The Collatz Conjecture, Pythagorean Triples, and the Riemann Hypothesis: Unveiling a Novel Connection Through Dropping Times

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Abstract

In the landscape of mathematical inquiry, where the ancient and the modern intertwine, few problems captivate the imagination as profoundly as the Collatz conjecture and the quest for Pythagorean triples. The former, a puzzle that has defied solution since its inception in the 1930s by Lothar Collatz, asks us to consider a simple iterative process: for any positive integer, if it is even, divide it by two; if it is odd, triple it and add one. Despite its apparent simplicity, the conjecture leads us into a labyrinth of diverse complexity, where patterns emerge and dissolve in an unpredictable dance. On the other hand, Pythagorean triples, sets of three integers that satisfy the ancient Pythagorean theorem, have been a cornerstone of geometry since the time of the ancient Greeks, embodying the harmony of numbers and the elegance of spatial relationships.

This exploratory paper embarks on an unprecedented journey to bridge these seemingly disparate domains of mathematics. At the heart of this exploration is the discovery of a novel connection between Collatz dropping times and Pythagorean triples. I will demonstrate how the dropping time of each odd number can be uniquely associated with a Pythagorean triple. As you will see, the triples seem to be encoding spatial information about Collatz trajectories. As we begin to work with triples, we'll be motivated to move from the number line to the complex plane where we find structure and behavior resembling that of the Riemann Zeta function and it's zeros.

1 Introduction

The Collatz Conjecture, often dubbed the “3n + 1 conjecture”, stands as one of the most notorious unsolved problems in the realm of mathematics. Originating from the musings of Lothar Collatz in 1937, this seemingly simple problem has defied solutions and resisted all attempts at a rigorous proof, all the while captivating the imaginations of amateur and professional mathematicians alike. The piece wise function is described as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases} \tag{1}$$

At first glance, there appears to be no reason to believe this function has any kind of connection to Pythagorean triples. A Pythagorean triple is a set of integers a, b, c such that:

$$a^2 + b^2 = c^2 \tag{2}$$

What I will attempt to demonstrate in this paper is that each integer n can be mapped to a unique Pythagorean triple via properties derived from iterating the Collatz function on n. Furthermore, I will demonstrate there is mathematical universe full of rich structure and hierarchy that only reveals itself when these two seemingly disparate ideas are bound together. Before jumping into the details, I lay out a few definitions the reader should be familiar with.

2 Definitions

If you are familiar with the Collatz Conjecture, you might already know these terms. Even so, revisiting them for a refresher might be beneficial.

Definition 1 (**Orbit**$_n$)**.** The sequence of numbers you get when you follow the Collatz rules from a starting number n until you reach the number 1.

Definition 2 (**Total Stopping Time**$_n$)**.** The number of steps needed to get to the number 1 when following the Collatz rules from a starting number.

Definition 3 (**Dropping Time**$_n$)**.** The minimum number of iterations required such that, after these iterations, the subsequent iteration results in a number less than the initial starting number n. The final Collatz step will always be division by 2. This measures the iteration count needed to effectively reduce the value below the starting point under a given set of rules or transformations.

Definition 4 (**Dropping Destination**$_n$)**.** The first number reached that via iteration of the Collatz function that is less than n.

Definition 5 (**Dropping Orbit**$_n$)**.** The sequence of numbers you get when you follow the Collatz rules from a starting number n until you reach a $2\times$ **Dropping Destination**$_n$.

3 Exploring Ordered Dropping Sets

In this section I will introduce the ideas of a **Dropping Set**. Let's begin by considering the implications of the Collatz Conjecture being proven to be true. This would mean that every number $n > 1$ eventually reaches a number lower than itself in some finite number of steps k. For the rest of this paper, I will refer to the ordered set of integers requiring k iterations before reaching a number lower than itself as belonging to **Dropping Set**$_k$.

Definition 6 (**Dropping Set**$_k$)**.** An ordered set of integers $\{a_1, a_2, \ldots, a_n\}$, where each element a_i in the set requires exactly k iterations of a specified function f such that $f^k(a_i) < a_i$, where f^k denotes the k-th iteration of function f, and $a_i \in \mathbb{Z}$. All possible values of k can be found in this sequence: A122437.

