A PROOF OF THE KAKEYA MAXIMAL FUNCTION CONJECTURE VIA BIG BUSH ARGUMENT

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ABSTRACT. In this paper we reduce the Kakeya maximal function conjecture to the tube sets of unit measure. We show that the Kakeya maximal function is essentially monotonic. So by adding tubes we can reduce the conjecture to the case of unit measure tube set if we allow the technicality that there are possibly two tubes on the same direction. Then we proof by Kakeya maximal function conjecture by a density argument.

CONTENTS

1. INTRODUCTION

We define a line l_i as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and for all} \quad t \in \mathbf{R} \qquad y = a + xt\}$$

We define the δ-tubes as δ-neighborhoods of lines on $B(0,1)$:

$$T_i^\delta := \{x \in \mathbf{R}^n | |x - y| < \delta, \quad y \in l_i\}.$$

The order of intersection is defined as the number of tubes intersecting in an intersection. We define $A \lesssim B$ to mean that there exists a constant C_n depending only on n such that $A \leq C_n B$. We define $A \lessapprox B$ to mean that for any $\epsilon > 0$ there exists a constant C_ϵ depending only on n and ϵ such that $A \leq C_\epsilon \delta^\epsilon$. We say that tubes are δ-separated if their angles are δ-separated. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,1)$ define a as it's center of mass. Define the Kakeya maximal function as
$f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T_\omega^\delta(a) \cap B(0,1)} \int_{T_\omega^\delta(a) \cap B(0,1)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$(1.1) \qquad ||f_\delta^*||_p \le C_\epsilon \delta^{-n/p+1-\epsilon}||f||_p,$$

for all $\epsilon > 0$ and some $n \le p \le \infty$. A very important reformulation of the problem by Tao is the following. A bound of the form (1.1) follows from a bound of the form

$$(1.2) \qquad ||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{p/(p-1)} \le C_\epsilon \delta^{-n/p+1-\epsilon},$$

for all $\epsilon > 0$, and for any set of $N \le \delta^{1-n}$ δ-separated of δ-tubes. See for example [2] or [1]. It's enough to consider the case $p = n$ and the rest of the cases will follow via interpolation [1, 2]. In this paper any constant can depend on dimension n. Our main lemma is the following:

Lemma 1.1. *Let there be a $N \sim \delta^{1-n}$ δ-tubes that are δ-separated. Then we have*

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \lesssim ||\sum_{\omega' \in \Omega'} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)},$$

where Ω' is almost δ-separated with two tubes of the same direction and

$$(1.3) \qquad |\bigcup_{\omega' \in \Omega'} T_{\omega'}^\delta| \sim \delta^{n-1}N.$$

We will proof that then we have:

Theorem 1.2. *Let there be a $N \sim \delta^{1-n}$ δ-tubes that are δ-separated. Then we have*

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \lesssim 1.$$

2. The proof of the lemma

We assume that $N \sim \delta^{1-n}$ We also drop the δ-upper index and the center points a_i so we have

$$1 \sim \delta^{n-1}N \sim \sum_{i=1}^N |T_i| = \int \sum_{i=1}^N 1_{T_i}.$$

We define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \le \sum_{i=1}^N 1_{T_i}(x) \le 2^{k+1}\}.$$

So we have

$$(2.1) \qquad \int_{E_{2^k}} \sum_{i=1}^N 1_{T_i} \sim \sum_{i=1}^N \int_{E_{2^k}} 1_{T_i} \sim \sum_k 2^k |E_{2^k}|.$$

However, we can also calculate

$$(2.2) \quad \sum_k \int_{E_{2^k}} \sum_{i=1}^N 1_{T_i} \sim \sum_k \sum_{i=1}^N \int_{E_{2^k}} 1_{T_i} \sim \sum_k \sum_{i=1}^N |E_{2^k} \cap T_i| \sim \sum_{i=1}^N |T_i| \sim \delta^{n-1}N.$$

