ZERNIKE EXPANSION OF CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

RICHARD J. MATHAR

ABSTRACT. The even Chebyshev Polynomials $T_i(x)$ can be expanded into sums of even Zernike Polynomials $R_n^0(x)$, and the odd Chebyshev Polynomials can be expanded into sums of odd Zernike Polynomials $R_n^1(x)$. This manuscripts provides closed forms for the rational expansion coefficients as products of Γ functions of integer and half-integer arguments.

1. Overview

Zernike Polynomials $R_n^m(x)$ are polynomials of degree n in the radial coordinate x, $0 \le x \le 1$ and defined for even nonnegative $n - m = 0, 2, 4, \ldots$.

Definition 1. *(Zernike Radial Polynomials)*

$$R_n^m(x) = (-1)^{(n-m)/2} x^m \sum_{s=0}^{(n-m)/2} \binom{(n+m)/2+s}{(n-m)/2-s} \binom{m+2s}{s} (-x^2)^s. \tag{1}$$

We shall call n the radial quantum number and m the azimuthal quantum number. The inverse relation expands monomials x^j for fixed m into R_n^m by defining coefficients h,

$$x^j = \sum_{n \equiv m \pmod 2}^{j} h_{j,n,m} R_n^m(x), \tag{2}$$

which turn out to be rational numbers [4]

$$h_{j,n,m} = \frac{n+1}{1+\frac{j+n}{2}} \binom{(j-m)/2}{(n-m)/2} \frac{1}{\binom{(j+n)/2}{(n-m)/2}} \tag{3}$$

where $j - m = 0, 2, 4, \ldots$.

Definition 2. *(Chebyshev Polynomials of the First Kind)* [1, 22.3]

$$T_0(x) = 1; \quad T_i = \frac{i}{2} \sum_{s=0}^{\lfloor i/2 \rfloor} (-)^s \frac{(i-s-1)!}{s!(i-2s)!} (2x)^{i-2s}, \quad i = 1, 2, 3, 4 \ldots \tag{4}$$

Date: March 18, 2024.
2020 *Mathematics Subject Classification.* Primary 26C05; Secondary 42C05.

Replacing the monomials in (4) via (2) is one approach of expanding T_i in Zernike Polynomials:

$$(5)\quad T_i(x) = i2^{i-1}\sum_{s=0}^{\lfloor i/2\rfloor}(-\frac{1}{4})^s\frac{(i-s-1)!}{s!(i-2s)!}\sum_{n=m,m+2,\ldots}^{i-2s} h_{i-2s,n,m}R_n^m(x)$$

$$= i2^{i-1}\sum_{s=0}^{\lfloor i/2\rfloor}(-\frac{1}{4})^s\frac{(i-s-1)!}{s!(i-2s)!}\sum_{n=m,m+2,\ldots}^{i-2s}\frac{n+1}{1+\frac{i-2s+n}{2}}\binom{(i-2s-m)/2}{(n-m)/2}\frac{1}{\binom{(i-2s+n)/2}{(n-m)/2}}R_n^m(x).$$

The major constraint of this formula is that the monomials x^{i-2s} can only be fully represented by series of $R_n^m(x)$ if the exponent $i-2s$ is $\geq m$. Since the sum of the Chebyshev Polynomials requires to represent all powers starting at x^0 or x^1 (depending on whether i is even or odd), (5) is correct only for even $i \geq 2$ and $m = 0$ or for odd $i \geq 1$ and $m = 1$.

Swapping the order of the two finite sums yields

$$(6)\quad T_i(x) = i2^{i-1}\sum_{n=m,i-n even}^{i}(n+1)R_n^m(x)$$

$$\times\sum_{s=0}^{\lfloor (i-n)/2\rfloor}(-\frac{1}{4})^s\frac{(i-s-1)!}{s!(i-2s)!}\frac{1}{1+\frac{i-2s+n}{2}}\binom{(i-2s-m)/2}{(n-m)/2}\frac{1}{\binom{(i-2s+n)/2}{(n-m)/2}}$$

$$= i2^{i-1}\sum_{n=m,i-n even}^{i}(n+1)C_{i,n,m}R_n^m(x).$$

Definition 3. *(Coupling coefficients)*
(7)

$$C_{i,n,m} \equiv \sum_{s=0}^{\lfloor (i-n)/2\rfloor}(-\frac{1}{4})^s\frac{(i-s-1)!}{s!(i-2s)!}\frac{1}{1+\frac{i-2s+n}{2}}\binom{(i-2s-m)/2}{(n-m)/2}\frac{1}{\binom{(i-2s+n)/2}{(n-m)/2}},\quad i\geq 1$$

for even $i-n \geq 0$ and even $n-m \geq 0$.