Below are a few tables to demonstrate concrete examples of **Dropping Sets**

n	**Dropping Orbit**$_n$	**Dropping Set**$_n$
5	[5, 16, 8]	3
9	[9, 28, 14]	3
13	[13, 40, 20]	3
17	[17, 52, 26]	3
25	[21, 64, 32]	3

Table 1: Example dropping orbits for dropping set k=3

n	**Dropping Orbit**$_n$	**Dropping Set**$_n$
3	[3, 10, 5, 16, 8, 4]	6
19	[19, 58, 29, 88, 44, 22]	6
35	[35, 106, 53, 160, 80, 40]	6
51	[51, 154, 77, 232, 116, 58]	6
61	[67, 202, 101, 304, 152, 76]	6

Table 2: Example dropping orbits for dropping set k=6

n	**Dropping Orbit**$_n$	**Dropping Set**$_n$
11	[11, 34, 17, 52, 26, 13, 40, 20]	8
23	[23, 70, 35, 106, 53, 160, 80, 40]	8
43	[43, 130, 65, 196, 98, 49, 148, 74]	8
55	[55, 166, 83, 250, 125, 376, 188, 94]	8
75	[75, 226, 113, 340, 170, 85, 256, 128]	8

Table 3: Example dropping orbits for dropping set k=8

Though identified by a single parameter k, **Dropping Sets** exhibit intricate internal and mutual structures. Empirical observations suggest that each **Dropping Set**$_k$ is an ordered set with infinite elements. For instance, **Dropping Set**$_1$ encompasses all even numbers, illustrating a clear and simple pattern. Similarly, **Dropping Set**$_3$ is comprised of numbers n for which $n \mod 4 = 1$, pointing to a more specific but still discernible structure.

As the parameter k increases, the underlying patterns within **Dropping Set**$_k$ become less intuitive and harder to predict. Despite this complexity, a consistent theme emerges: every **Dropping Orbit** contained within a **Dropping Set**$_k$ features precisely k odd numbers. This observation hints at a deeper, underlying order, even amidst the increasing unpredictability of the sets' compositions as k escalates. To formalize this observation, I introduce the concept of **Orbital Oddity**, defined as the count of odd integers within a **Dropping Orbit**. This refinement enables us to quantify the "distance" between integers based on their Collatz behavior, encapsulated by their **Orbital Oddity**.

Definition 7 (**Orbital Oddity**)**.** The **Orbital Oddity** of a **Dropping Orbit** is defined to be the count of odd integers in the orbit.

These results form the basis of my first conjecture:

Conjecture 1. *All* ***Dropping Orbits*** *existing in* ***Dropping Set***$_k$ *share the same* ***Orbital Oddity****.*

This conjecture suggests a fundamental characteristic of **Dropping Sets**, proposing a specific structural consistency that holds regardless of the complexity or unpredictability introduced by higher values of k. As k increases past $k \geq 8$, more intricate inner structures begin to emerge within the elements of **Dropping Sets**. Interestingly, the elements of **Dropping Set**$_1$, **Dropping Set**$_3$, and **Dropping Set**$_6$ exhibit linearity, meaning each member of these sets can be generated through a linear formula, highlighting a surprising simplicity amidst the broader complexity.

- The elements of **Dropping Set**$_1$ can be generated with the formula $a_n = 2n$, representing all even numbers.
- The elements of **Dropping Set**$_3$ can be generated with the formula $a_n = 4n + 1$, indicating a subset of numbers congruent to 1 modulo 4 (excluding 1).
- The elements of **Dropping Set**$_6$ can be generated with the formula $a_n = 16n - 3$, revealing a more complex pattern within this particular set.

However, for **Dropping Sets**$_k$ where $k >= 8$, there is not a single simple linear pattern. Each **Dropping Set** seems to have an overlap of multiple linear patterns. For example, let's take a look at **Dropping Set**$_8$ again, but this time I will add another column representing the **Dropping Modulus**. A **Dropping Modulus** acts to classify which inner set within **Dropping Set**$_k$ a particular integer n resides. **Dropping Set**$_8$ has two such inner sets.

