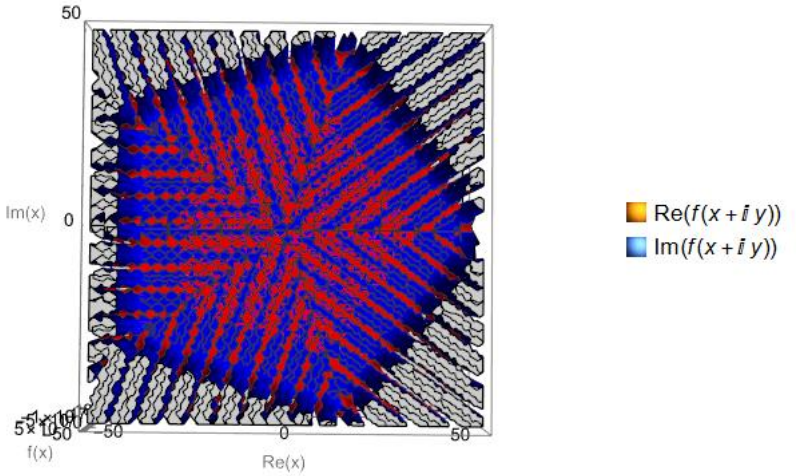
$$-1.2566371 = -\frac{1}{5}(2\pi) \rightarrow \frac{4}{5}(2\pi)$$

$$0 \rightarrow \frac{5}{5}(2\pi)$$

You can check it in the graph below (see Annex 3) which plots the function:

$$f(x) = 1 - \frac{x^5}{5!} + \frac{x^{10}}{10!} - \frac{x^{15}}{15!} + \cdots$$





The Annex 4 graphs the equivalent following function. But the program overflows and does not graph it properly, as you can check in the annex.

$$f(x) = \frac{1}{5}\left(e^{-x\cdot e^{\frac{1}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{4}{5}(2\pi i)}} + e^{-x\cdot e^{\frac{5}{5}(2\pi i)}}\right)$$

Continuing with the solutions, again, we have:

{{18.706788, -2.5132741}, {18.706788,
2.5132741}, {13.361992, -2.5132741}, {13.361992,
2.5132741}, {8.0171451, -2.5132741}, {8.0171451,
2.5132741}, {2.6072512, -2.5132741}, {2.6072512,
2.5132741}, {2.6072512, -1.2566371}, {2.6072512,
1.2566371}, {8.0171451, -1.2566371}, {8.0171451,
1.2566371}, {2.6072512, 0.}, {13.361992, -1.2566371}, {13.361992,
1.2566371}, {18.706788, -1.2566371}, {18.706788,
1.2566371}, {8.0171451, 0.}, {13.361992, 0.}, {18.706788, 0.}}

Which correspond to:

$$2.6072512 \cdot e^{\frac{1}{5}(2\pi i)}, 2.6072512 \cdot e^{\frac{2}{5}(2\pi i)}, 2.6072512 \cdot e^{\frac{3}{5}(2\pi i)}, 2.6072512 \cdot e^{\frac{4}{5}(2\pi i)}, 2.6072512 \cdot e^{\frac{5}{5}(2\pi i)}$$

$$8.0171451 \cdot e^{\frac{1}{5}(2\pi i)}, 8.0171451 \cdot e^{\frac{2}{5}(2\pi i)}, 8.0171451 \cdot e^{\frac{3}{5}(2\pi i)}, 8.0171451 \cdot e^{\frac{4}{5}(2\pi i)}, 8.0171451 \cdot e^{\frac{5}{5}(2\pi i)}$$

$$13.361992 \cdot e^{\frac{1}{5}(2\pi i)}, 13.361992 \cdot e^{\frac{2}{5}(2\pi i)}, 13.361992 \cdot e^{\frac{3}{5}(2\pi i)}, 13.3619921 \cdot e^{\frac{4}{5}(2\pi i)}, 13.361992 \cdot e^{\frac{5}{5}(2\pi i)}$$

$$18.706788 \cdot e^{\frac{1}{5}(2\pi i)}, 18.706788 \cdot e^{\frac{2}{5}(2\pi i)}, 18.706788 \cdot e^{\frac{3}{5}(2\pi i)}, 18.706788 \cdot e^{\frac{4}{5}(2\pi i)}, 18.706788 \cdot e^{\frac{5}{5}(2\pi i)}$$

If we want to obtain the period of the solutions, we see:

$$8.0171451 - 2.6072512 = 5.4098938$$

$$13.361992 - 8.0171451 = 5.3438469$$

$$18.706788 - 13.361992 = 5.344796$$

We can see that the function tends to a periodicity of ~5.344 that very probably corresponds the number π multiplied by a coefficient, as we have seen in the previous chapter 5. Anyhow, we are not even interested in calculating this number as we have the same problem as in the previous chapter 5. The function is not periodic. It tends to a period of ~5.344 but it is not really periodic.

So, we cannot apply the "Basel problem" we have seen in chapter 3. As we will have the same problem as in chapter 5, we will obtain a solution for ζ(5) "near" the real solution but not exact in any case. The only way to obtain the real solution would be to discover/calculate a function g(x) similar to f(x) but perfectly periodic (not asymptotically periodic as f(x)).

This g(x) would have a form similar to this (where r(n) is a function still to be calculated/discovered:

$$g(x) = 1 - r(5) \cdot \frac{x^5}{5!} + r(10) \cdot \frac{x^{10}}{10!} - r(15) \cdot \frac{x^{15}}{15!} + \cdots$$

Check chapter 5 for more details of what I am talking about.

7. General study regarding the values of the Riemann Zeta function at odd integers

As you can imagine we could continue with different functions to try to find the result of a general ζ(k).But we would find the same problem as in the chapters 5 and 6. We need to find functions that not only "resemble" the cosine function with different elements but also that have periodic solutions. This is, to find something like:

$$g(x) = 1 - r(1 \cdot k) \cdot \frac{x^{(1 \cdot k)}}{(1 \cdot k)!} + r(2 \cdot k) \cdot \frac{x^{(2 \cdot k)}}{(2 \cdot k)!} - r(3 \cdot k) \cdot \frac{x^{(3 \cdot k)}}{(3 \cdot k)!} + \cdots$$

Or equivalently:

$$g(x) = \frac{1}{k}\left(e^{-h(x)x\cdot e^{\frac{1}{k}(2\pi i)}} + e^{-h(x)x\cdot e^{\frac{2}{k}(2\pi i)}} + \cdots + e^{-h(x)x\cdot e^{\frac{k}{k}(2\pi i)}}\right)$$

$$g(x) = \frac{1}{k}\left(t(x,1)\cdot e^{-x\cdot e^{\frac{1}{k}(2\pi i)}} + t(x,2)e^{-x\cdot e^{\frac{2}{k}(2\pi i)}} + \cdots + t(x,3)e^{-x\cdot e^{\frac{k}{k}(2\pi i)}}\right)$$

Where r(n), h(x) and t(x,n) are functions to be calculated/discovered, so the g(x) function has "periodic" zeros. If we discovered them, we could calculate ζ(k) using the same method as in chapters 5 and 6.

Until now I have always considered k as an odd integer. If we dream a little more, we could consider k as an even integer, and using above procedure would give as a closed equation of the ζ(k) when k is even.

This is, we would have a closed equation for the Bernoulli numbers using the following expression [1] (having already calculated ζ(k) using above procedure)::

$$\zeta(k) = (-1)^{\frac{k}{2}+1}\frac{(2\pi)^k B_n}{2(n!)}$$

Dreaming even further, we could consider k even a general complex number. Specifically if we consider:

$$k = \frac{1}{2} + it$$

We could even obtain an expression for the non-trivial zeros of the Riemann Zeta function.

8. Conclusions

The goal of this paper was to calculate ζ(3) (the Riemann Zeta function with s=3) emulating the "Basel problem". But instead of using a sine or cosine function, using functions similar to these:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

$$f(x) = \frac{1}{3}\left(e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

We have discovered that the process itself seems ok, but with a problem. The solutions of the above functions are not periodic, so we cannot emulate the "Basel problem" perfectly, obtaining the following value:

$$\zeta(3) = \frac{\frac{\pi^3}{3!\left(\sqrt{3}\right)^3}}{1-\frac{1}{2^3}} = 1.1366020 \neq 1.202056903$$

With a small correction, we could arrive to:

$$\zeta(3) = \frac{\frac{\pi^3}{3!(\sqrt{3})^3}\, e^{\frac{2}{\sqrt{3}}}}{1-\frac{1}{2^3}}\frac{}{3} = 1.202173775 \approx 1.202056903 \text{ [7]}$$

But not getting the correct value anyhow. The only way of obtaining the correct value would be to find a function of the form:

$$g(x) = 1 - r(3)\cdot\frac{x^3}{3!} + r(6)\cdot\frac{x^6}{6!} - r(9)\cdot\frac{x^9}{9!} + \cdots$$

That has periodic zeros. Where r(n) is an unknown function to be calculate/discovered.

We have also generalized this study to calculate a general $\zeta(k)$ where k can be higher odd numbers, or even numbers. Having $\zeta(k)$ for even numbers would lead to obtaining a closed equation for the Bernoulli numbers.

We could go further and consider k a general complex number. In this case, considering k=½+it we could obtain a closed function for the zeros of the Riemann Zeta function.

Gorliz, 4th April 2024 (viXra-v1)

9. Acknowledgements

To my family and friends. To Paco Menéndez and Juan Delcán. To Cine3d. To Arash Majidan [10], To the current gods of Geometric Algebra: David Hestenes, Chris Doran, Anthony Lasenby, Garret Sobczyk, Joy Christian and many others. To A. Garret Lisi and J.O Weatherall.

