Ideals of the Algebra

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Abstract We construct an algebra $\boldsymbol{A}$ such that $\boldsymbol{A}$ has a nonempty finite set Δ of associative and commutative binary operations. Then we may define an ideal with respect to a nonempty subset of Δ. If some hypotheses are satisfied, then we have that a union of the ideals is an ideal. An ideal M is maximal with respect to a subset of Δ if there is not an ideal $J \neq \boldsymbol{A}$ such that J contains M. And an algebra is local with respect to a subset of Δ if it has a unique maximal ideal. Suppose that the algebra $\boldsymbol{A}$ is local with respect to Φ and Ψ, M and N are the maximal ideals, respectively, and J is an ideal with respect to $\Phi \cup \Psi$. Then we have that $J \subseteq M \cap N$ if some conditions hold. Let $\boldsymbol{A}$ be a local algebra with respect to Φ, M the maximal ideal. For all Ψ with $\Phi \subseteq \Psi \subset \Delta$, if M is an ideal with respect to Ψ, then $\boldsymbol{A}$ is local with respect to Ψ. A preimage of an ideal with respect to Φ under a homomorphism is an ideal with respect to Φ.

Contents

1. Introduction

Let $\Delta := \{\beta_1, \ldots, \beta_n\}$ be a set of binary operation symbols. Suppose that the ordered pair $\mathfrak{A} := \langle \Delta, \sigma \rangle$ is an algebraic language such that $\mathfrak{A}$ contains binary operations which are commutative and associative, and suppose that $\boldsymbol{A}$ is an algebra of the language $\mathfrak{A}$, see notation 3.1 and convention 3.1 for the details.

We may define an ideal with respect to a nonempty subset of Δ in an algebra $\boldsymbol{A}$, see definition 3.1 and examples 3.1 and 3.2 for more details.

If subalgebras I and J are ideals with respect to $\Phi \subset \Delta$ and $\Psi \subset \Delta$ in $\boldsymbol{A}$, respectively, then the subset $I \cup J$ is a subalgebra, see propositions 3.1 and 3.3 and corollary 3.1.1 for more details. The subalgebra $I \cup J$ is an ideal if the hypotheses of propositions 3.2 and 3.4 and corollary 3.2.1 are satisfied.

In definition 3.2, we define a maximal ideal with respect to a subset of Δ in $\boldsymbol{A}$. Let M be a maximal ideal with respect to Φ. We have that if M is an ideal with respect

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to Ψ then M is maximal with respect to Ψ for all Ψ with $\Phi \subseteq \Psi$, see proposition 3.5 for more details.

And if an algebra $\boldsymbol{A}$ contains a unique maximal ideal with respect to a subset of Δ, then the algebra $\boldsymbol{A}$ is called local, see definition 3.3 for the details.

Suppose that an algebra $\boldsymbol{A}$ is local with respect to $\Phi \subset \Delta$, and M is the maximal ideal. If $J \neq \boldsymbol{A}$ is an ideal with respect to $\Psi \subset \Delta$ and $\Phi \subseteq \Psi$, then $J \subseteq M$, see proposition 3.6 and corollary 3.6.1 for more details. Thus if $J = M$, then the algebra $\boldsymbol{A}$ is local with respect to Ψ. This is discussed in corollary 3.6.2.

Suppose that an algebra $\boldsymbol{A}$ is local with respect to Φ and Ψ, M and N are the maximal ideals, respectively, and $M \cap N \neq \varnothing$. If $J \neq \boldsymbol{A}$ is an ideal with respect to $\Phi \cup \Psi$, then $J \subseteq M \cap N$. And we have that $J = M \cap N$ implies that the algebra $\boldsymbol{A}$ is local with respect to $\Phi \cup \Psi$, see corollaries 3.6.3 to 3.6.5 for more details.

Let $f\colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism of algebras of the language $\mathfrak{A}$. We have that Φ makes the subalgebra $f^{-1}(J)$ to be an ideal if J is an ideal with respect to Φ in $\boldsymbol{B}$, see proposition 3.7 for more details.

2. Preliminaries

Recall some definitions in universal algebra.

Definition 2.1 ([4, 5]). An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) **language** provided that

- L is a nonempty set,
- $\sigma\colon L \to \mathbb{Z}$ is a mapping.