Definition 8 (**Dropping Modulus**)**.** A postive integer used to classify integers n into distinct inner subsets within a **Dropping Set**$_k$. Each **Dropping Set**$_k$ may comprise one or more such inner subsets, determined by specific modular relationships or other criteria. For instance, **Dropping Set**$_8$ is known to contain two distinct inner sets. The **Dropping Modulus** essentially acts as a key to identifying the particular inner subset to which an integer n belongs within the broader structure of **Dropping Set**$_k$. Every **Dropping Set**$_k$ has a total number of distinct **Dropping Moduluses** which can be found in this sequence: A100982.

Below is a table demonstrating an ordered n column with staggered **Dropping Modulus**.

n	**Dropping Orbit**$_n$	**Dropping Set**$_n$	**Dropping Modulus**$_n$
11	[11, 34, 17, 52, 26, 13, 40, 20]	8	0
23	[23, 70, 35, 106, 53, 160, 80, 40]	8	1
43	[43, 130, 65, 196, 98, 49, 148, 74]	8	0
55	[55, 166, 83, 250, 125, 376, 188, 94]	8	1
75	[75, 226, 113, 340, 170, 85, 256, 128]	8	0
87	[87, 262, 131, 394, 197, 592, 296, 148]	8	1

Table 4: Example dropping orbits and dropping moduli for dropping set k=8

Within **Dropping Set**$_8$, the classification of elements according to their **Dropping Modulus** can be explicitly described by two linear formulas, each corresponding to a distinct **Dropping Modulus** value:

- For elements having a **Dropping Modulus** of 0, the generating formula is $32x - 21$. This formula delineates the subset of elements within **Dropping Set**$_8$ that align with this specific classification criterion, indicating a particular inner structural pattern governed by the modulus value of 0.
- Conversely, elements with a **Dropping Modulus** of 1 are generated by the formula $32x - 9$. This differentiates another distinct subset within **Dropping Set**$_8$, highlighting the nuanced internal organization and the role of the **Dropping Modulus** in segregating the set's elements based on their modular properties.

These formulations not only categorize elements within **Dropping Set**$_8$ efficiently but also emphasize the critical function of the **Dropping Modulus** in delineating the set's detailed internal configurations. It is observed that all **Dropping Orbits** associated with the same **Dropping Set** and **Dropping Modulus** exhibit a consistent structural pattern, especially regarding the positioning of odd numbers within these orbits.

To further elucidate the structure of Collatz orbits and enhance my analysis, one additional concept is introduce: the **Dropping Index**.

3.1 Dropping Index

As we've seen above, each integer n can be mapped to a unique **Dropping Set**$_k$ and **Dropping Modulus**$_m$. The next logical classification would be to index each integer by ordering all integers within **Dropping Set**$_k$ and **Dropping Modulus**$_m$ by magnitude. This allows us to speak of the **Dropping Index** of integer n.

Definition 9 (**Dropping Index**)**.** Denoted as i in the array notation $a[i]$, it specifies the position of an integer within a **Dropping Set**$_k$ and a particular **Dropping Modulus**$_m$, when all members of the set are ordered by the magnitude of n. In this context, $a[i]$ represents the i-th integer in the ordered sequence within the set and modulus. This indexing facilitates a structured examination of their distribution and relationships within the set and across different moduli.

With this final definition under our belt, we can uniquely classify each integer's dropping time behavior among all other integers.

3.2 Dropping Genus

Motivated to compact these classifications into a single "object" for meaningful analysis of Collatz dropping times, we can assign each number a **Dropping Genus**.

Definition 10 (**Dropping Genus**)**.** The **Dropping Genus** represents a unique combination of three critical parameters: the **Dropping Set**, **Dropping Modulus**, and **Dropping Index**. This triadic classification system enables a refined and comprehensive understanding of individual elements' behaviors and patterns within the broader framework of Collatz orbits.

For example, consider the integer 3 with the following properties:

- Its dropping time is 6.
- Its dropping modulus is 0.
- Its dropping index is 0.

Therefore, it has a **Dropping Genus** $= (6, 0, 0)$, uniquely identifying its place and behavior within the study of Collatz Dropping times. To gain a fuller understanding of what a **Dropping Genus** represents, here are a few more examples:

- The integer 15 has a **Dropping Genus** $= (11, 1, 0)$.
- The integer 635 has a **Dropping Genus** $= (13, 3, 2)$.
- The integer 5183 has a **Dropping Genus** $= (19, 7, 1)$.

