Solution to the ground state of the spin glass model

— Mentioning the solution to the P=NP problem —

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Summary

We present a method for determining the order parameters of the spin glass Ising model (a general Ising model) in its ground state. This solution is valid specifically for the ground state, revealing the final outcomes of interactions and providing a solution to combinatorial optimization problems. The solution is presented through differential equations related to the inverse temperature, which can be solved using Euler's method. If the tracing of states through inverse temperature allows for the determination of state variables in a practically finite time, it becomes relevant to the P=NP problem. Furthermore, the set of equations obtained is also shown to be equivalent to those used in Boltzmann machines.

Introduction

The Ising model is the simplest and most fundamental model for interactions, where spins (nodes) that can take on states of either -1 or 1 change over time due to interactions at a temperature T, eventually settling into a state known as the ground state at T=0. In this ground state, the system's energy is at its lowest, and the order parameters, which are the expected values of each state, take on values of either -1 or 1. These parameters indicate the final state of the system as a result of the interactions. In complex systems, this state results from interactions, while in combinatorial optimization problems, it represents the solution. The outcome of these interaction systems and combinatorial optimization solutions has potential applications across a wide range of fields in the real world. The spin glass Ising model, a variant of the Ising model, incorporates both positive and negative individual interactions [1], and can model various interaction systems and combinatorial optimization problems. This text will describe a method to obtain the ground state of the spin glass Ising model. Specifically, it involves deriving a set of simultaneous equations for the expected values of the state variables, which include derivatives with respect to t=1/(Tk) (where T is temperature and k is the Boltzmann constant) and tracing the solution from t=0 using Euler's method.

Results

At t=1/(Tk)=∞ (in the ground state), the following results were obtained.

$$x_i = \frac{1}{1 + e^{-\xi_i t}} \quad (1)$$

$$x_i = \langle n_i \rangle \,, \quad \xi_i = \sum_{j=1 \neq i}^{N} \varepsilon_{i,j} x_j \,, \quad t = \frac{1}{k_B T}$$

Take the derivative.

$$\dot{x}_i - x_i(1-x_i)t\dot{\xi}_i = x_i(1-x_i)\xi_i \quad (2)$$

Multiply by the inverse matrix.

$$\dot{x}_i = A^{-1} \cdot \vec{b} = f(x) \quad (3)$$

$$A_{i,j} = \begin{cases} 1 & (for\ j = i) \\ -x_i(1-x_i)t\varepsilon_{i,j} & (for\ j \neq i) \end{cases}, \quad \vec{b}_i = x_i(1-x_i)\xi_i$$

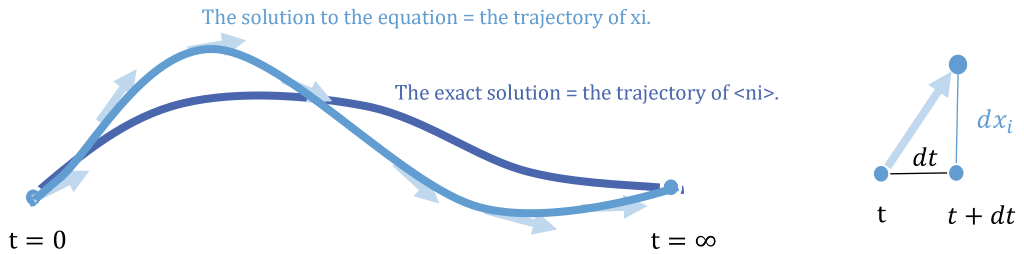
Use Euler's method.

$$\dot{x}_i = f_i(x_j)$$

$$\tilde{x}_i^{(t+dt)} = x_i^{(t)} + f_i\left(x_j^{(t)}\right)dt$$

$$x_i^{(t+dt)} = x_i^{(t)} + \frac{1}{2}\left(f_i\left(x_j^{(t)}\right) + f_i\left(\tilde{x}_i^{(t+dt)}\right)\right)dt \quad (4)$$

By tracing x from an initial value of 0.5, it is possible to find the ground state at t=∞. Equation (1) is 0.5 at t=0, so if there is continuity between the initial value and the final ground state, regardless of the different paths taken in between, adding up the variations of Equation (3) will lead to the ground state.