We also notice that the number of k is less than $\sim \ln N$. Now, we have from (2.2) that

$$(2.3) \qquad \delta^{n-1} N \sim \sum_k 2^k |E_{2^k}|.$$

Next we use our big bush argument. We consider N δ-tubes that are δ-separated. Moreover, all the center points of the tubes are in the origin. This set $\bigcup_{j=1}^N T_j^\delta(0)$ is the so called big bush. It's clear that

$$|\bigcup_{j=1}^N T_j^\delta(0)| \sim \delta^{n-1} N,$$

because if $N \sim \delta^{1-n}$, the big bush covers the unit ball. However the number of tubes N only doubles if take the union with the original tube set! So we take the union

$$E' := \bigcup_{i=1}^N T_i^\delta(a_i) \cup \bigcup_{j=1}^N T_j^\delta(0),$$

and do another dyadic decomposition. We have then

$$(2.4) \qquad \delta^{n-1} N \sim \sum_m 2^m |E'_{2^m}|.$$

Now if $x \in E_{2^k}$ then $x \in \bigcup_{m \geq k} E'_{2^m}$! This is the monotonicity condition. So we have the key inequality

$$(2.5) \qquad 2^k |E_{2^k}|^{(n-1)/n} \lesssim \left(\sum_{m \geq k} 2^{mn/(n-1)} |E'_{2^m}| \right)^{(n-1)/n}.$$

It's clear via dyadic decomposition that

$$\| \sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)} \|_{n/(n-1)} \sim \left(\sum_k 2^{kn/(n-1)} |E_{2^k}| \right)^{(n-1)/n}.$$

So we have from (2.5) that

(2.6)
$$\| \sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)} \|_{n/(n-1)} \sim \left(\sum_m 2^{kn/(n-1)} |E_{2^k}| \right)^{(n-1)/n} \lesssim (\ln N)^{(n-1)/n} \max_k 2^k |E_{2^k}|^{n/(n-1)}$$
$$\lesssim (\ln N)^{(n-1)/n} \sum_{m \geq k} (2^{mn/(n-1)} |E'_{2^m}|)^{(n-1)/n}.$$

So we are done proving our main lemma 1.1, because we can combine the above (2.6) with

$$\left(\sum_{m \geq k} 2^{mn/(n-1)} |E'_{2^m}| \right)^{(n-1)/n} \lesssim \left(\sum_m 2^{mn/(n-1)} |E'_{2^m}| \right)^{(n-1)/n} \sim \| \sum_{\omega' \in \Omega} 1_{B(0,1)} 1_{T'_\omega(a_{\omega'})} \|_{n/(n-1)}.$$

3. The proof of the theorem

Next we use the density argument to proof our theorem 1.2. We assume

$$1 \sim \delta^{n-1} N.$$

We follow the exposition in [3]. We have

$$\| \sum_{i=1}^N 1_{B(0,1)} 1_{T_i(a_i)} \|_{n/(n-1)} \gtrsim 1,$$

which is by Hölder equivalent to

$$\sum_{i=1}^{N} \int_{T_i} F_i \le || \sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)} F_i ||_n || \sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)} ||_{n/(n-1)} \lessapprox || \sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i(a_i)} F_i ||_n,$$

Then we break F to level sets via dyadic decomposition and obtain equivalently

$$\sum_{i=1}^{N} 2^k |A \cap T_i| \lesssim \sum_{i=1}^{N} \int_{T_i} F_i \lessapprox ||F||_n \gtrapprox 2^k |A|^{1/n},$$

so we have equivalently

$$\sum_{i=1}^{N} |A \cap T_i| \lessapprox |A|^{1/n}.$$

However we have

$$|A| \sim | \bigcup_{i=1}^{N} T_i \cap A | \sim \sum_{1=i}^{N} |A \cap T_i|,$$

by the big bush condition (1.3) of the lemma 1.1. So we have

$$|A| \lesssim |A|^{1/n}$$

because

$$|A| \lesssim 1$$

and we are done proving the theorem 1.2 via lemma 1.1.

References

[1] E.Kroc, *The Kakeya problem*, available at

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[3] T. Tao *Blog Post*, available at https://terrytao.wordpress.com/2009/05/15/the-two-ends-reduction-for-the-kakeya-maximal-conjecture/ (2009)

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