The sum covers nonnegative integer excesses:

Definition 4. *(Parameter Excess)*

$$(8)\qquad \epsilon \equiv (i-n)/2 \geq 0.$$

The manuscript is devoted to reduce the coupling coefficients to simpler terms of the factorial class.

2. Reduction to Generalized Hypergeometric Function

Handling (7) follows the standard recipe [5]: expressing all factors as Γ-functions, conversion to Pochhammer Symbols, duplication formulas for these [6], and compiling a generalized hypergeometric function:

$$C_{i,n,m} = \sum_{s=0}^{\lfloor (i-n)/2\rfloor}(-\frac{1}{4})^s\frac{\Gamma(i-s)\Gamma(1+i/2-s-m/2)\Gamma(1+i/2-s+m/2)}{s!\Gamma(i-2s+1)\Gamma(1+i/2-s-n/2)\Gamma(2+i/2-s+n/2)}$$

Definition 5. *(Pochhammer Symbol)*[1, 6.1.22]

$$(9)\qquad (x)_0 \equiv 1;\quad (x)_n = x(x+1)(x+2)\cdots(x+n-1),\quad n\geq 1$$

$$
\begin{aligned}
(10)\quad C_{i,n,m} &= (-)\sum_{s=0}^{\lfloor (i-n)/2\rfloor}(-\frac{1}{4})^s\frac{1}{s!}\frac{\Gamma(i)\Gamma(1+i/2-m/2)\Gamma(1+i/2+m/2)}{\Gamma(i+1)\Gamma(1+i/2-n/2)\Gamma(1+i/2+n/2)}\\
&\times\frac{(i)_{-s}(1+i/2-m/2)_{-s}(1+i/2+m/2)_{-s}}{(-1-i/2+s-n/2)(i+1)_{-2s}(1+i/2-n/2)_{-s}(1+i/2+n/2)_{-s}}\\
&= (-)\sum_{s=0}^{\lfloor (i-n)/2\rfloor}(-\frac{1}{4})^s\frac{1}{s!}\frac{\Gamma(1+i/2-m/2)\Gamma(1+i/2+m/2)}{i(-1-i/2-n/2)\Gamma(1+i/2-n/2)\Gamma(1+i/2+n/2)}\\
&\times\frac{(i)_{-s}(1+i/2-m/2)_{-s}(1+i/2+m/2)_{-s}(-1-i/2-n/2)_s}{(-i/2-n/2)_s(i+1)_{-2s}(1+i/2-n/2)_{-s}(1+i/2+n/2)_{-s}}\\
&= (-)\sum_{s=0}^{\lfloor (i-n)/2\rfloor}(\frac{1}{4})^s\frac{1}{s!}\frac{\Gamma(1+i/2-m/2)\Gamma(1+i/2+m/2)}{i(-1-i/2-n/2)\Gamma(1+i/2-n/2)\Gamma(1+i/2+n/2)}\\
&\times\frac{(-i)_{2s}(-i/2+n/2)_s(-i/2-n/2)_s(-1-i/2-n/2)_s}{(1-i)_s(-i/2+m/2)_s(-i/2-m/2)_s(-i/2-n/2)_s}\\
&= \sum_{s=0}^{\lfloor (i-n)/2\rfloor}(\frac{1}{4})^s\frac{1}{s!}\frac{\Gamma(1+i/2-m/2)\Gamma(1+i/2+m/2)}{i(1+i/2+n/2)\Gamma(1+i/2-n/2)\Gamma(1+i/2+n/2)}\\
&\times\frac{(-i)_{2s}(-i/2+n/2)_s(-1-i/2-n/2)_s}{(1-i)_s(-i/2+m/2)_s(-i/2-m/2)_s}\\
&= \sum_{s=0}^{\lfloor (i-n)/2\rfloor}(\frac{1}{4})^s\frac{1}{s!}\frac{\Gamma(1+i/2-m/2)\Gamma(1+i/2+m/2)}{i\Gamma(1+i/2-n/2)\Gamma(2+i/2+n/2)}\\
&\times\frac{(-i/2)_s(-i/2+1/2)_s 2^{2s}(-i/2+n/2)_s(-1-i/2-n/2)_s}{(1-i)_s(-i/2+m/2)_s(-i/2-m/2)_s}\\
&= \frac{\Gamma(i/2-m/2+1)\Gamma(i/2+m/2+1)}{i\Gamma(i/2-n/2+1)\Gamma(2+i/2+n/2)}\\
&\times {}_4F_3\left(\begin{matrix}-i/2,-i/2+n/2,-i/2+1/2,-1-i/2-n/2\\ -i+1,-i/2-m/2,-i/2+m/2\end{matrix}\;\middle|\;1\right).
\end{aligned}
$$

