Division by Zero is Incoherent and Contradictory

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Abstract

A number of authors have claimed that Division by Zero and in particular the Division of Zero by Zero (0/0) can be computed and has a definite value (Mwang 2018, Saitoh & Saitoh 2024). I refute these claims. This is trivial, but despite its elementary standing, some peripheral or recreational mathematicians make claims about 0/0 or k/0 having some value, or in some cases, several values in different contexts, according to the author's whim. Division by zero is undefined and attempts to define it lead to contradiction.

A number of authors have claimed that Division by Zero and in particular the Division of Zero by Zero (0/0) can be computed and has a definite value (Mwang 2018, Saitoh & Saitoh 2024).

I want to refute these claims, for the following reasons. (Those versed in elementary mathematical structures should skip this exposition and go direct to the next section).

Division is defined partially in the number systems N and Z (partially because they contain only whole numbers), and fully in Q, R and C (excluding division by 0).

Consider the number system Q. It is a triple <Q, +, *> that comprises the rationals, colloquially called fractions.

Addition and multiplication are defined by induction for whole numbers as follows

$n+0 =_{def} n$, $n+1 =_{def} Sn$, and $n+(Sk) =_{def} S(n+k)$, where S is the successor function

$n*0 =_{def} 0$, $n*1 =_{def} n$, and $n*(Sk) =_{def} n*k + n$.

For rationals of the form p/q we extend addition

The addition operation in the domain of rational numbers Q is written +.

Let: a=p/q, b=r/s

where: p,q∈Z, r,s ≠0 ∈Z

Then a+b is defined as: (ps+rq)/qs

This definition follows from the definition of and proof of existence of the field of quotients of any integral domain, of which the set of integers is one example.

If the multiplication operation in the domain of rational numbers Q is written as *.

And a and b are as before, then a*b is defined as:

$a*b = (p/q)*(r/s) =_{def} (p*r)/(q*s)$

Notice that 0 is definitionally excluded as denominator in Z.

Subtraction and division are defined in terms of inverses

Additive inverse of a is that number b such that a+b = 0.
It (b) is designated ^{-}a so $a + {}^{-}a = 0$

Multiplicative inverse of a is that number b such that a*b = 1.
It (b) is designated a^{-1} so that $a*a^{-1} = 1$. By definition 0 is excluded.

Additive and multiplicative inverses are demonstrably unique. So, if a-b = a-c then b=c. Likewise for division.

The claim: 'Division by Zero is possible'

If we put the exclusion of zero from division operations to one side, let us see what happens with the assumption 'Division by Zero is possible'.

The claim that 0 could serve as its own inverse (it is a self-inverse) would mean that $0*0^{-1} = 1$, that is $0=0^{-1}$, and thus 0/0 = 1.

But this leads to a contradiction. For assuming that the normal rules of arithmetic apply (including 0+0=0) then

0/0 = (0+0)/0 = 0/0 + 0/0 = 1+1 = 2, but since 0/0 = 1, 2=1. A basic and disallowable contradiction

If, however, one rejects 0+0=0, then there is some value k such that 0+0 = k, so 0/0 = 0+0/0 = k/0 = k. But since 0/0 = 1, this implies that 1 = k. Another contradiction. QED.

The inverse of 0 (0^{-1}) is not defined because it does not have a unique value, and hence definitionally **cannot be** a function or operator.

All of this is trivial, but despite its elementary standing, some peripheral or recreational mathematicians make claims about 0/0 or k/0 having some value, or in some cases, several values in different contexts, according to the author's whim.

Thus, searching the web for 'Division by Zero' one comes up with many references, especially to S. Saitoh. References to S. Saitoh's publications on Division by Zero are widespread across the web, but as far as I have ascertained are not in respectable peer refereed publications. Because of the many, varied, incoherent and contradiction producing claims about 'division by zero' found on the web these may confuse the unschooled and unwary.

This note of correction is not intended to be unkind. It is merely to point out the most elementary of errors. Because it is such an elementary error and misconception the web literature seems to be devoid of corrections. I feel obliged to publish this little piece.

See my fuller discussion of infinity and the associated controversies in Ernest (2023).

References

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Note: the author is unwilling to enter into discussion about the validity of the main claims of this short paper.