AAAAÁBCCCDEEIIILLLLLMMMOOOPSTU

If you consider this helpful, do not hesitate to drop your BTC here:

bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f

10. References

[1] https://en.wikipedia.org/wiki/Riemann_zeta_function

[2] https://en.wikipedia.org/wiki/Basel_problem

[3] https://en.wikipedia.org/wiki/Taylor_series

[4] https://en.wikipedia.org/wiki/Weierstrass_factorization_theorem

[5] https://en.wikipedia.org/wiki/Geometric_series#Sum

[6] https://en.wikipedia.org/wiki/Sine_and_cosine

[7] https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function

[8]https://www.researchgate.net/publication/376352610_Embedding_the_Einstein_tensor_in_the_Klein-Gordon_Equation_using_Geometric_Algebra_Cl_30

[9]https://www.researchgate.net/publication/376894587_Energy-momentum_relation_in_curved_space-time

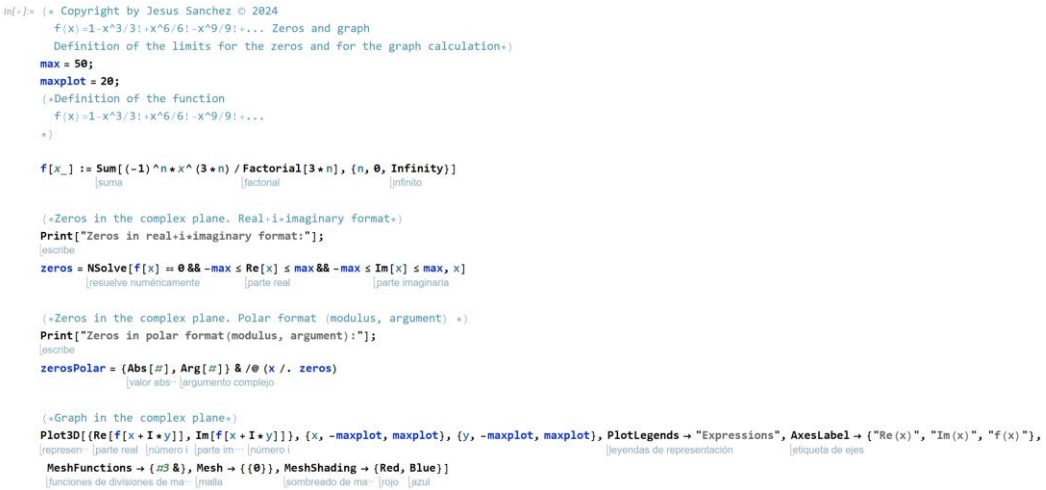
[10] https://www.researchgate.net/profile/Arash-Majidian

A1. Annex A1. Mathematica program for f(x) of three elements (1st version)

This is the Mathematica program to solve and plot:

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots$$

```
(* Copyright by Jesus Sanchez © 2024
f(x)=1-x^3/3!+x^6/6!-x^9/9!+... Zeros and graph
Definition of the limits for the zeros and for the graph calculation*)
max = 50;
maxplot = 20;
(*Definition of the function
f(x)=1-x^3/3!+x^6/6!-x^9/9!+...
*)

f[x_] := Sum[(-1)^n*x^(3*n)/Factorial[3*n], {n, 0, Infinity}]

(*Zeros in the complex plane. Real+i*imaginary format*)
Print["Zeros in real+i*imaginary format:"];
zeros = NSolve[
   f[x] == 0 && -max <= Re[x] <= max && -max <= Im[x] <= max, x]

(*Zeros in the complex plane. Polar format (modulus, argument) *)
Print["Zeros in polar format(modulus, argument):"];
zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)

(*Graph in the complex plane*)
Plot3D[{Re[f[x + I*y]], Im[f[x + I*y]]}, {x, -maxplot,
   maxplot}, {y, -maxplot, maxplot}, PlotLegends -> "Expressions",
 AxesLabel -> {"Re(x)", "Im(x)", "f(x)"}, MeshFunctions -> {#3 &},
 Mesh -> {{0}}, MeshShading -> {Red, Blue}]
```

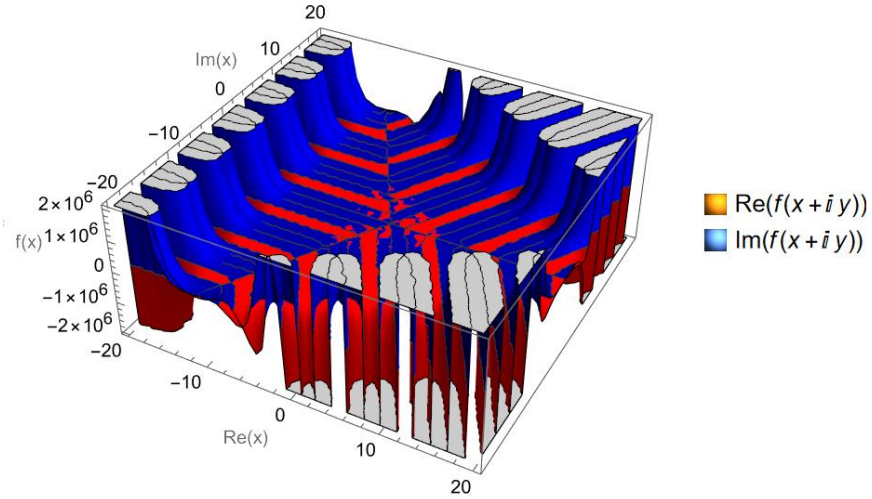


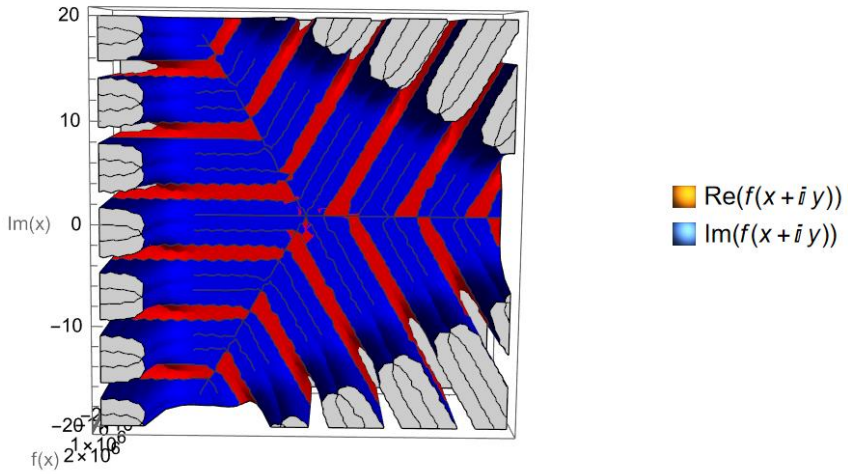
Zeros in real+i*imaginary format:

Out[•]= {{x → −28.11389 − 48.694686 i}, {x → −28.11389 + 48.694686 i}, {x → −26.300091 − 45.553093 i}, {x → −26.300091 + 45.553093 i},
{x → −24.486291 − 42.411501 i}, {x → −24.486291 + 42.411501 i}, {x → −22.672492 − 39.269908 i}, {x → −22.672492 + 39.269908 i}, {x → −20.858693 − 36.128316 i},
{x → −20.858693 + 36.128316 i}, {x → −19.044893 − 32.986723 i}, {x → −19.044893 + 32.986723 i}, {x → −17.231094 − 29.84513 i}, {x → −17.231094 + 29.84513 i},
{x → −15.417295 − 26.703538 i}, {x → −15.417295 + 26.703538 i}, {x → −13.603495 − 23.561945 i}, {x → −13.603495 + 23.561945 i}, {x → −11.789696 − 20.420352 i},
{x → −11.789696 + 20.420352 i}, {x → −9.9758965 − 17.27876 i}, {x → −9.9758965 + 17.27876 i}, {x → −8.1620971 − 14.137167 i}, {x → −8.1620971 + 14.137167 i},
{x → −6.3482978 − 10.995574 i}, {x → −6.3482978 + 10.995574 i}, {x → −4.5344988 − 7.8539823 i}, {x → −4.5344988 + 7.8539823 i}, {x → −2.7206167 − 4.7122463 i},
{x → −2.7206167 + 4.7122463 i}, {x → −0.9249064 − 1.6019849 i}, {x → −0.9249064 + 1.6019849 i}, {x → 1.8498128 + 0. i}, {x → 5.4412334 + 0. i},
{x → 9.0689975 + 0. i}, {x → 12.696596 + 0. i}, {x → 16.324194 + 0. i}, {x → 19.951793 + 0. i}, {x → 23.579392 + 0. i}, {x → 27.20699 + 0. i},
{x → 30.834589 + 0. i}, {x → 34.462188 + 0. i}, {x → 38.089787 + 0. i}, {x → 41.717385 + 0. i}, {x → 45.344984 + 0. i}, {x → 48.972583 + 0. i}}

Zeros in polar format(modulus, argument):

Out[•]= {{56.22778, −2.0943951}, {56.22778, 2.0943951}, {52.600182, −2.0943951}, {52.600182, 2.0943951}, {48.972583, −2.0943951}, {48.972583, 2.0943951},
{45.344984, −2.0943951}, {45.344984, 2.0943951}, {41.717385, −2.0943951}, {41.717385, 2.0943951}, {38.089787, −2.0943951}, {38.089787, 2.0943951},
{34.462188, −2.0943951}, {34.462188, 2.0943951}, {30.834589, −2.0943951}, {30.834589, 2.0943951}, {27.20699, −2.0943951}, {27.20699, 2.0943951},
{23.579392, −2.0943951}, {23.579392, 2.0943951}, {19.951793, −2.0943951}, {19.951793, 2.0943951}, {16.324194, −2.0943951}, {16.324194, 2.0943951},
{12.696596, −2.0943951}, {12.696596, 2.0943951}, {9.0689975, −2.0943951}, {9.0689975, 2.0943951}, {5.4412334, −2.0943951}, {5.4412334, 2.0943951},
{1.8498128, −2.0943951}, {1.8498128, 2.0943951}, {1.8498128, 0.}, {5.4412334, 0.}, {9.0689975, 0.}, {12.696596, 0.}, {16.324194, 0.}, {19.951793, 0.},
{23.579392, 0.}, {27.20699, 0.}, {30.834589, 0.}, {34.462188, 0.}, {38.089787, 0.}, {41.717385, 0.}, {45.344984, 0.}, {48.972583, 0.}}