These examples showcase the nuanced and unique classifications of integers within the context of Collatz Dropping times, demonstrating the diversity and specificity of the **Dropping Genus**. While the **Dropping Set** has been chosen as the first parameter in the **Dropping Genus** representation, it might be advantageous to replace this parameter with one that transitions more smoothly between distinct genuses. If **Conjecture 1** proves true, the **Dropping Set** parameter in the **Dropping Genus** could be substituted with a value representing the number of odd integers found within any member's **Dropping Orbit** of **Dropping Set**$_k$. Empirical evidence suggests that **Dropping Sets** can be orderly classified using this heuristic. The following table illustrates this concept:

Dropping Set$_k$	**Count of Odd Integers in Dropping Orbits**
1	0
3	1
6	2
8	3
11	4
13	5
16	6

Table 5: Dropping Sets can be smoothly ordered by the number of odd integers in their Dropping Orbits.

If we consider replacing the **Dropping Set** parameter in the **Dropping Genus** with the count of odd integers of the orbit (it's **Orbital Oddity**), this approach can be thought of as defining a novel metric. Specifically, this metric quantifies the "distance" between orbits based on their behavior under the Collatz function. Formally, for any two integers x and y, this metric $d(x, y)$ could be conceptualized as reflecting the difference in complexity or structure of their respective paths in the Collatz sequence. Such a metric would enable us to map the integers to unique positions within $\mathbb{R}^3$, based on their **Orbital Oddity**, **Dropping Modulus**, and **Dropping Index**. This innovative approach to categorizing Collatz behaviors not only aligns with the principles of metric spaces but also opens new pathways for analyzing the intricate dynamics of Collatz orbits.

All this foundational work is laid out to motivate the reader that there is significant structure to be discovered by investigating Collatz Dropping times. In the subsequent section, we will scrutinize **Dropping Orbits** and **Dropping Destinations** for specific integers in greater detail. This examination will serve to connect the relationship between odd integers with Pythagorean Triples, bridging a fascinating link between these mathematical concepts.

4 On Pythagorean Triples

A Pythagorean triple consists of three positive integers a, b, and c, satisfying the equation $a^2 + b^2 = c^2$. This condition is fundamental in the study of right triangles, where a and b represent the lengths of the legs and c the length of the hypotenuse.

One of the most well-known methods for generating Pythagorean triples is through Euclid's Formula, which provides a systematic way to find an infinite number of triples based on two parameters. Given any pair of positive integers m and n with $m > n > 0$, Euclid's Formula generates a Pythagorean triple (a, b, c) as follows:

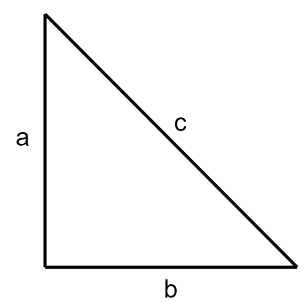

Figure 1: Classic Pythagorean Triangle

- $a = m^2 - n^2$,
- $b = 2mn$,
- $c = m^2 + n^2$.

These equations arise from the geometric properties of right triangles and offer a powerful tool for exploring the relationship between the sides of these triangles. By selecting different values of m and n, one can generate a diverse set of Pythagorean triples, showcasing the richness of this area of number theory. To relate **Dropping Orbits** to Pythagorean triples, we can choose m and n such that they have properties related to **Dropping Orbits**, I demonstrate in the following section.

4.1 Orbital Triple Mapping: From Dropping Orbits to Pythagorean Triples

Here I will introduce a novel mapping, termed **Orbital Triple Mapping**, which establishes a direct correlation between **Dropping Orbits** and Pythagorean triples. This mapping leverages the structural insights gained from analyzing **Dropping Orbits** to identify corresponding Pythagorean triples, revealing an underlying geometric representation.

To map each integer j to a Pythagorean triple T, let d denote the **Dropping Destination** of j, which is the first integer less than j reached by iterating the Collatz function. We can find a unique Pythagorean Triple for j using the following formulas.