A language $\langle L, \sigma \rangle$ is denoted by $\mathfrak{L}$. If $f \in \mathfrak{L}$ and $\sigma(f) \geq 0$ then f is called an **operation symbol**, and $\sigma(f)$ is called the **arity** of f. If $r \in \mathfrak{L}$ and $\sigma(r) < 0$, then r is called a **relation symbol**, and $-\sigma(r)$ is called the **arity** of r. A language is said to be **algebraic** if it has no relation symbols.

Definition 2.2 ([4]). Let X be a nonempty class and n a nonnegative integer. Then an n-ary **partial operation** on X is a mapping from a subclass of X^n to X. If the domain of the mapping is X^n, then it is called an n-ary **operation**. And an n-ary **relation** is a subclass of X^n where $n > 0$. An operation(relation) is said to be **unary**, **binary** or **ternary** if the arity of the operation(relation) is 1, 2 or 3, respectively. And an operation is called **nullary** if the arity is 0.

Definition 2.3 ([4]). An ordered pair $\boldsymbol{A} := \langle A, \mathfrak{L} \rangle$ is said to be a **structure** of a language $\mathfrak{L}$ if A is a nonempty class and there exists a mapping which assigns to every n-ary operation symbol $f \in \mathfrak{L}$ an n-ary operation f^A on $\boldsymbol{A}$ and assigns to every n-ary relation symbol $r \in \mathfrak{L}$ an n-ary relation r^A on $\boldsymbol{A}$. If all operation on $\boldsymbol{A}$ are partial operations, then $\boldsymbol{A}$ is called a **partial structure**. A (partial)structure $\boldsymbol{A}$ is said to be a (**partial**)**algebra** if the language $\mathfrak{L}$ is algebraic.

Definition 2.4 ([4, 5])**.** Let $\boldsymbol{A}, \boldsymbol{B}$ be (partial)structures of a language $\mathfrak{L}$. A mapping $\varphi\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is said to be a **homomorphism** provided that

$$\begin{aligned} \varphi(f^A(a_1,\dots,a_n)) &= f^B(\varphi(a_1),\dots,\varphi(a_n)) \text{ for every } n\text{-ary operation } f;\\ r^A(a_1,\dots,a_n) &\Longrightarrow r^B(\varphi(a_1),\dots,\varphi(a_n)) \text{ for every } n\text{-ary relation } r. \end{aligned}$$

Definition 2.5 (cf. [4, 5])**.** Let X be a nonempty set. Suppose that β is a binary operation on X. Then the 2-ary operation β is **associative** provided that

$$\beta(a, \beta(b, c)) = \beta(\beta(a, b), c) \text{ for every } a, b, c \in X.$$

Definition 2.6 (cf. [4, 5])**.** With the notations of definition 2.5, the 2-ary operation β is **commutative** provided that

$$\beta(a, b) = \beta(b, a) \text{ for every } a, b \in X.$$

3. Ideals of the Algebras

Convention 3.1. *We assume that all binary operations are associative[definition 2.5] and commutative[definition 2.6] in this paper.*

Notation 3.1. Let $\Delta := \{\beta_1, \beta_2, \dots, \beta_n\}$ be a set of operation symbols for $n > 0$, and $\sigma\colon \Delta \longrightarrow \mathbb{Z}$ a map which assigns to β_i 2 for all $\beta_i \in \Delta$. Then the ordered pair $\mathfrak{A} := \langle \Delta, \sigma \rangle$ is an algebraic language[definition 2.1]. It is clear that all operations of the language $\mathfrak{A}$ are binary operations. Suppose that $\boldsymbol{A}$ is an algebra[definition 2.3] of the language $\mathfrak{A}$.

Definition 3.1. Let the notations be as in notation 3.1, and $\Phi \subseteq \Delta$ a nonempty subset of 2-ary operations on $\boldsymbol{A}$. A nonempty subalgebra J is said to be an **ideal with respect to** Φ provided that $\beta_i \in \Phi$ implies $\beta_i(a, x) \in J$ for all $a \in J, x \in \boldsymbol{A}$. In this case, we say that the nonempty subset $\Phi \subseteq \Delta$ makes the subalgebra J to be an ideal.