{{56.22778, -2.0943951}, {56.22778,
2.0943951}, {52.600182, -2.0943951}, {52.600182,
2.0943951}, {48.972583, -2.0943951}, {48.972583,
2.0943951}, {45.344984, -2.0943951}, {45.344984,
2.0943951}, {41.717385, -2.0943951}, {41.717385,
2.0943951}, {38.089787, -2.0943951}, {38.089787,
2.0943951}, {34.462188, -2.0943951}, {34.462188,
2.0943951}, {30.834589, -2.0943951}, {30.834589,
2.0943951}, {27.20699, -2.0943951}, {27.20699,
2.0943951}, {23.579392, -2.0943951}, {23.579392,

```
2.0943951}, {19.951793, -2.0943951}, {19.951793,
2.0943951}, {16.324194, -2.0943951}, {16.324194,
2.0943951}, {12.696596, -2.0943951}, {12.696596,
2.0943951}, {9.0689975, -2.0943951}, {9.0689975,
2.0943951}, {5.4412334, -2.0943951}, {5.4412334,
2.0943951}, {1.8498128, -2.0943951}, {1.8498128,
2.0943951}, {1.8498128, 0.}, {5.4412334, 0.}, {9.0689975,
0.}, {12.696596, 0.}, {16.324194, 0.}, {19.951793, 0.}, {23.579392,
0.}, {27.20699, 0.}, {30.834589, 0.}, {34.462188, 0.}, {38.089787,
0.}, {41.717385, 0.}, {45.344984, 0.}, {48.972583, 0.}}
```

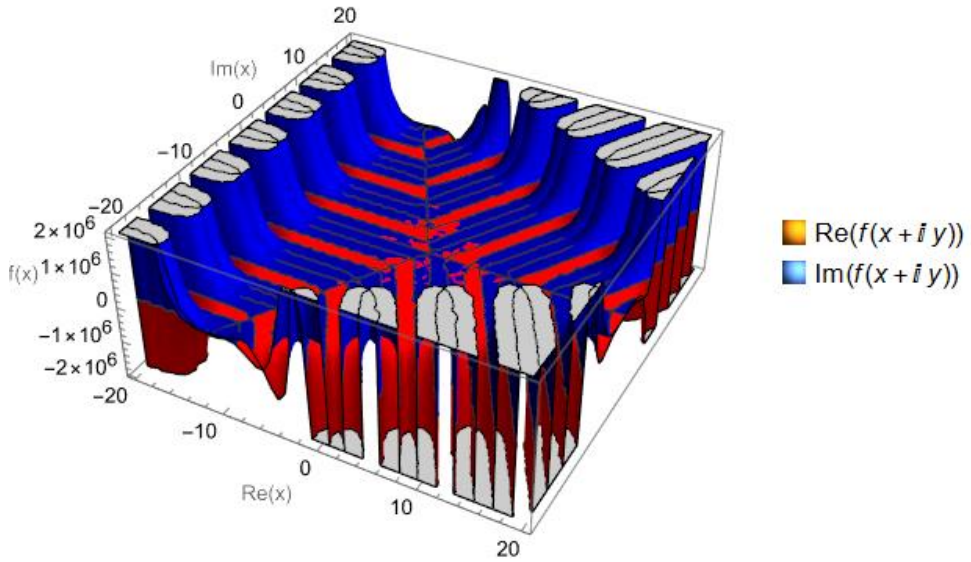
A2. Annex A2. Mathematica program for f(x) of three elements (2nd version)

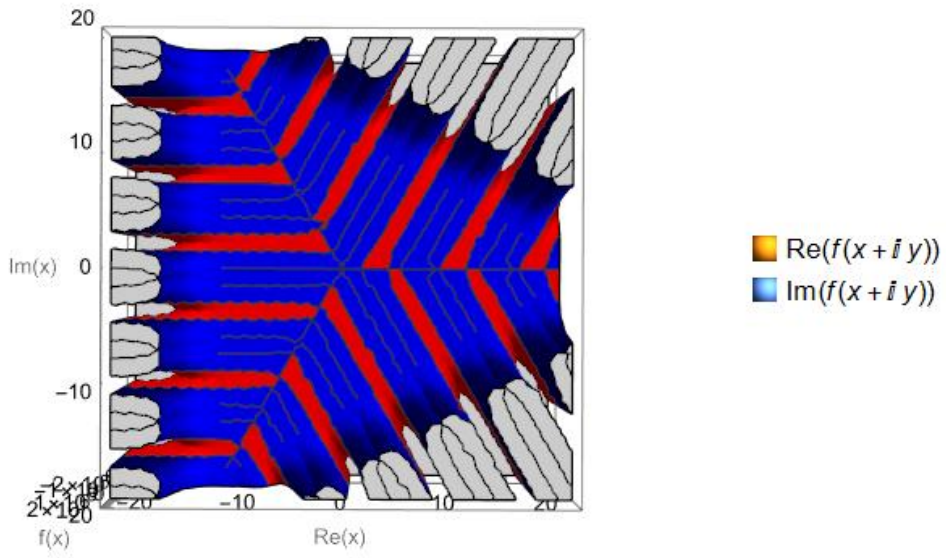
This is the Mathematica program to solve and plot:

$$f(x) = \frac{1}{3}\left(e^{-x\cdot e^{\frac{1}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{2}{3}(2\pi i)}} + e^{-x\cdot e^{\frac{3}{3}(2\pi i)}}\right)$$

```
(* Copyright by Jesus Sanchez © 2024
f(x)=1/3(e^(e^(-x*(1/3)*2pi.i))+e^(e^(-x*(2/3)*2pi.i))+e^(e^(-x*(3/3)*\
2pi.i)))
Zeros and graph
Definition of the limits for the zeros and for the graph calculation*)
max = 50;
maxplot = 20;
(*Definition of the function

f(x)=1/3(e^(e^(-x*(1/3)*2pi.i))+e^(e^(-x*(2/3)*2pi.i))+e^(e^(-x*(3/3)*\
2pi.i)))
*)

f[x_] :=
  1/3 (Exp[Exp[2*Pi*I/3]*x*(-1)] + Exp[Exp[4*Pi*I/3]*x*(-1)] +
      Exp[Exp[6*Pi*I/3]*x*(-1)])

(*Zeros in the complex plane. Real+i*imaginary format*)
Print["Zeros in real+i*imaginary format:"];
zeros = NSolve[
   f[x] == 0 && -max <= Re[x] <= max && -max <= Im[x] <= max, x]

(*Zeros in the complex plane. Polar format (modulus, argument) *)
Print["Zeros in polar format(modulus, argument):"];
zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)

(*Graph in the complex plane*)
Plot3D[{Re[f[x + I*y]], Im[f[x + I*y]]}, {x, -maxplot,
   maxplot}, {y, -maxplot, maxplot}, PlotLegends -> "Expressions",
  AxesLabel -> {"Re(x)", "Im(x)", "f(x)"}, MeshFunctions -> {#3 &},
  Mesh -> {{0}}, MeshShading -> {Red, Blue}]
```

```
In[•]:= (* Copyright by Jesus Sanchez © 2024
  f(x)=1/3(e^(e^(-x*(1/3)*2pi.i))+e^(e^(-x*(2/3)*2pi.i))+e^(e^(-x*(3/3)*2pi.i)))
  Zeros and graph
  Definition of the limits for the zeros and for the graph calculation*)
max = 50;
maxplot = 20;
(*Definition of the function

  f(x)=1/3(e^(e^(-x*(1/3)*2pi.i))+e^(e^(-x*(2/3)*2pi.i))+e^(e^(-x*(3/3)*2pi.i)))
*)

f[x_] := 1 / 3 (Exp[Exp[2 * Pi * I / 3] * x * (-1)] + Exp[Exp[4 * Pi * I / 3] * x * (-1)] + Exp[Exp[6 * Pi * I / 3] * x * (-1)])
                 |ex··  |expon··  |··  |número i          |ex··  |expon··  |··  |número i          |ex··  |expon··  |··  |número i

(*Zeros in the complex plane. Real+i*imaginary format*)
Print["Zeros in real+i*imaginary format:"];
|escribe
zeros = NSolve[f[x] == 0 && -max ≤ Re[x] ≤ max && -max ≤ Im[x] ≤ max, x]
                |resuelve numéricamente  |parte real  |parte imaginaria

(*Zeros in the complex plane. Polar format (modulus, argument) *)
Print["Zeros in polar format(modulus, argument):"];
|escribe
zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)
              |valor abs··  |argumento complejo

(*Graph in the complex plane*)
Plot3D[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, -maxplot, maxplot}, {y, -maxplot, maxplot}, PlotLegends → "Expressions", AxesLabel → {"Re(x)", "Im(x)", "f(x)"},
|represen··  |parte real  |número i  |parte im···  |número i                                  |leyendas de representación  |etiqueta de ejes
 MeshFunctions → {#3 &}, Mesh → {{0}}, MeshShading → {Red, Blue}]
 |funciones de divisiones de ma··  |malla  |sombreado de ma··  |rojo  |azul

Zeros in real+i*imaginary format:

Out[•]= {{x → -28.11389 - 48.694686 i}, {x → -28.11389 + 48.694686 i}, {x → -26.300091 - 45.553093 i}, {x → -26.300091 + 45.553093 i}, {x → -24.486291 - 42.411501 i}, {x → -24.486291 + 42.411501 i},
  {x → -22.672492 - 39.269908 i}, {x → -22.672492 + 39.269908 i}, {x → -20.858693 - 36.128316 i}, {x → -20.858693 + 36.128316 i}, {x → -19.044893 - 32.986723 i},
  {x → -19.044893 + 32.986723 i}, {x → -17.231094 - 29.84513 i}, {x → -17.231094 + 29.84513 i}, {x → -15.417295 - 26.703538 i}, {x → -15.417295 + 26.703538 i},
  {x → -13.603495 - 23.561945 i}, {x → -13.603495 + 23.561945 i}, {x → -11.789696 - 20.420352 i}, {x → -11.789696 + 20.420352 i}, {x → -9.9758965 - 17.27876 i},
  {x → -9.9758965 + 17.27876 i}, {x → -8.1620971 - 14.137167 i}, {x → -8.1620971 + 14.137167 i}, {x → -6.3482978 - 10.995574 i}, {x → -6.3482978 + 10.995574 i},
  {x → -4.5344988 - 7.8539823 i}, {x → -4.5344988 + 7.8539823 i}, {x → -2.7206167 - 4.7122463 i}, {x → -2.7206167 + 4.7122463 i}, {x → -0.9249064 - 1.6019849 i},
  {x → -0.9249064 + 1.6019849 i}, {x → 1.8498128 + 0. i}, {x → 5.4412334 + 0. i}, {x → 9.0689975 + 0. i}, {x → 12.696596 + 0. i}, {x → 16.324194 + 0. i}, {x → 19.951793 + 0. i},
  {x → 23.579392 + 0. i}, {x → 27.20699 + 0. i}, {x → 30.834589 + 0. i}, {x → 34.462188 + 0. i}, {x → 38.089787 + 0. i}, {x → 41.717385 + 0. i}, {x → 45.344984 + 0. i}, {x → 48.972583 + 0. i}}

Zeros in polar format(modulus, argument):

Out[•]= {{56.22778, -2.0943951}, {56.22778, 2.0943951}, {52.600182, -2.0943951}, {52.600182, 2.0943951}, {48.972583, -2.0943951}, {48.972583, 2.0943951}, {45.344984, -2.0943951},
  {45.344984, 2.0943951}, {41.717385, -2.0943951}, {41.717385, 2.0943951}, {38.089787, -2.0943951}, {38.089787, 2.0943951}, {34.462188, -2.0943951}, {34.462188, 2.0943951},
  {30.834589, -2.0943951}, {30.834589, 2.0943951}, {27.20699, -2.0943951}, {27.20699, 2.0943951}, {23.579392, -2.0943951}, {23.579392, 2.0943951}, {19.951793, -2.0943951},
  {19.951793, 2.0943951}, {16.324194, -2.0943951}, {16.324194, 2.0943951}, {12.696596, -2.0943951}, {12.696596, 2.0943951}, {9.0689975, -2.0943951}, {9.0689975, 2.0943951},
  {5.4412334, -2.0943951}, {5.4412334, 2.0943951}, {1.8498128, -2.0943951}, {1.8498128, 2.0943951}, {1.8498128, 0.}, {5.4412334, 0.}, {9.0689975, 0.}, {12.696596, 0.},
  {16.324194, 0.}, {19.951793, 0.}, {23.579392, 0.}, {27.20699, 0.}, {30.834589, 0.}, {34.462188, 0.}, {38.089787, 0.}, {41.717385, 0.}, {45.344984, 0.}, {48.972583, 0.}}
```