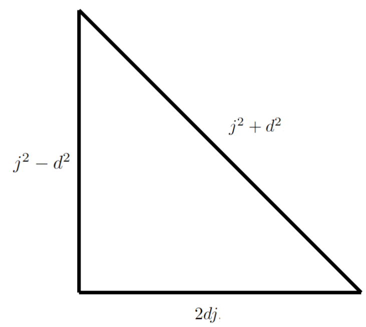

Figure 2: Geometrization of Dropping Behavior

- $a = j^2 - d^2$,
- $b = 2dj$,
- $c = j^2 + d^2$.

It should be noted that whenever j is an even number, we are left with a degenerate triangle. That is, it does not satisfy the triangle inequality, which states that for any triangle with sides of lengths a, b, and c,

$$a + b > c, \quad a + c > b, \quad \text{and} \quad b + c > a.$$

For odd numbers, however, empirical evidence seems to indicate that there are two more properties that the **Orbital Triple Mapping** preserve that relate the starting integer n to T. I have tested all of these constraints up to integers less than or equal to 10,000,000 and have verified that they hold. I will name and investigate each constraint in the following sections.

4.2 The Collatz-Pythagorean Convergence

Directly following the principles of the **Orbital Triple Mapping**, we encounter a compelling and consistent property that manifests across all mapped Pythagorean triples. This property, encapsulated by the equation

$$b + c = (2dj) + (j^2 + d^2) = n^2$$

where n is the starting integer in the Collatz sequence and (a, b, c) represents the Pythagorean triple derived via this mapping, has been termed the **Collatz-Pythagorean Convergence**.

The **Collatz-Pythagorean Convergence** not only underscores a tangible link between the Collatz sequences and Pythagorean triples but also reflects a profound structural resonance between these mathematical realms. The relationship $b+c = n^2$ signifies a hitherto unseen bridge between the iterative processes of the Collatz Conjecture and the geometric rigor of Pythagorean theorem, suggesting an underlying order and connectivity within mathematical phenomena.

The implications of this convergence are multifaceted, offering a fresh perspective on the behavior of numbers within the Collatz sequence and illuminating their geometric counterparts in the world of Pythagorean triples. It suggests that the path a number takes through the Collatz sequence, traditionally viewed through the lens of arithmetic operations, can also be interpreted geometrically in terms of right triangles.

4.3 The Collatz Circle Proportionality

Within the framework of my exploration, a fascinating geometric relationship has emerged, linking the dynamics of Dropping Orbits with the geometry of the triangles formed by Pythagorean triples through **Orbital Triple Mapping**. Specifically, I have identified a novel proportionality involving the centers of the in-circle and circumcircle of these triangles. This relationship, which I have termed **Collatz Circle Proportionality**, is expressed as follows:

Given **in_circle** as the center of the in-circle and **out_circle** as the center of the circumcircle for the triangle corresponding to a Pythagorean triple T, n is the starting integer, and d is the **Dropping Location** of n, then the following equivalence holds

$$\frac{\text{in\_circle}.x}{(\text{out\_circle}.y \cdot 2/n)} = n - d,$$

This proportionality not only underscores a direct linkage between the arithmetic progression of numbers through the Collatz sequence and the geometric properties of their corresponding triangles but also hints at a deeper, underlying mathematical structure that governs these relationships. Below is a diagram for clarification of what I mean by in-circle and out-circle.