Remark 3.1. We have an immediate consequence of definition 3.1. For all nonempty subset $\Psi \subset \Phi$, if J is an ideal with respect to Φ, then J is an ideal with respect to Ψ. And the converse need not hold.

Example 3.1 (cf. [1–3])**.** Let $\mathfrak{R} := \langle \{+, \cdot, 0, 1\}, \sigma \rangle$ where the map σ is given by assigning 2 to + and $\cdot$. Then a commutative ring $\boldsymbol{R}$ is an algebra of the language $\mathfrak{R}' := \langle \{+, \cdot\}, \sigma \rangle$. Hence an ideal in the ring $\boldsymbol{R}$ is an ideal with respect to $\{\cdot\}$ in $\boldsymbol{R}$(as an algebra of the language $\mathfrak{R}'$).

Example 3.2 (cf. [4, 5])**.** Let $\boldsymbol{B} := \langle B, \vee, \wedge, ', 0, 1 \rangle$ be a boolean algebra. Hence the boolean algebra $\boldsymbol{B}$ can be regarded as an algebra of the language $\mathfrak{B} := \langle \{\vee, \wedge\}, \sigma \rangle$. Then an ideal in $\boldsymbol{B}$ is an ideal with respect to $\{\wedge\}$ in $\boldsymbol{B}$(as an algebra of the language $\mathfrak{B}$), and a filter in $\boldsymbol{B}$ is an ideal with respect to $\{\vee\}$.

Proposition 3.1. *Let the notations be as in notation 3.1, $\beta_i \neq \beta_j \in \Delta$. Suppose that subalgebras I and J are ideals with respect to $\Delta \setminus \{\beta_i\}$ and $\Delta \setminus \{\beta_j\}$ in $\boldsymbol{A}$, respectively. Then the subset $I \cup J$ is a subalgebra of $\boldsymbol{A}$.*

Proof. It suffices to prove that $\beta(x,y) \in I \cup J$ for $\beta \in \Delta, x \in I, y \in J$, since I and J are subalgebras. For every $x \in I, y \in J$, we have that

$$\beta_k(x,y) \in \begin{cases} I & \text{if } \beta_k = \beta_j, \\ J & \text{if } \beta_k = \beta_i, \\ I \cap J & \text{otherwise.} \end{cases}$$

Observe that $I \cap J$ is not empty. Therefore, the subset $I \cup J$ is a subalgebra. □

Remark 3.2. Let the notations be as in notation 3.1, I and J ideals with respect to Φ and Ψ, respectively. Then we have that $\Phi \cap \Psi \neq \emptyset$ implies $I \cap J \neq \emptyset$, since we have that $\beta(x,y) \in I \cap J$, for all $x \in I, y \in J$, and all $\beta \in \Phi \cap \Psi$.

Corollary 3.1.1. *Let the notations be as in notation 3.1, and $\Phi, \Psi \subset \Delta$ with $\Psi \cap \Phi = \emptyset$. If subalgebras I and J are ideals with respect to $\Delta \setminus \Phi$ and $\Delta \setminus \Psi$ in $\boldsymbol{A}$, respectively, then the subset $I \cup J$ is a subalgebra.*

Proof. Obviously. □

Proposition 3.2. *With the same hypotheses as in proposition 3.1, if $\{\beta_i, \beta_j\} \neq \Delta$ then the subalgebra $I \cup J$ is an ideal with respect to $\Delta \setminus \{\beta_i, \beta_j\}$.*

Proof. By remark 3.1, we have that I and J are ideals with respect to $\Delta \setminus \{\beta_i, \beta_j\}$, since $\{\beta_i\}^C \cap \{\beta_j\}^C = \left(\{\beta_i\} \cup \{\beta_j\}\right)^C$. Hence we have that $\beta_k \in \Delta \setminus \{\beta_i, \beta_j\}$ implies $\beta_k(x,y) \in I \cup J$, for every $x \in I \cup J, y \in \boldsymbol{A}$. It follows that the subalgebra $I \cup J$ is an ideal with respect to $\Delta \setminus \{\beta_i, \beta_j\}$. □

Corollary 3.2.1. *With the hypotheses of corollary 3.1.1, if $\Phi \cup \Psi \neq \Delta$ then the subalgebra $I \cup J$ is an ideal with respect to $\Delta \setminus (\Phi \cup \Psi)$.*

Proof. Obviously. □

The two following propositions are just restatements of corollaries 3.1.1 and 3.2.1, respectively.