Remark 1. *It is a standard in some computer algebra systems to cancel common constants in the list of the upper and lower coefficients of the ${}_4F_3$, in particular to cancel common negative integers, which may lead to unexpected side effects. The correct interpretation is that ${}_4F_3$ is a* terminating *generalized hypergeometric series with the number of terms implied by the (arithmetically) largest negative integer in the upper coefficients, here $-i/2-n/2$.*

3. Azimuthal $m = 0$

As argued above, the only interesting cases for the Chebyshev expansion are $m = 0$ for even i and $m = 1$ for odd i. For $m = 0$ and even i, (10) is

(11)

$$
C_{i,n,0} = \frac{[(i/2)!]^2}{i(\frac{i-n}{2})!\Gamma(2+i/2+n/2)}\,{}_3F_2\left(\begin{matrix}1/2-i/2,-1-i/2-n/2,-i/2+n/2\\ 1-i,-i/2\end{matrix}\;\middle|\;1\right).
$$

If in addition $i-n\equiv 2 \pmod 4$, which means if $i-n$ is not a multiple of 4, then $i=n+4k+2$ for some $k\ge 0$ and by definition

$$C_{i,n,0}=\sum_{s=0}^{2k+1}(-\frac{1}{4})^s\frac{\Gamma(n+4k+2-s)\Gamma^2(2+n/2+2k-s)}{s!\Gamma(n+4k+3-2s)\Gamma(2+2k-s)\Gamma(3+n+2k-s)} \tag{12}$$

$$=-\frac{\pi(n+2k+1)}{2k(n+2+4k)}\frac{\Gamma(n+4k+2)\Gamma^2(2+n/2+2k)\Gamma^2(-2k-n/2)}{\Gamma(n+3+4k)\Gamma(2k+2)\Gamma(3+n+2k)k\Gamma^2(1/2-n/2-k)\Gamma^2(-k)}.$$

Considering the set of poles of the Γ-functions in numerator and denominator the result for odd excess ϵ is

$$C_{i,n,0}=0,\quad i-n\equiv 2 \pmod 4. \tag{13}$$

For the other cases, $i-n\equiv 0 \pmod 4$, apply Lavoie's equation [3, p. 270] to (11),

$$C_{i,n,0}=\frac{[(i/2)!]^2}{i(\frac{i-n}{2})!\Gamma(2+i/2+n/2)} \tag{14}$$
$$\times\frac{\Gamma(1/2)\Gamma(-i/2)\Gamma(1-i/2)\Gamma(3/2)}{\Gamma(-i/4-n/4)\Gamma(-i/4+n/4+1/2)\Gamma(3/2-i/4+n/4)\Gamma(1-i/4-n/4)}$$

Repeated use of the reflection and duplication formula of the Γ-function reduces this to the more convenient arguments which are positive and integer or half-integer:

$$C_{i,n,0}=-\frac{1}{2^{3+i}}\frac{\Gamma(-\frac{1}{2}+\frac{i-n}{4})\Gamma(\frac{i+n}{4})}{\Gamma(1+\frac{i-n}{4})\Gamma(\frac{3}{2}+\frac{i+n}{4})},\quad i-n\equiv 0 \pmod 4, i\ge n, i>1. \tag{15}$$

Due to the fundamental equation of the Γ-function, the non-vanishing coefficients can be evaluated recursively via

$$C_{i,i,0}=\frac{[(i/2)!]^2}{i(\frac{i-n}{2})!(1+i/2+n/2)!};\quad C_{i,n-4,0}=\frac{(2+i+n)(-2+i-n)}{(i+n-4)(4+i-n)}C_{i,n,0}. \tag{16}$$

4. Azimuthal $m=1$

For odd i and $m=1$, Eq. (10) is

$$C_{i,n,1}=\frac{\Gamma(i/2+1/2)\Gamma(i/2+3/2)}{i\Gamma(i/2-n/2+1)\Gamma(2+i/2+n/2)} \tag{17}$$
$$\times\,{}_3F_2\left(\begin{matrix}-i/2,-i/2+n/2,-1-i/2-n/2\\ -i+1,-i/2-1/2\end{matrix}\ \middle|\ 1\right).$$

For odd $i=n$ the ${}_3F_2$ series consists only of the first term,

$$C_{i,i,1}=\frac{\Gamma^2(\frac{i+1}{2})}{2ii!},\quad i=1,3,5,7\ldots \tag{18}$$

where $\Gamma(i/2+1/2)=(\frac{i-1}{2})!$.