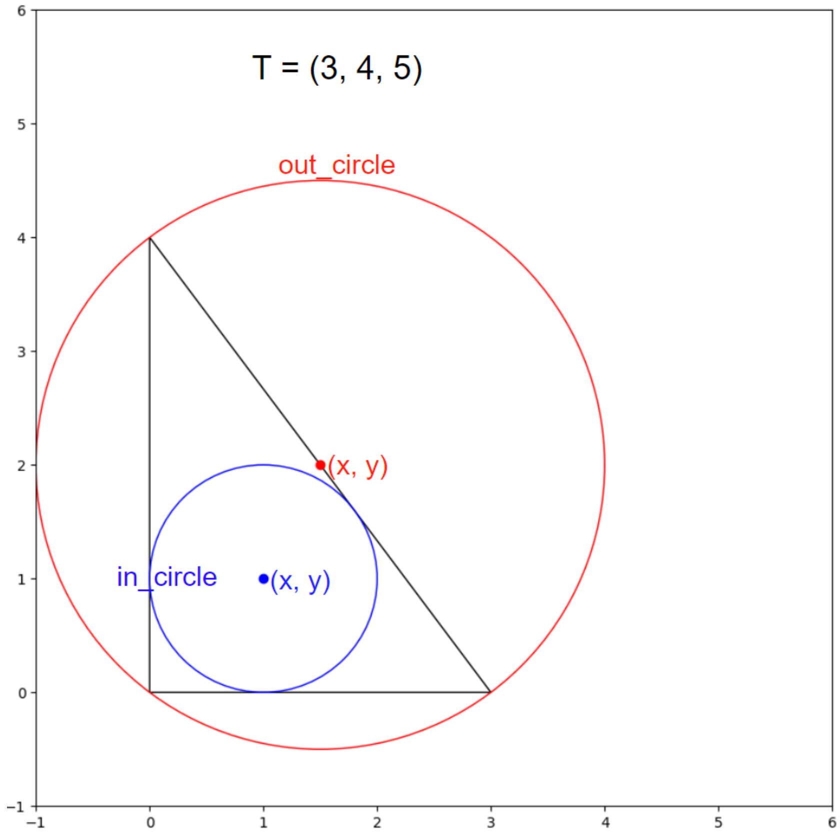

Figure 3: Example in-circle and out-circle.

The significance of the *Collatz Circle Proportionality* extends beyond its immediate mathematical elegance; it suggests that the journey of numbers through the Collatz sequence might inherently encode geometric information, specifically relating to the properties of circles associated with their corresponding Pythagorean triangles. This discovery opens new avenues for inquiry, proposing a potentially rich interplay between the arithmetic sequences of the Collatz Conjecture and the geometric principles underlying classical Euclidean geometry.

5 Bridging Dropping Orbits, Pythagorean Triples, and the Riemann Hypothesis through the Complex Plane

In the culmination of this exploration, I unveil a striking connection that bridges the dynamic arithmetic sequences of the Collatz Conjecture, the geometric elegance of Pythagorean Triples, and the profound complexities of the Riemann Hypothesis, all through the lens of complex plane transformations.

5.1 Complex Plane Mapping of Dropping Orbits

Our journey begins with the mapping of the starting integer n and its Dropping Location d from a Dropping Orbit to points in the complex plane, defined as follows:

$$z \to n + ni,$$
$$z' \to (n - d) + di.$$

This mapping transforms the arithmetic progression captured by n and d into geometric positions within the complex plane, providing a new perspective on the behavior of numbers as they traverse through the Collatz sequence. Below is a table with the calculated values of z_0 for the first 8 odd integers $n >= 3$, as well as a graph plotting all z_0 values for odd integers $n < 1,000,000$.

n	$\mathbf{z_0}$
3	$\frac{1}{2} + \frac{1}{6}i$
5	$\frac{1}{2} + \frac{3}{10}i$
7	$\frac{1}{2} + \frac{3}{14}i$
9	$\frac{1}{2} + \frac{5}{18}i$
11	$\frac{1}{2} + \frac{9}{22}i$
13	$\frac{1}{2} + \frac{7}{26}i$
15	$\frac{1}{2} + \frac{5}{30}i$
17	$\frac{1}{2} + \frac{9}{34}i$