Proposition 3.3. *Let the notations be as in notation 3.1, I and J ideals with respect to $\Phi \subset \Delta$ and $\Psi \subset \Delta$ in $\boldsymbol{A}$, respectively. We have that the subset $I \cup J$ is a subalgebra of $\boldsymbol{A}$ if $\Phi \cup \Psi = \Delta$.*

Proof. Let $\beta \in \Delta$, $x,y \in I \cup J$. Since I and J are subalgebras, and $\Phi \cup \Psi = \Delta$. It suffices to show that $\beta(x,y) \in I \cup J$ for all $x \in I, y \in J, \beta \in \Delta$. By definition 3.1, we have that $\beta \in \Phi$ or $\beta \in \Psi$ implies $\beta(x,y) \in I$ or $\beta(x,y) \in J$, respectively, for all $x \in I, y \in J$. It follows that $I \cup J$ is a subalgebra. □

Proposition 3.4. *Let the notations be as in proposition 3.3. If $\Phi \cup \Psi = \Delta$ and $\Psi \cap \Phi \neq \emptyset$, then the subalgebra $I \cup J$ is an ideal with respect to $\Psi \cap \Phi$.*

Proof. It is clear that $\beta(x,a) \in I \cup J$ for every $x \in \boldsymbol{A}, a \in I \cup J$, and every $\beta \in \Psi \cap \Phi$. Hence the proposition is an immediate consequence of definition 3.1. □

Remark 3.3. Let the notations be as in proposition 3.3. It is clear that the subset $\Psi \cap \Phi$ makes the subalgebra $I \cap J$ to be an ideal if $\Phi \cap \Psi \neq \emptyset$. And the subalgebra $I \cap J$ need not be an ideal with respect to $\Phi \cup \Psi$ if $\Phi \neq \Psi$, since there may be $x \in \boldsymbol{A}, y \in I \cap J$ such that $\beta(x, y) \in I$ but $\beta(x, y) \notin I \cap J$ for some $\beta \in \Phi \setminus \Psi$.

Definition 3.2 (cf. [1, 4, 5])**.** Let the notations be as in notation 3.1, and $\Phi \subseteq \Delta$. An ideal M with respect to Φ in $\boldsymbol{A}$ is said to be **maximal** if $M \neq \boldsymbol{A}$ and for every ideal N with respect to Φ such that $M \subset N \subset \boldsymbol{A}$, either $M = N$ or $N = \boldsymbol{A}$.

Remark 3.4. Let the notations be as in notation 3.1, M a maximal ideal with respect to Φ. The ideal M need not be maximal with respect to Ψ for $\Psi \subsetneq \Phi$.

Proposition 3.5. *Let the notations be as in notation 3.1. Suppose that M is a maximal ideal with respect to Φ of the algebra $\boldsymbol{A}$. For all Ψ with $\Phi \subseteq \Psi$, we have that if M is an ideal with respect to Ψ then M is maximal with respect to Ψ. And there is no an ideal $N \neq \boldsymbol{A}$ with respect to Ψ such that $M \subset N$, i.e., if Ψ makes $N \neq \boldsymbol{A}$ to be an ideal, then we have $M \not\subset N$, for all Ψ with $\Phi \subseteq \Psi$.*

Proof. We assume that $N \neq \boldsymbol{A}$ is a maximal ideal with respect to Ψ, and $M \subsetneq N$. By remark 3.1, we have that N is an ideal with respect to Φ. This is a contradiction. Hence we have $M = N$ or $M \not\subset N$. Therefore, the proposition holds. □

Definition 3.3 (cf. [1–3])**.** Let the notations be as in notation 3.1, and $\Phi \subseteq \Delta$. The algebra $\boldsymbol{A}$ is **local with respect to** Φ provided that $\boldsymbol{A}$ has a unique maximal ideal with respect to Φ.