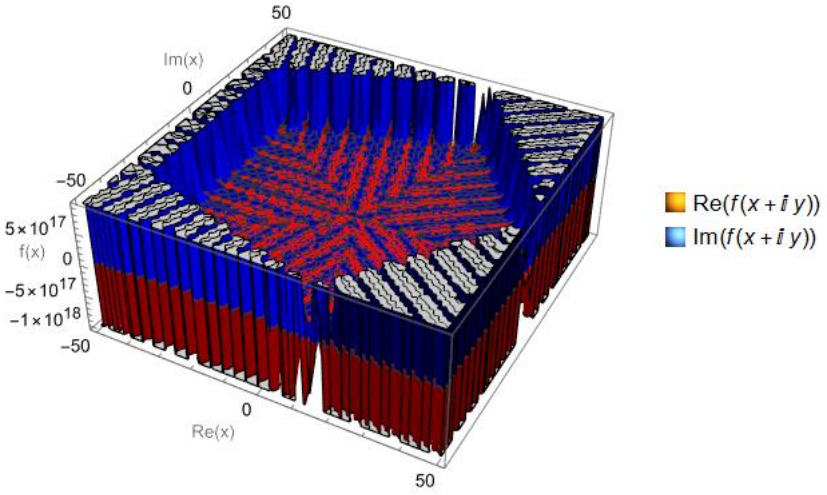
A3. Annex A3. Mathematica program for f(x) of five elements (1st version)

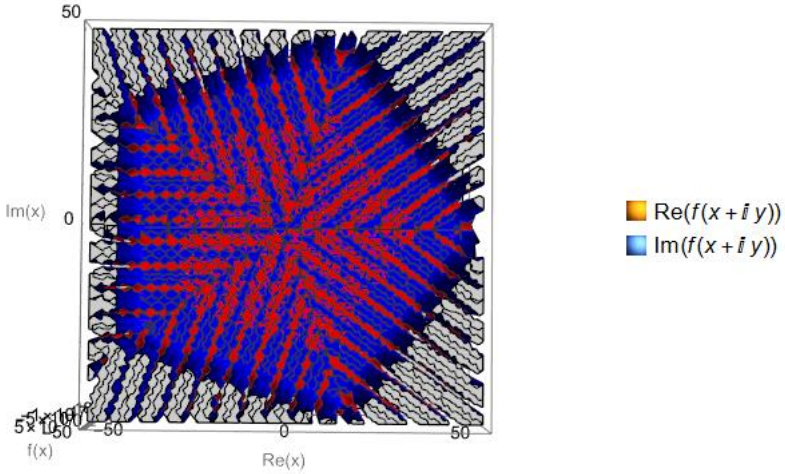
This is the Mathematica program to solve and plot:

$$f(x) = 1 - \frac{x^5}{5!} + \frac{x^{10}}{10!} - \frac{x^{15}}{15!} + \cdots$$

```
(* Copyright by Jesus Sanchez © 2024
f(x)=1-x^5/5!+x^10/10!-x^15/15!+... Zeros and graph
Definition of the limits for the zeros and for the graph calculation*)
max = 50;
maxplot = 50;
(*Definición de la función
f(x)=1-x^5/5!+x^10/10!-x^15/15!+...
*)

f[x_] := Sum[(-1)^n*x^(5*n)/Factorial[5*n], {n, 0, Infinity}]

(*Zeros in the complex plane. Real+i*imaginary format*)
Print["Zeros in real+i*imaginary format:"];
zeros = NSolve[
   f[x] == 0 && -max <= Re[x] <= max && -max <= Im[x] <= max, x]

(*Zeros in the complex plane. Polar format (modulus, argument) *)
Print["Zeros in polar format(modulus, argument):"];
zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)

(*Graph in the complex plane*)
Plot3D[{Re[f[x + I*y]], Im[f[x + I*y]]}, {x, -maxplot,
   maxplot}, {y, -maxplot, maxplot}, PlotLegends -> "Expressions",
  AxesLabel -> {"Re(x)", "Im(x)", "f(x)"}, MeshFunctions -> {#3 &},
  Mesh -> {{0}}, MeshShading -> {Red, Blue}]
```

```
In[•]:= (* Copyright by Jesus Sanchez © 2024
          f(x)=1-x^5/5!+x^10/10!-x^15/15!+... Zeros and graph
          Definition of the limits for the zeros and for the graph calculation*)
        max = 50;
        maxplot = 50;
        (*Definición de la función
          f(x)=1-x^5/5!+x^10/10!-x^15/15!+...
        *)

        f[x_] := Sum[(-1)^n * x^(5 * n) / Factorial[5 * n], {n, 0, Infinity}]
                 suma                  factorial                 infinito

        (*Zeros in the complex plane. Real+i*imaginary format*)
        Print["Zeros in real+i*imaginary format:"];
        escribe
        zeros = NSolve[f[x] == 0 && -max ≤ Re[x] ≤ max && -max ≤ Im[x] ≤ max, x]
                resuelve numéricamente   parte real                parte imaginaria

        (*Zeros in the complex plane. Polar format (modulus, argument) *)
        Print["Zeros in polar format(modulus, argument):"];
        escribe
        zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)
                     valor abs··  argumento complejo

        (*Graph in the complex plane*)
        Plot3D[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, -maxplot, maxplot}, {y, -maxplot, maxplot}, PlotLegends → "Expressions", AxesLabel → {"Re(x)", "Im(x)", "f(x)"},
        represen··  parte real  número i  parte im··  número i                                        leyendas de representación   etiqueta de ejes
         MeshFunctions → {#3 &}, Mesh → {{0}}, MeshShading → {Red, Blue}]
         funciones de divisiones de ma··  malla   sombreado de ma··  rojo  azul

        Zeros in real+i*imaginary format:
Out[•]= {{x → -15.13411 - 10.995574 i}, {x → -15.13411 + 10.995574 i}, {x → -10.810079 - 7.853982 i}, {x → -10.810079 + 7.853982 i},
         {x → -6.4860066 - 4.7123596 i}, {x → -6.4860066 + 4.7123596 i}, {x → -2.1093106 - 1.5325038 i}, {x → -2.1093106 + 1.5325038 i}, {x → 0.80568494 - 2.4796433 i},
         {x → 0.80568494 + 2.4796433 i}, {x → 2.4774341 - 7.6247581 i}, {x → 2.4774341 + 7.6247581 i}, {x → 2.6072512 + 0. i}, {x → 4.1290827 - 12.70801 i},
         {x → 4.1290827 + 12.70801 i}, {x → 5.7807155 - 17.791213 i}, {x → 5.7807155 + 17.791213 i}, {x → 8.0171451 + 0. i}, {x → 13.361992 + 0. i}, {x → 18.706788 + 0. i}}

        Zeros in polar format(modulus, argument):
Out[•]= {{18.706788, -2.5132741}, {18.706788, 2.5132741}, {13.361992, -2.5132741}, {13.361992, 2.5132741}, {8.0171451, -2.5132741}, {8.0171451, 2.5132741},
         {2.6072512, -2.5132741}, {2.6072512, 2.5132741}, {2.6072512, -1.2566371}, {2.6072512, 1.2566371}, {8.0171451, -1.2566371}, {8.0171451, 1.2566371}, {2.6072512, 0.},
         {13.361992, -1.2566371}, {13.361992, 1.2566371}, {18.706788, -1.2566371}, {18.706788, 1.2566371}, {8.0171451, 0.}, {13.361992, 0.}, {18.706788, 0.}}
```





```
{{18.706788, -2.5132741}, {18.706788,
   2.5132741}, {13.361992, -2.5132741}, {13.361992,
   2.5132741}, {8.0171451, -2.5132741}, {8.0171451,
   2.5132741}, {2.6072512, -2.5132741}, {2.6072512,
   2.5132741}, {2.6072512, -1.2566371}, {2.6072512,
   1.2566371}, {8.0171451, -1.2566371}, {8.0171451,
   1.2566371}, {2.6072512, 0.}, {13.361992, -1.2566371}, {13.361992,
   1.2566371}, {18.706788, -1.2566371}, {18.706788,
   1.2566371}, {8.0171451, 0.}, {13.361992, 0.}, {18.706788, 0.}}
```