Definition 6. *(Double Factorial)*

$$n!!\equiv n(n-2)(n-4)\cdots=\prod_{i=0}^{\lfloor(n-1)/2\rfloor}(n-2i),\quad n\ge 1 \tag{19}$$

and equal to 1 if $n<0$.

Remark 2. *For odd n,*

$$n!! = n(n-2)(n-4)\cdots 1 = 2^{(n+1)/2}\frac{n}{2}(\frac{n}{2}-1)\cdots\frac{1}{2} = 2^{(n+1)/2}(1/2)_{n/2+1/2} = 2^{(n+1)/2}\frac{\Gamma(1+n/2)}{\Gamma(1/2)}. \tag{20}$$

For even n,

$$n!! = n(n-2)(n-4)\cdots 2 = 2^{n/2}\frac{n}{2}(\frac{n}{2}-1)\cdots 1 = 2^{n/2}(1)_{n/2} = 2^{n/2}\Gamma(1+n/2). \tag{21}$$

The contiguous relations [2] of the hypergeometric series yield recurrences for even and odd ϵ:

$$C_{i,i-2\epsilon,1} = \begin{cases} \frac{i-\epsilon+2}{\epsilon}C_{i,i-2\epsilon+2,1}, & \epsilon \equiv 0 \pmod 2; \\ \frac{\epsilon-2}{i-\epsilon}C_{i,i-2\epsilon+2,1}, & \epsilon \equiv 1 \pmod 2. \end{cases} \tag{22}$$

Stacking all these factors yields for even ϵ

$$\begin{aligned} C_{i,i-2\epsilon,1} &= \frac{i(i-2)\cdots(i-\epsilon+2)(-1)1\cdot 3\ldots\cdot(\epsilon-3)}{\epsilon!!(i-1)(i-3)\cdots(i-\epsilon+1)}C_{i,i,1} \\ &= -\frac{(\epsilon-3)!!(i-\epsilon-1)!!}{\epsilon!![(i-1)!!]^2(i-\epsilon)!!}\frac{\Gamma^2(\frac{i+1}{2})}{2i} \\ &= -\frac{\Gamma(1+i/2)\Gamma(1+\frac{i-n}{2})\Gamma(\frac{1}{2}+\frac{i+n}{4})\Gamma(\frac{i+1}{2})}{2^{(i-n)/2}ii![i-n-2]\Gamma^2(1+\frac{i-n}{4})\Gamma(1+\frac{i+n}{4})} \end{aligned} \tag{23}$$

and for odd ϵ

$$\begin{aligned} C_{i,i-2\epsilon,1} &= \frac{i(i-2)\cdots(i-\epsilon+3)(-1)1\cdot 3\ldots\cdot(\epsilon-2)}{(\epsilon-1)!!(i-1)(i-3)\cdots(i-\epsilon)}C_{i,i,1} \\ &= -\frac{(\epsilon-2)!!(i-\epsilon-2)!!}{(\epsilon-1)!!(i-\epsilon+1)!![(i-1)!!]^2}\frac{\Gamma^2(\frac{i+1}{2})}{2i} \\ &= -\frac{\Gamma(\frac{i-n}{4})\Gamma(\frac{i+n}{4})}{i\Gamma(\frac{i-n}{4}+\frac{1}{2})\Gamma(\frac{i+n}{4}+\frac{3}{2})2^{i+2}}. \end{aligned} \tag{24}$$

Replacing ϵ, this gives for $i-n$ multiples or non-multiples of 4

$$C_{i,n,1} = -\frac{\Gamma^2(\frac{i+1}{2})}{2i[(i-1)!!]^2} \times \begin{cases} \frac{(\frac{i-n}{2}-3)!!(\frac{i+n}{2}-1)!!}{\frac{i-n}{2}!!(\frac{i+n}{2})!!}, & i-n \equiv 0 \pmod 4; \\ \frac{(\frac{i-n}{2}-2)!!(\frac{i+n}{2}-2)!!}{(\frac{i-n}{2}-1)!!(1+\frac{i+n}{2})!!}, & i-n \equiv 2 \pmod 4). \end{cases} \tag{25}$$

In numerical practice in conjunction with the T_i-expansion this formula will not be used, because the evaluation is wanted for all odd n, and tabulation starting at (18) for a ladder of decreasing n (increasing ϵ) via the recurrences (22) is faster.