Table 6: Complex multipliers for selected odd integers

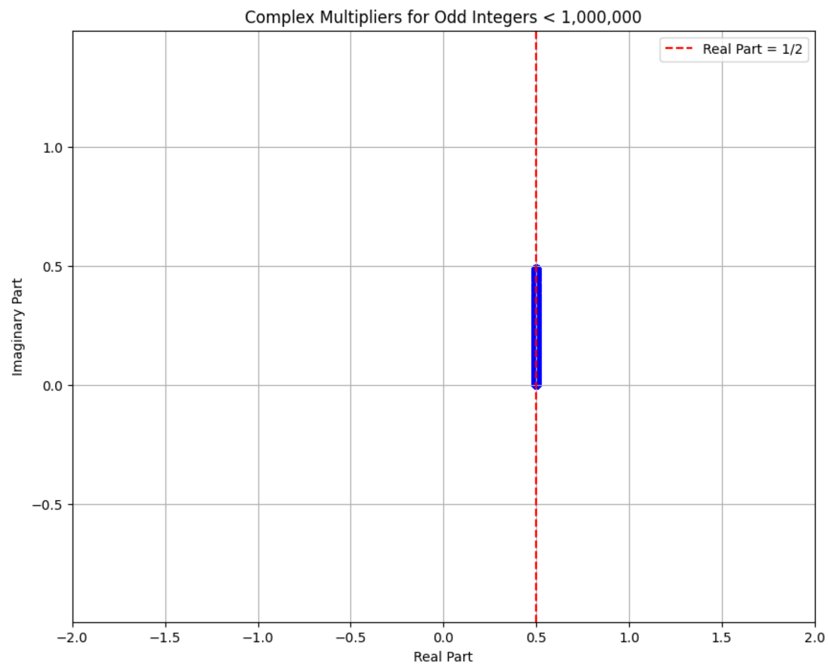

Figure 4: Select **Stopping Points** from **Stopping Class**$_8$

5.2 Transformation and the Critical Interval

Crucially, I discovered a complex transformation that connects z to z', through a complex multiplier z_0, such that $z \cdot z_0 = z'$. Astonishingly, following two properties seem to hold as true for all transformations.

- The real part of z_0 is $\frac{1}{2}$
- The imaginary part of z_0 is $\frac{d}{2n}$ where d is the **Dropping Location** and n is the starting integer.
- The imaginary part of z_0 lies somewhere in the interval $(0, \frac{1}{2})$

The real part of z_0 consistently equals $1/2$, mirroring the critical line's real part postulated by the Riemann Hypothesis for non-trivial zeros of the Riemann zeta function. This observation does not merely appear to be a numerical coincidence but suggests a deeper, potentially foundational link between the nature of arithmetic sequences in the Collatz Conjecture and the distribution of primes as encoded in the zeros of the Riemann zeta function.

5.3 Transformation Example

To demonstrate what one of these transformations looks like when plotted on the complex plane, we will take one of the more famous Collatz orbits where $n = 27$, which has a **Dropping Genus** of $(96, 0, 0)$, **Orbital Oddity** of 37, and a z_0 value of $\frac{1}{2} + \frac{19}{54}i$. The image can be seen below.

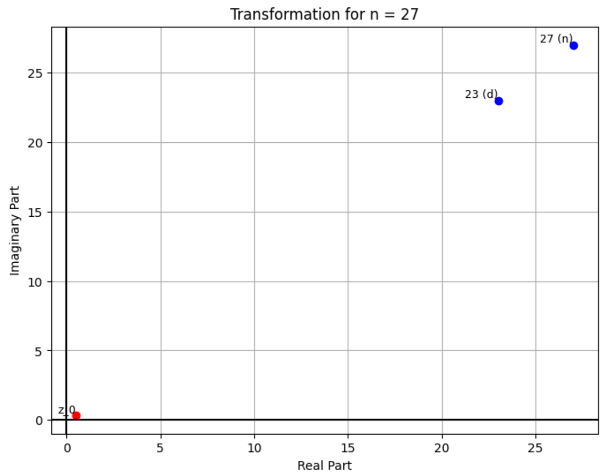

Figure 5: Transformation for n = 27

5.4 Implications and Insights

The consistent appearance of the $1/2$ value in the real part of z_0, akin to the critical line of the Riemann Hypothesis, invites us to ponder the existence of a universal structure or symmetry underlying both the arithmetic progressions of natural numbers and the prime number theorem's intricacies. This link hints at a unified mathematical framework that could offer insights into the seemingly chaotic behavior of numbers under the Collatz Conjecture, the geometric properties of Pythagorean Triples, and the distribution of primes, all through the transformative power of complex analysis.

5.5 Concluding Remarks

The convergence of these distinct mathematical narratives around the critical value of $1/2$ in the complex plane not only underscores the elegance and interconnectedness of mathematical theory but also propels us towards a future where the boundaries between number theory, geometry, and complex analysis are further blurred. These findings suggest that the answers to some of mathematics' most enduring questions may lie in the unexplored connections between these fields, offering a tantalizing glimpse into the unity underlying the mathematical universe.

6 Select References

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