A4. Annex A4. Mathematica program for f(x) of five elements (2nd version)

This is the Mathematica program to solve and plot:

$$f(x)=\frac{1}{5}\left(e^{-x\cdot e^{\frac{1}{5}(2\pi i)}}+e^{-x\cdot e^{\frac{2}{5}(2\pi i)}}+e^{-x\cdot e^{\frac{3}{5}(2\pi i)}}+e^{-x\cdot e^{\frac{4}{5}(2\pi i)}}+e^{-x\cdot e^{\frac{5}{5}(2\pi i)}}\right)$$

The program overflows, the solutions and the graph are not reliable:

```
(* Copyright by Jesus Sanchez © 2024
f(x)=1/5(e^(e^(-x*(1/5)*2pi.i))+e^(e^(-x*(2/5)*2pi.i))+e^(e^(-x*(3/5)*\
2pi.i))+e^(e^(-x*(4/5)*2pi.i))+e^(e^(-x*(5/5)*2pi.i)))
Zeros and graph
Definition of the limits for the zeros and for the graph calculation*)
max = 50;
maxplot = 50;
(*Definition of the function*)
(* f(x)=1/5(e^(e^(-x*(1/5)*2pi.i))+e^(e^(-x*(2/5)*2pi.i))+e^(e^(-x*(3/\
5)*2pi.i))+e^(e^(-x*(4/5)*2pi.i))+e^(e^(-x*(5/5)*2pi.i))) *)

f[x_] :=
  1/5 (Exp[Exp[2*Pi*I/5]*x*(-1)] + Exp[Exp[4*Pi*I/5]*x*(-1)] +
       Exp[Exp[6*Pi*I/5]*x*(-1)] +
       Exp[Exp[8*Pi*I/5]*x*(-1) + Exp[Exp[10*Pi*I/5]*x*(-1)]])

(*Zeros in the complex plane. Real+i*imaginary format*)
Print["Zeros in real+i*imaginary format:"];
zeros = NSolve[
    f[x] == 0 && -max <= Re[x] <= max && -max <= Im[x] <= max, x]

(*Zeros in the complex plane. Polar format (modulus, argument) *)
Print["Zeros in polar format(modulus, argument):"];
zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)

(*Graph in the complex plane*)
Plot3D[{Re[f[x + I*y]], Im[f[x + I*y]]}, {x, -maxplot,
   maxplot}, {y, -maxplot, maxplot}, PlotLegends -> "Expressions",
  AxesLabel -> {"Re(x)", "Im(x)", "f(x)"}, MeshFunctions -> {#3 &},
  Mesh -> {{0}}, MeshShading -> {Red, Blue}]
```

```
In[•]:= (* Copyright by Jesus Sanchez © 2024
  f(x)=1/5(e^(e^(-x*(1/5)*2pi.i))+e^(e^(-x*(2/5)*2pi.i))+e^(e^(-x*(3/5)*2pi.i))+e^(e^(-x*(4/5)*2pi.i))+e^(e^(-x*(5/5)*2pi.i)))
  Zeros and graph
  Definition of the limits for the zeros and for the graph calculation*)
max = 50;
maxplot = 50;
(*Definition of the function*)
(* f(x)=1/5(e^(e^(-x*(1/5)*2pi.i))+e^(e^(-x*(2/5)*2pi.i))+e^(e^(-x*(3/5)*2pi.i))+e^(e^(-x*(4/5)*2pi.i))+e^(e^(-x*(5/5)*2pi.i))) *)

f[x_] := 1 / 5 (Exp[Exp[2 * Pi * I / 5] * x * (-1)] + Exp[Exp[4 * Pi * I / 5] * x * (-1)] + Exp[Exp[6 * Pi * I / 5] * x * (-1)] + Exp[Exp[8 * Pi * I / 5] * x * (-1)] + Exp[Exp[10 * Pi * I / 5] * x * (-1)])

(*Zeros in the complex plane. Real+i*imaginary format*)
Print["Zeros in real+i*imaginary format:"];
zeros = NSolve[f[x] == 0 && -max ≤ Re[x] ≤ max && -max ≤ Im[x] ≤ max, x]

(*Zeros in the complex plane. Polar format (modulus, argument) *)
Print["Zeros in polar format(modulus, argument):"];
zerosPolar = {Abs[#], Arg[#]} & /@ (x /. zeros)

(*Graph in the complex plane*)
Plot3D[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, -maxplot, maxplot}, {y, -maxplot, maxplot}, PlotLegends → "Expressions", AxesLabel → {"Re(x)", "Im(x)", "f(x)"},
 MeshFunctions → {#3 &}, Mesh → {{0}}, MeshShading → {Red, Blue}]
```