Proposition 3.6. *Let the notations be as in notation 3.1, and $\beta_i \in \Delta$. Suppose that $\boldsymbol{A}$ is local with respect to $\{\beta_i\}$, and M is the maximal ideal. For all $\beta_j \in \Delta$, if $J \neq \boldsymbol{A}$ is an ideal with respect to $\{\beta_i, \beta_j\}$ then $J \subseteq M$.*

Proof. Observe remark 3.1, we have that the subset $\{\beta_i\} \subset \{\beta_i, \beta_j\}$ makes J to be an ideal. Therefore, we have $J \subseteq M$. □

Corollary 3.6.1. *Let the notations be as in notation 3.1, and $\Phi \subset \Delta$. Suppose that $\boldsymbol{A}$ is local with respect to Φ, and M is the maximal ideal. For all subset $\Psi \subset \Delta$ with $\Phi \subset \Psi$, if $J \neq \boldsymbol{A}$ is an ideal with respect to Ψ, then $J \subseteq M$.*

Proof. Obviously. □

Corollary 3.6.2. *Let the notations be as in notation 3.1. Suppose that $\boldsymbol{A}$ is local with respect to Φ, and M is the maximal ideal. For all Ψ with $\Phi \subseteq \Psi \subset \Delta$, we have that if M is an ideal with respect to Ψ then the algebra $\boldsymbol{A}$ is local with respect to Ψ and M is the unique maximal ideal.*

Proof. This is an immediate consequence of proposition 3.5 and corollary 3.6.1. □

Remark 3.5. For $\Psi \subsetneq \Phi$, the algebra $\boldsymbol{A}$ defined in corollary 3.6.2 need not be local with respect to Ψ, since the ideal M need not be unique maximal with respect to Ψ, cf. remarks 3.1 and 3.4.

Corollary 3.6.3. *Let the notations be as in notation 3.1, $\Phi \neq \Psi \subset \Delta$. Suppose that $\boldsymbol{A}$ is local with respect to Φ and Ψ, M and N are the maximal ideals, respectively, and $M \cap N \neq \emptyset$. If $J \neq \boldsymbol{A}$ is an ideal with respect to Θ then $J \subseteq M \cap N$, for all Θ with $\Phi \cup \Psi \subseteq \Theta \subset \Delta$.*

Proof. By corollary 3.6.1, we have $J \subseteq M$ and $J \subseteq N$. It follows that $J \subseteq M \cap N$. □

Remark 3.6. The subalgebra $M \cap N$ need not be an ideal, cf. remark 3.3. But we have the two following corollaries which are consequences of corollaries 3.6.2 and 3.6.3.

Corollary 3.6.4. *With the hypotheses of corollary 3.6.3, for all Θ with $\Phi \cup \Psi \subseteq \Theta \subset \Delta$, if Θ makes $M \cap N$ to be an ideal, then the algebra $\boldsymbol{A}$ is local with respect to Θ.*

Proof. For all ideal J with respect to Θ, we have that $J \subseteq M \cap N$ by corollary 3.6.3. This suffices to prove that if Θ makes $M \cap N$ to be an ideal, then $M \cap N$ is a unique maximal ideal with respect to Θ by corollary 3.6.2. Thus the algebra $\boldsymbol{A}$ is local with respect to Θ. □

Corollary 3.6.5. *With the hypotheses of corollary 3.6.3, for all Θ with $\Phi \cup \Psi \subseteq \Theta \subset \Delta$, if the subset $M \cap N = \emptyset$, then there is no an ideal J with respect to Θ such that $J \neq \boldsymbol{A}$.*

Proof. Obviously. □

Proposition 3.7. *Let the notations be as in notation 3.1, $\boldsymbol{B}$ an algebra of the language $\mathfrak{A}$, and J an ideal with respect to $\Phi \subset \Delta$ in $\boldsymbol{B}$. Suppose that $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism[definition 2.4]. We have that the inverse image $f^{-1}(J)$ is an ideal with respect to Φ.*

Proof. Let $I := f^{-1}(J)$. It is clear that I is a subalgebra of $\boldsymbol{A}$. It suffices to prove that $\beta^A(a,x) \in I$. For all $a \in I, x \in \boldsymbol{A}$ and all $\beta \in \Phi$, we have that $\beta^B(f(a), f(x)) \in J$ implies that $\beta^A(a,x) \in I$, since we have $\beta^B(f(a), f(x)) = f(\beta^A(a,x))$. This completes the proof. □

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