5. Summary

The expansion coefficients $C_{i,n,0}$ for even $i \geq 2$ and even $i-n$ are given by (13) if $i-n$ is not a multiple of 4 and by (15) if $i-n$ is a multiple of 4. The values of $C_{i,n,1}$ for odd $i \geq 1$ and even $i-n$ are given by (25).

i	n	m	$C_{i,n,m}$	$i2^{i-1}(n+1)C_{i,n,m}$
1	1	1	1/2	1
2	0	0	0	0
2	2	0	1/12	1
2	2	2	1/6	2
3	1	1	-1/72	-1/3
3	3	1	1/36	4/3
3	3	3	1/12	4
4	0	0	-1/96	-1/3
4	2	0	0	0
4	2	2	-1/48	-2
4	4	0	1/120	4/3
4	4	2	1/80	2
4	4	4	1/20	8
5	1	1	-1/480	-1/3
5	3	1	-1/1200	-4/15
5	3	3	-9/400	-36/5
5	5	1	1/300	8/5
5	5	3	1/150	16/5
5	5	5	1/30	16
6	0	0	0	0
6	2	0	-1/960	-3/5
6	2	2	1/480	6/5
6	4	0	0	0
6	4	2	-1/720	-4/3
6	4	4	-1/45	-64/3
6	6	0	1/840	8/5
6	6	2	1/630	32/15
6	6	4	1/252	16/3
6	6	6	1/42	32
7	1	1	-1/13440	-1/15
7	3	1	-1/3360	-8/15
7	3	3	17/3360	136/15
7	5	1	-1/11760	-8/35
7	5	3	-3/1960	-144/35
7	5	5	-25/1176	-400/7
7	7	1	1/1960	64/35
7	7	3	1/1176	64/21
7	7	5	1/392	64/7
7	7	7	1/56	64
8	0	0	-1/15360	-1/15
8	2	0	0	0
8	2	2	-1/1920	-8/5
8	4	0	-1/6720	-16/21
8	4	2	1/13440	8/21
8	4	4	5/672	800/21
8	6	0	0	0
8	6	2	-1/6720	-16/15
8	6	4	-1/672	-32/3
8	6	6	-9/448	-144
8	8	0	1/5040	64/35
8	8	2	1/4032	16/7
8	8	4	1/2016	32/7
8	8	6	1/576	16
8	8	8	1/72	128

i	n	m	$C_{i,n,m}$	$i2^{i-1}(n+1)C_{i,n,m}$
9	1	1	-1/69120	-1/15
9	3	1	-1/120960	-8/105
9	3	3	-53/40320	-424/35
9	5	1	-1/20160	-24/35
9	5	3	1/3360	144/35
9	5	5	19/2016	912/7
9	7	1	-1/90720	-64/315
9	7	3	-1/6048	-64/21
9	7	5	-25/18144	-1600/63
9	7	7	-49/2592	-3136/9
9	9	1	1/11340	128/63
9	9	3	1/7560	64/21
9	9	5	1/3240	64/9
9	9	7	1/810	256/9
9	9	9	1/90	256
10	0	0	0	0
10	2	0	-1/107520	-1/7
10	2	2	1/10752	10/7
10	4	0	0	0
10	4	2	-1/26880	-20/21
10	4	4	-1/420	-1280/21
10	6	0	-1/40320	-8/9
10	6	2	0	0
10	6	4	1/2240	16
10	6	6	1/90	3584/9
10	8	0	0	0
10	8	2	-1/50400	-32/35
10	8	4	-1/6300	-256/35
10	8	6	-1/800	-288/5
10	8	8	-4/225	-4096/5
10	10	0	1/27720	128/63
10	10	2	1/23100	256/105
10	10	4	1/13200	64/15
10	10	6	1/4950	512/45
10	10	8	1/1100	256/5
10	10	10	1/110	512

References

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642
2. W. N. Bailey, *Contiguous hypergeometric functions of the type* ${}_3f_2$, Proc. Glasg. Math. Ass. **2** (1954), no. 2, 62–65. MR 0064918
3. J. L. Lavoie, *Some summation formulas for the series* ${}_3f_2(1)$, Math. Comput. **49** (1987), no. 179, 269–274. MR 0890268
4. Richard J. Mathar, *Zernike basis to cartesian transformations*, Serb. Astr. J. **179** (2009), 107–120.
5. Ranjan Roy, *Binomial identities and hypergeometric series*, Amer. Math. Monthly **94** (1987), no. 1, 36–46. MR 0873603

6. Lucy Joan Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR 0201688

URL: https://www.mpia-hd.mpg.de/homes/mathar
Email address: mathar@mpia.de

Max-Planck Institute for Astronomy, Königstuhl 17, 69117 Heidelberg, Germany