The program overflows, the solutions and the graph are not reliable:

```
Zeros in real+i*imaginary format:
Out[•]= {{x → -49.439369 - 35.919804 i}, {x → -46.766971 - 33.978193 i}, {x → -38.749776 - 28.15336 i}, {x → -30.732581 - 22.328527 i}, {x → -22.715386 - 16.503694 i}, {x → -20.042987 - 14.562083 i},
 {x → -18.68986 - 32.986723 i}, {x → -16.750486 - 39.26991 i}, {x → -14.698189 - 10.67886 i}, {x → -13.879265 + 39.269838 i}, {x → -13.473103 - 32.986761 i}, {x → -13.419633 - 32.986763 i},
 {x → -13.418484 - 32.98663 i}, {x → -13.248918 - 32.98677 i}, {x → -12.532562 - 26.703354 i}, {x → -12.299943 - 14.137253 i}, {x → -12.285501 - 32.986409 i}, {x → -11.393199 - 14.137368 i},
 {x → -9.832688 - 20.418268 i}, {x → -9.4859789 - 39.264239 i}, {x → -8.9373974 - 20.42264 i}, {x → -8.709113 - 32.990309 i}, {x → -8.6729299 - 7.8514255 i}, {x → -8.0385587 - 14.128493 i},
 {x → -7.9458983 - 14.142132 i}, {x → -7.8425709 - 20.426709 i}, {x → -7.7002469 - 7.8591745 i}, {x → -7.443547 - 39.28313 i}, {x → -7.4035642 - 20.429914 i}, {x → -6.8309656 - 14.150965 i},
 {x → -6.797263 - 14.107194 i}, {x → -6.7430101 + 39.181919 i}, {x → -6.7294436 - 26.724134 i}, {x → -6.6322811 - 45.584677 i}, {x → -6.6164937 - 39.298914 i}, {x → -6.4808374 - 26.625844 i},
 {x → -6.4577935 - 7.8697768 i}, {x → -6.3430423 - 29.813601 i}, {x → -6.3030498 - 39.308961 i}, {x → -6.2875382 + 1.5652597 i}, {x → -6.2643648 + 7.8256522 i}, {x → -6.2289499 + 14.084325 i},
 {x → -6.0956622 - 39.101574 i}, {x → -5.9640006 - 39.323776 i}, {x → -5.8870393 - 14.169724 i}, {x → -5.8760599 + 1.5624552 i}, {x → -5.8643508 - 20.310458 i}, {x → -5.8504641 - 4.7383352 i},
 {x → -5.7977986 - 7.882337 i}, {x → -5.4869594 - 7.891259 i}, {x → -5.4180296 - 42.316297 i}, {x → -5.4178077 - 45.653723 i}, {x → -5.4175857 - 1.5576408 i}, {x → -5.4100101 - 45.158863 i},
 {x → -5.3920122 - 39.362589 i}, {x → -5.3839599 - 45.147962 i}, {x → -5.3607162 - 1.5568772 i}, {x → -5.3090162 - 14.005004 i}, {x → -5.3079241 - 26.781183 i},
 {x → -5.302349 + 45.111578 i}, {x → -5.299287 + 42.830612 i}, {x → -5.243704 - 26.785973 i}, {x → -5.206792 + 26.424533 i}, {x → -5.1782847 + 42.88941 i}, {x → -4.9914395 - 7.9114756 i},
 {x → -4.9733374 - 1.5503705 i}, {x → -4.9387174 - 43.038172 i}, {x → -4.8992762 - 17.191291 i}, {x → -4.8664091 - 17.53858 i}, {x → -4.8589334 - 23.453479 i}, {x → -4.8541124 - 7.9187717 i},
 {x → -4.7904062 - 26.829354 i}, {x → -4.6374512 - 43.335651 i}, {x → -4.6134106 - 39.464476 i}, {x → -4.610591 - 45.77147 i}, {x → -4.557151 - 26.860006 i}, {x → -4.55249 - 42.19458 i},
 {x → -4.549879 - 39.4767 i}, {x → -4.547189 - 45.785302 i}, {x → -4.540134 - 35.933608 i}, {x → -4.5281416 - 29.673179 i}, {x → -4.5093827 + 19.98864 i}, {x → -4.5052871 - 7.9416435 i},
 {x → -4.4993101 - 14.250883 i}, {x → -4.4946838 - 10.89508 i}, {x → -4.4268972 - 14.258523 i}, {x → -4.4213928 - 20.570118 i}, {x → -4.4127817 - 33.194176 i}, {x → -4.3926141 + 13.806358 i},
 {x → -4.318183 - 13.780397 i}, {x → -4.3060456 - 1.6130791 i}, {x → -4.2510777 - 17.781621 i}, {x → -4.2359755 - 19.842792 i}, {x → -4.2246363 - 4.8476922 i}, {x → -4.1684096 - 10.861979 i},
 {x → -4.1650809 + 1.5256356 i}, {x → -4.0928496 - 48.316377 i}, {x → -4.0776295 - 7.9807239 i}, {x → -4.0672639 - 10.849695 i}, {x → -3.9974126 - 23.322004 i}, {x → -3.9945684 - 2.9022238 i},
 {x → -3.9568392 - 14.32232 i}, {x → -3.8374735 - 8.0097644 i}, {x → -3.7135649 - 17.023906 i}, {x → -3.6916308 - 8.0305333 i}, {x → -3.6844214 - 10.792283 i}, {x → -3.6785482 - 27.063483 i},
 {x → -3.6715498 - 35.679263 i}, {x → -3.6588613 - 4.5633795 i}, {x → -3.6308545 - 29.443496 i}, {x → -3.6184895 - 48.079626 i}, {x → -3.5911041 - 23.211686 i}, {x → -3.536938 - 39.835871 i},
 {x → -3.520884 - 8.0583918 i}, {x → -3.5111552 - 20.769975 i}, {x → -3.4446936 - 7.3888136 i}, {x → -3.4021363 - 5.0370998 i}, {x → -3.3440481 - 1.6769091 i}, {x → -3.3161218 - 10.716107 i},
 {x → -3.3094977 - 14.470543 i}, {x → -3.2706717 - 29.271417 i}, {x → -3.2607436 - 7.2928158 i}, {x → -3.239204 - 46.486828 i}, {x → -3.2312442 - 41.579198 i}, {x → -3.2016921 - 23.053906 i},
 {x → -3.1855523 - 40.117956 i}, {x → -3.1646065 - 47.618928 i}, {x → -3.1334279 - 16.846133 i}, {x → -3.098476 - 46.726696 i}, {x → -3.0712332 - 20.958088 i}, {x → -3.0677506 - 10.649123 i},
 {x → -3.0568238 - 41.361343 i}, {x → -3.0560461 - 8.1590554 i}, {x → -3.0230979 - 47.030151 i}, {x → -3.0116771 - 40.36115 i}, {x → -2.9367926 - 35.111188 i}, {x → -2.925815 - 33.97901 i},
 {x → -2.9241876 - 41.057249 i}, {x → -2.90288 - 40.693006 i}, {x → -2.8454186 - 27.577722 i}, {x → -2.8452291 - 16.711087 i}, {x → -2.8019435 - 28.871937 i}, {x → -2.7773694 - 34.764334 i},
 {x → -2.7771578 - 21.156353 i}, {x → -2.7694161 - 34.339206 i}, {x → -2.7290372 - 14.71739 i}, {x → -2.7078668 - 8.267833 i}, {x → -2.6493524 - 22.64935 i}, {x → -2.6218399 - 27.960413 i},
 {x → -2.6031854 - 28.468719 i}, {x → -2.5939578 + 6.6253222 i}, {x → -2.4766064 - 16.453239 i}, {x → -2.462762 - 21.543266 i}, {x → -2.4305422 + 1.3417981 i}, {x → -2.3883327 - 22.168936 i},
 {x → -2.3056012 - 15.069086 i}, {x → -2.186601 - 8.5257078 i}, {x → -2.1058545 - 15.862145 i}, {x → -2.0870853 - 4.211694 i}, {x → -1.8455609 - 1.1932395 i}, {x → -1.683869 - 9.5424331 i},
 {x → -0.82311206 - 0.72449397 i}, {x → -0.6694682 - 3.2940716 i}, {x → 0.26773409 - 2.1503594 i}, {x → 0.79299281 - 2.5211581 i}, {x → 2.4765895 - 7.5532706 i}, {x → 2.4775426 - 7.6247488 i},
 {x → 2.5965357 - 0.0023210079 i}, {x → 4.1290825 - 12.70801 i}, {x → 4.141264 - 12.702126 i}, {x → 5.7807155 - 17.791213 i}, {x → 5.7826464 - 17.792985 i}, {x → 7.4321651 - 22.874886 i},
 {x → 7.4323485 - 22.874417 i}, {x → 8.0171455 - 3.4982314 × 10^-6 i}, {x → 9.083885 - 27.95762 i}, {x → 9.0839815 - 27.95762 i}, {x → 10.735608 - 33.040807 i}, {x → 10.735614 - 33.040824 i},
 {x → 12.387247 + 38.124028 i}, {x → 12.38725 - 38.124025 i}, {x → 13.361992 + 6.1439362 × 10^-14 i}, {x → 14.03888 + 43.207231 i}, {x → 14.038881 - 43.207232 i}, {x → 15.690513 - 48.290435 i},
 {x → 15.690513 + 48.290435 i}, {x → 18.706788 + 0. i}, {x → 24.051585 + 0. i}, {x → 29.396382 + 0. i}, {x → 34.741178 + 0. i}, {x → 40.085975 + 0. i}, {x → 45.430772 + 0. i}}

Out[•]= {{61.110421, -2.5132741}, {57.807155, -2.5132741}, {47.897357, -2.5132741}, {37.987559, -2.5132741}, {28.077761, -2.5132741}, {24.774495, -2.5132741},
 {37.913517, -2.0862853}, {42.693145, -1.9739771}, {18.167961, -2.5132741}, {41.65038, 1.9105264}, {35.632161, -1.9585571}, {35.611979, -1.9571671}, {35.611423, 1.9571386},
 {35.548007, -1.9527186}, {29.498038, 2.0096045}, {18.73901, -2.2868089}, {35.172112, 1.9251904}, {18.156822, -2.2491199}, {22.662467, 2.0195861}, {40.393864, 1.8078473},
 {22.292629, -1.9833092}, {34.120509, -1.8288981}, {11.698914, 2.4058684}, {16.255237, 2.0880802}, {16.221504, -2.0826995}, {21.880502, -1.9373791}, {11.002746, -2.3459806},
 {39.98213, -1.7580608}, {21.730029, -1.9184647}, {15.713431, -2.0205253}, {15.659365, 2.0198022}, {39.757904, 1.7412219}, {27.558388, -1.8174792}, {46.064628, -1.7152762},
 {39.852009, -1.7375953}, {27.403227, 1.8095574}, {10.180201, -2.2579616}, {30.480895, -1.7804273}, {39.81109, -1.7297893}, {6.4794425, 2.8976059}, {10.024126, 2.2458366},
 {15.400261, 1.987196}, {39.573858, 1.7254446}, {39.773466, -1.7213132}, {15.343999, -1.964564}, {6.0802423, 2.8817044}, {21.140135, 1.851887}, {7.5285955, 2.4608396},
 {9.7849735, -2.2049806}, {9.6113835, -2.1783752}, {42.66174, -1.69814}, {45.974069, -1.6889157}, {5.6370631, 2.8616284}, {45.481767, 1.6900276}, {39.73018, -1.7069322},
 {45.467851, 1.6894873}, {5.5822168, 2.8589448}, {14.97751, 1.9331391}, {27.302121, -1.7664567}, {45.422124, 1.687798}, {43.157198, 1.6938973}, {27.29441, -1.7641146},
 {26.932631, 1.7653478}, {43.200881, 1.6909505}, {9.3544596, -2.1336352}, {5.209389, 2.8394036}, {43.320609, 1.6850486}, {17.875777, -1.8484225}, {18.201201, 1.8414563},
 {23.951512, -1.7750796}, {9.2881296, -2.1207112}, {27.253665, -1.7474853}, {43.583077, 1.677403}, {39.733216, -1.6871685}, {46.003098, -1.6711884}, {27.243854, -1.738859},
 {42.43946, -1.6782733}, {39.738033, -1.6855448}, {46.010551, -1.6697872}, {36.219291, -1.6964782}, {30.016689, -1.7222286}, {20.49098, 1.7926792}, {9.1305702, -2.0868239},
 {14.944279, -1.8766134}, {11.785795, -1.9620682}, {14.929933, -1.8718342}, {21.039926, -1.7825175}, {33.486206, -1.7029598}, {14.488291, 1.8788282}, {14.441124, 1.8744617},
 {4.5982663, -2.7831657}, {18.282716, 1.8054631}, {20.289896, 1.7811161}, {6.4302155, 2.2876256}, {11.634356, -1.9372261}, {4.4357032, 2.7904782}, {48.489418, -1.6553039},
 {8.9620877, -2.0431534}, {11.586998, -1.9294561}, {23.662104, -1.7405477}, {4.9375581, -2.5132741}, {14.85885, -1.8403436}, {8.8815837, -2.0175841},
 {17.424234, -1.7855701}, {8.8384163, -2.0016869}, {11.403874, -1.8997839}, {27.312339, -1.7058914}, {35.867674, -1.6733397}, {5.8490768, -2.2466264},
 {29.666523, -1.6934929}, {48.215598, -1.6459151}, {23.487836, -1.7242904}, {39.992581, -1.6593519}, {8.7939924, -1.982721}, {21.064664, -1.7382626},
 {8.1523298, 2.0070434}, {6.0783967, 2.1648315}, {3.7409467, -2.6767771}, {11.217469, -1.870902}, {14.84417, -1.7956352}, {29.453576, -1.6820708}, {7.9885926, 1.9912504},
 {46.599545, -1.6403639}, {41.704564, -1.6483534}, {23.275167, -1.7087922}, {40.244231, -1.6500347}, {47.723967, -1.6371557}, {17.135069, -1.7546975},
 {46.829314, -1.63701}, {21.181924, -1.7163023}, {11.082189, -1.8512776}, {41.474147, -1.6445675}, {8.7126117, -1.9291801}, {47.127213, -1.634988}, {40.473357, -1.6452765},
 {35.233794, -1.6542447}, {34.104743, -1.656691}, {41.161251, -1.6418985}, {40.796415, -1.6420118}, {27.724126, -1.6736107}, {16.951587, -1.739445}, {29.007579, -1.6675407},
 {34.875102, -1.6505183}, {21.33785, -1.7013183}, {34.4507, -1.651271}, {14.968274, -1.7541433}, {8.6999774, -1.8873043}, {22.803774, -1.6872397}, {28.083069, -1.6642926},
 {28.587489, -1.6619829}, {7.1150201, 1.9439726}, {16.638589, -1.7201987}, {21.683577, -1.6846192}, {2.7763208, 2.6371715}, {22.297216, -1.6781157}, {15.244447, -1.722621},
 {8.8016429, -1.8218565}, {16.001321, -1.7027843}, {4.7004565, -2.0308738}, {2.1977068, 2.5676496}, {9.689863, -1.7454594}, {1.0965423, 2.4198314}, {3.3614127, -1.7712998},
 {2.1669626, -1.4469271}, {2.6429294, 1.2660585}, {7.948924, -1.253959}, {8.0171698, 1.2566238}, {2.5965368, 0.00089388613}, {13.361992, 1.2566371}, {13.360168, -1.2556338},
 {18.706788, 1.2566371}, {18.70907, -1.2565682}, {24.051974, -1.2566503}, {24.051585, 1.2566371}, {8.0171455, -4.3634376 × 10^-9}, {29.396352, -1.2566402},
 {29.396382, 1.2566371}, {34.74116, -1.2566371}, {34.741178, 1.2566371}, {40.085975, 1.2566371}, {40.085973, -1.256637}, {13.361992, 4.598069 × 10^-15}, {45.430772, 1.2566371},
 {45.430772, -1.2566371}, {50.775568, -1.2566371}, {50.775568, 1.2566371}, {18.706788, 0.}, {24.051585, 0.}, {29.396382, 0.}, {34.741178, 0.}, {40.085975, 0.}, {45.430772, 0.}}

··· General: Overflow occurred in computation.
··· General: Overflow occurred in computation.
··· General: Overflow occurred in computation.
··· General: Further output of General::ovfl will be suppressed during this calculation.
```

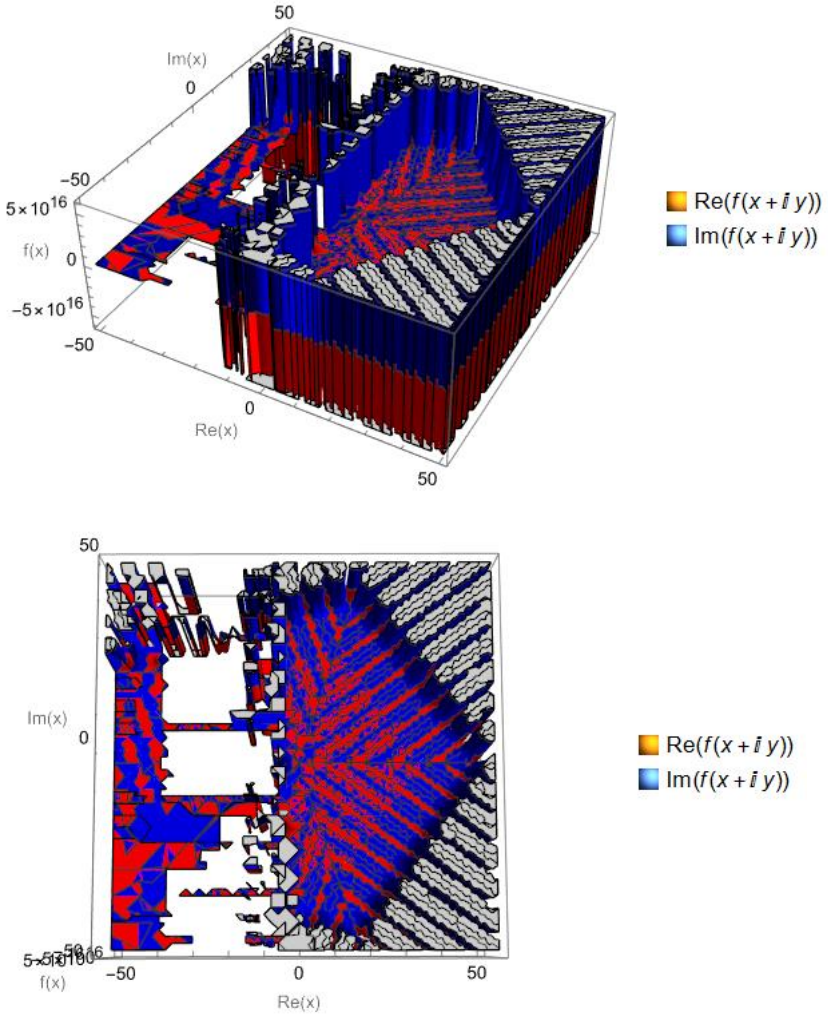
The program overflows, the solutions and the graph are not reliable. The graph should be equal to the graph in Annex 3. It should be symmetrical in fifth parts of the complex plane. And this graph is not. It is not correct.

General: Overflow occurred in computation.

General: Overflow occurred in computation.

General: Overflow occurred in computation.

General: Further output of General::ovfl will be suppressed during this calculation.



A5. Annex A5. Infinite sums and products from -infinity to +infinity

We are accustomed to infinite series that start at one point and finish at infinity. The result of the series (the sum or the product) normally is a number that has a number related to the series.

What it is not so common is to check series that start at -infinity and finished at +infinity. This means, they do not have a beginning and they do not have an end. We will see that the result of these series surprisingly is zero when it is a sum and 1 when it is a product. As we will see now.

We start with the geometric series [5]. The standard geometric series has the following form. It has a start at one point and then the series is infinite. This way:

$$S = \sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots = a + r(a + ar + ar^2 + \cdots) = a + rS$$

The solution to this series is well known [5] and it can be found like this:

$$S = a + rS$$

$$S - rS = a$$

$$S(1 - r) = a$$

$$S = \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

But if instead of that, we find a geometric series that has no beginning and no end in the following way (it starts at -infinity and finished at +infinity):

$$S = \sum_{k=-\infty}^{\infty} ar^k = \cdots + ar^{-3} + ar^{-2} + ar^{-1} + a + ar + ar^2 + ar^3 + \cdots =$$

We can divide it in two series, one with the negative exponents (S_) and one with the zero and positive exponents (S_+):

$$S_- \equiv \sum_{k=-\infty}^{-1} ar^k = \cdots + ar^{-3} + ar^{-2} + ar^{-1} = ar^{-1} + ar^{-2} + ar^{-3} + \cdots$$

$$S_+ \equiv \sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

So, the sum of both is S:

$$S = \sum_{k=-\infty}^{\infty} ar^k = \sum_{k=-\infty}^{-1} ar^k + \sum_{k=0}^{\infty} ar^k = S_- + S_+$$

Calculating S_:

$$S_- = ar^{-1} + ar^{-2} + ar^{-3} + \cdots = ar^{-1} + r^{-1}(ar^{-1} + ar^{-2} + \cdots) = ar^{-1} + rS_-$$

$$S_- = \frac{ar^{-1}}{1 - r^{-1}} = \frac{\frac{a}{r}}{1 - \frac{1}{r}} = \frac{\frac{a}{r}}{\frac{r-1}{r}} = \frac{a}{r - 1}$$

And S_+:

$$S_+ \equiv a + ar + ar^2 + ar^3 + \cdots = a + r(a + ar + ar^2 + \cdots) = a + rS_+$$

$$S_+ = a + rS_+$$

$$S_+ - rS_+ = a$$

$$S_+(1-r) = a$$

$$S_+ = \frac{a}{1-r}$$

Now, if we sum both to get S:

$$S = S_- + S_+ = \frac{a}{r-1} + \frac{a}{1-r} = \frac{a}{r-1} - \frac{a}{r-1} = 0$$

$$S = \sum_{k=-\infty}^{\infty} ar^k = \cdots + ar^{-1} + a + ar + ar^2 + \cdots = 0$$

We see that the result of geometric series that start at -infinity and finishes at +infinity is zero (independently of the r (ratio) and of a (the scalation). If the geometric series starts at -infinity and finishes at +infinity, its value is always zero.

Now, let's do another sum. If we sum all the 1s starting at one point (a first 1) and finishing at infinity, we have:

$$S = 1 + 1 + 1 + 1 + \cdots$$

This is the Riemann Zeta Function [1] with s=0. And we know [1] that its value is -1/2.

$$\xi(0) = \sum_{k=1}^{\infty} k^0 = \sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = S = -\frac{1}{2}$$

$$S = \sum_{k=1}^{\infty} 1 = -\frac{1}{2}$$

But now, we want to make the sum starting at -infinity and finishing at +infinity.

$$S = \sum_{k=-\infty}^{\infty} 1 = \ldots + 1 + 1 + 1 + 1 + 1 + \cdots$$

To do this, we can define a 1 that is in the "middle" of the sum (let's say it is in the position zero, we call it 1_0).

$$S = \cdots 1_{-3} + 1_{-2} + 1_{-1} + 1_0 + 1_1 + 1_2 + 1_3 \ldots$$

Let's put it in green to visualize it better. This 1_0 is in the "middle" (in fact any 1 can be this 1_0, we just select one of them). The 1s at the right of it (blue ones) are a series of 1s that have a beginning and ends at infinity. And the ones at the left of 1_0 (red ones) are another series with a beginning and ends at -infinity.

$$S = \cdots 1_{-3} + 1_{-2} + 1_{-1} + 1_0 + 1_1 + 1_2 + 1_3 \ldots$$

$$S = \cdots + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots$$

So, we have this:

$$S = \sum_{k=-\infty}^{-1} 1 + 1 + \sum_{k=1}^{\infty} 1$$

Only for the red sum, let's make a change of variable this way:

$$m = -k$$

$$k = -1 \rightarrow m = 1$$

$$k = -\infty \rightarrow m = \infty$$

Leading to:

$$S = \sum_{m=1}^{\infty} 1 + 1 + \sum_{k=1}^{\infty} 1$$

So, making the sum:

$$S = \xi(0) + 1 + \xi(0) = -\frac{1}{2} + 1 - \frac{1}{2} = 0$$

We obtain the infamous zero. So, summing up, the sum of 1s when we start at -infinity and finish at +infinity is zero:

$$S = \sum_{k=-\infty}^{\infty} 1 = \ldots + 1 + 1 + 1 + 1 + 1 + \cdots = 0$$

Knowing this, calculating an infinite product is trivial (we can use logarithms to convert any product into sums). We start with a standard infinite product that starts at one point and finishes at infinity:

$$P = \prod_{k=1}^{\infty} r = r \cdot r \cdot r \cdot r \cdot r \cdot \ldots$$

$$\ln P = \ln r + \ln r + \ln r + \ln r + \cdots = \ln r\,(1 + 1 + 1 + 1 + \cdots) = \ln r\,\big(\xi(0)\big)$$

$$\ln P = \ln r\,\big(\xi(0)\big) = -\frac{1}{2}\ln r = \ln r^{-\frac{1}{2}}$$

$$P = r^{-\frac{1}{2}}$$

We obtain this:

$$P = \prod_{k=1}^{\infty} r = r \cdot r \cdot r \cdot r \cdot r \cdot \ldots = r^{-\frac{1}{2}}$$

But if we start the product at -infinity and finish at +infinity:

$$P = \prod_{k=-\infty}^{\infty} r = \ldots \cdot r \cdot r \cdot r \cdot r \cdot r \cdot r \cdot \ldots$$

$$\ln P = \cdots + \ln r + \ln r + \ln r + \ln r + \ln r + \cdots = \ln r\,(\ldots + 1 + 1 + 1 + 1 + \cdots)$$

$$\ln P = \ln r\,(\ldots + 1 + 1 + 1 + 1 + \cdots) = \ln r \cdot (0) = 0$$

$$P = e^{0} = 1$$

We obtain that the infinite product starting from -infinity to +infinity is 1 (independently of the value of r).

$$P = \prod_{k=-\infty}^{\infty} r = \ldots \cdot r \cdot r \cdot r \cdot r \cdot r \cdot r \cdot \ldots = 1$$

A6. Annex A6. Future developments

I want to take this opportunity to say that very probably I will not publish papers during a time. Some pending studies from my side:

- Weak interaction in Geometric Algebra $Cl_{3,0}$.
- In the paper "Embedding the Einstein tensor in the Klein-Gordon Equation using Geometric Algebra Cl 3,0" [8] the operations are calculated in an orthogonal basis to facilitate the calculations. Even if this option is ok in most of the cases (even Schwarzschild metric is orthogonal) it is not a general solution that should be found.
- The Annex A1 of the paper "Energy-momentum relation in curved space-time" [9] is the beginning of a more detailed study. Normally it is considered that in the "quantum mechanics" world, 8 degrees of freedom apply (corresponding to the eight parameters of the Dirac spinors or the eight parameters of the Geometric Algebra created by three spatial dimensions- scalars, 3 vectors, 3 bivectors and 1 trivector-). But in the "macro" world only 4 dimensions apply. This is not correct. The issue is that in a flat space-time when two spinors are multiplied (or one squared), the symmetries of the flat space-time make that four dimensions are cancelled due to symmetries (some terms are equal with different sign, so they are cancelled, you can check annexes A1 to A4 in [8] to check it). This does not happen in high curvature (big gravity) space-time areas, where these symmetries do not exist. So, in these areas the eight dimensions should be still considered, not only the 4 left that we normally experience in our "macro" world. The studies for high curvature space-time areas should be done considering the 8 dimensions (or 8 degrees of freedom or 8 free parameters if you want -scalars, 3 vectors, 3 bivectors and 1 trivector-) not only 4. To make this study in high curvature space-time areas considering all the 8 dimensions is still to be done. In fact, Arash Majidan [10] contacted me to make this study (specifically in black holes). I commented that was in my pending list, but of course, everyone is invited to do it, as far as they credit my work if it is based on it.
- To be able to do the study regarding the two previous points, a program (in Mathematica, Matlab, Python or similar) should be done that can perform all these calculations with the 8 dimensions (8 different elements: scalars, 3 vectors, 3 bivectors and 1 trivector). If an analytical calculation is not possible, a program of finite element or solutions with polynomials should be considered for the calculations. But always considering all the 8 elements existing in the Geometric Algebra $Cl_{3,0}$ (scalars, 3 vectors, 3 bivectors and 1 trivector).
- Try to justify that the 3 group of masses of each family of elemental particles is related to the 3 original linear dimensions that expand to $Cl_{3,0}$. This is somehow "commented" in [8] and [9] but never demonstrated or justified.
- Try to justify that that the universe has three linear dimensions, that expand to 8 degrees of freedom in $Cl_{3,0}$, is somehow the only solution (or one of the few possible solutions) that have the necessary symmetries to make the universe, its processes, or contents stable. Justifying that this is the reason that our universe has three linear dimensions.
- In this paper, try to find the equation g(x) for different cases with the goal of obtaining $\zeta(3)$, $\zeta(5)$ or even a $\zeta(k)$ valid for certain k's (odd, even, general etc…).

A7. Annex A7. Logarithm as a limit of an exponential

In this annex we will demonstrate the following equation. This transforms the logarithm, which is normally a function difficult to deal with, in an exponential. The price to pay is that we have to take a limit of a parameter ε to 0:

$$\ln(x) = \lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon} - 1}{\varepsilon}\right)$$

You can find attached the error we make with this equation with different parameters ε chosen.

		$\ln(x) = \lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon} - 1}{\varepsilon}\right)$ Epsilon	
x	**Ln**	**0,00000001**	**Error**
0,25	-1,386294361	-1,3862943	0,000%
0,5	-0,693147181	-0,6931472	0,000%
0,75	-0,287682072	-0,2876821	0,000%
1	0	0	0,000%
1,5	0,405465108	0,40546511	0,000%
2	0,693147181	0,69314718	0,000%
2,718281828	1	0,99999999	0,000%
3	1,098612289	1,09861229	0,000%
5	1,609437912	1,60943792	0,000%
10	2,302585093	2,30258512	0,000%
100	4,605170186	4,60517029	0,000%
1000	6,907755279	6,90775552	0,000%
10000	9,210340372	9,2103408	0,000%
100000	11,51292546	11,5129261	0,000%
1000000	13,81551056	13,8155115	0,000%

		$\ln(x) = \lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon} - 1}{\varepsilon}\right)$ Epsilon	
x	**Ln**	**0,0001**	**Error**
0,25	-1,386294361	-1,3861983	0,007%
0,5	-0,693147181	-0,6931232	0,003%
0,75	-0,287682072	-0,2876779	0,001%
1	0	0	0,000%
1,5	0,405465108	0,40547333	0,002%
2	0,693147181	0,6931712	0,003%
2,718281828	1	1,00005	0,005%
3	1,098612289	1,09867264	0,005%
5	1,609437912	1,60956743	0,008%
10	2,302585093	2,30285021	0,012%
100	4,605170186	4,60623073	0,023%
1000	6,907755279	6,91014168	0,035%
10000	9,210340372	9,21458319	0,046%
100000	11,51292546	11,5195554	0,058%
1000000	13,81551056	13,8250584	0,069%

		$\ln(x)=\lim_{\varepsilon\to 0}\left(\frac{x^{\varepsilon}-1}{\varepsilon}\right)$ Epsilon	
x	**Ln**	**0,01**	**Error**
0,25	-1,3862944	-1,3767296	0,690%
0,5	-0,6931472	-0,6907505	0,346%
0,75	-0,2876821	-0,2872687	0,144%
1	0	0	0,000%
1,5	0,40546511	0,40628823	0,203%
2	0,69314718	0,69555501	0,347%
2,71828183	1	1,00501671	0,502%
3	1,09861229	1,10466919	0,551%
5	1,60943791	1,62245913	0,809%
10	2,30258509	2,32929923	1,160%
100	4,60517019	4,71285481	2,338%
1000	6,90775528	7,15193052	3,535%
10000	9,21034037	9,64781961	4,750%
100000	11,5129255	12,2018454	5,984%
1000000	13,8155106	14,8153621	7,237%

The demonstration is as follows. First, we calculate the integral of 1/x that we know is the logarithm.

$$\int \frac{1}{x}dx = \int x^{-1}dx = \ln(x) + C_1$$

Then, we take the following integral that is very similar to 1/x but with a parameter ε that later we will make it tend to zero.

$$\int \frac{1}{x^{1-\varepsilon}}dx = \int x^{-1+\varepsilon}dx = \frac{x^{-1+\varepsilon+1}}{-1+\varepsilon+1} + C_2 = \frac{x^{\varepsilon}}{\varepsilon} + C_2$$

Now, we can make ε tend to zero and make the result equal between both integrals (as they should be equal).

$$\lim_{\varepsilon\to 0}\int \frac{1}{x^{1-\varepsilon}}dx = \lim_{\varepsilon\to 0}\left(\frac{x^{\varepsilon}}{\varepsilon} + C_2\right) = \ln(x) + C_1$$

$$\lim_{\varepsilon\to 0}\left(\frac{x^{\varepsilon}}{\varepsilon} + C_2\right) - C_1 = \ln(x)$$

As C_1 does not depend on ε we can introduce it in the limit.

$$\lim_{\varepsilon\to 0}\left(\frac{x^{\varepsilon}}{\varepsilon} + C_2 - C_1\right) = \ln(x)$$

We define a new constant C as:

$$C \equiv C_2 - C_1$$

$$\lim_{\varepsilon\to 0}\left(\frac{x^{\varepsilon}}{\varepsilon} + C\right) = \ln(x)$$

To calculate C, we take the specific case where x=1:

$$\lim_{\varepsilon \to 0}\left(\frac{1^{\varepsilon}}{\varepsilon} + C\right) = \ln(1)$$

$$\lim_{\varepsilon \to 0}\left(\frac{1}{\varepsilon} + C\right) = 0$$

For the limit to be zero, C has to be:

$$C = -\frac{1}{\varepsilon}$$

Now, returning to the general equation, we substitute the C with the obtained value:

$$\lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon}}{\varepsilon} + C\right) = \ln(x)$$

$$\lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon}}{\varepsilon} - \frac{1}{\varepsilon}\right) = \ln(x)$$

$$\lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon} - 1}{\varepsilon}\right) = \ln(x)$$

And we obtain the desired equation.

$$\ln(x) = \lim_{\varepsilon \to 0}\left(\frac{x^{\varepsilon} - 1}{\varepsilon}\right)$$

Higher above you can find comparisons of the real value of the logarithm with this equation. You can check that for values of ε=0.00000001 and x<1000000 we are in levels of error of 0.000%.

As homework, you can demonstrate that if in the following expression, you expand the Newton binomial that is to the power of ε, and take the limit when ε tends to zero, you can obtain the Taylor expansion series of the Logarithm of (1+x):

$$\ln(1 + x) = \lim_{\varepsilon \to 0}\left(\frac{(1 + x)^{\varepsilon} - 1}{\varepsilon}\right)$$