This implies that by adding up variations step by step, one can obtain the solution for an Ising model with any interaction. This is equivalent to obtaining the solution for combinatorial optimization problems. Therefore, the number of steps of adding variations needed to ensure the practical accuracy of the solution suggests that, practically, P=NP. (To obtain an exact solution, an infinite number of steps would be necessary.)

Theory

Assume there are state variables for each node i=1 to N, each having the following state.

$$n_i = 0\ or\ 1 \quad (5)$$

A model whose system Hamiltonian can be represented as follows is called the Ising model [1]. (While the Ising model generally uses n=-1 or 1, it can be simplified for calculations by treating n as 0 or 1.)

$$-H = \sum_{i \leq j} \varepsilon_{i,j} n_i n_j \quad (6)$$

However, εij can take any value within the following range (spin glass Ising model [1]).

$$\varepsilon_{i,j} = -1 \sim 1 \quad (7)$$

The order parameter <ni> is determined as follows.

$$\langle n_i \rangle = \frac{Z_i}{Z} \quad (8)$$

$$Z = \sum_{\{n\}} e^{-Ht} \;, Z_i = \sum_{\{n\}} n_i e^{-Ht} \;, t = \frac{1}{k_B T} \quad (9)$$

From equations (8) and (9),

$$\langle n_i \rangle = \langle \overline{n_i} e^{\varepsilon_{i,i} t} \prod_{j=1 \neq i}^{N} e^{\varepsilon_{i,j} n_j t} \rangle \quad (10)$$

Given nj=0 or 1, the following is equivalent.

$$\langle n_i \rangle = \langle \overline{n_i} e^{\varepsilon_{i,i} t} \prod_{j=1 \neq i}^{N} \left((e^{\varepsilon_{i,j} t} - 1) n_j + 1 \right) \rangle \quad (11)$$

The right-hand side becomes the expected value of the multiplication of nj, but at t=∞,

$$\langle n_a n_b \rangle = \langle n_a \rangle \langle n_b \rangle$$

Therefore,

$$\langle n_i \rangle = \langle \overline{n_i} \rangle e^{\varepsilon_{i,i} t} \prod_{j=1 \neq i}^{N} \left((e^{\varepsilon_{i,j} t} - 1) \langle n_j \rangle + 1 \right) \quad (12)$$

$$\langle \overline{n_i} \rangle = 1 - \langle n_i \rangle$$

$$n_j = 0 or 1$$

More

$$\langle n_i \rangle = (1 - \langle n_i \rangle)\, e^{\varepsilon_{i,i} t} \prod_{j=1 \neq i}^{N} e^{\varepsilon_{i,j} \langle n_j \rangle t} \quad (13)$$

Here,

$$\xi_i = \varepsilon_{ii} + \sum_{j \neq i} \varepsilon_{ij} \langle n_j \rangle \quad (14)$$

Using the above, it can be expressed as,

$$\langle n_i \rangle = (1 - \langle n_i \rangle)\, e^{\xi_i t} \quad (15)$$

Summarizing <nj> from here,

$$\langle n_i \rangle = \frac{e^{\xi_i t}}{1 + e^{\xi_i t}} \quad (16)$$

Therefore, the following is obtained.

$$\langle n_i \rangle = \frac{1}{1 + e^{-\xi_i t}} \quad (17)$$

$$\xi_i = \varepsilon_{ii} + \sum_{j \neq i} \varepsilon_{ij} \langle n_j \rangle$$

Equation (17) holds for all i=1 to N and is a system of simultaneous equations for N values of <ni>. In other words, it was possible to represent the order parameter <ni> at t=∞ (ground state) as a system of simultaneous equations.

Furthermore, the above equation involves taking the sum of the weights εij applied to each site and then passing it through an activation function to binarize it to 0 or 1, which is the structure of a perceptron in machine learning. Originally, this equation is derived from the mathematical development of statistical mechanics interaction systems (the spin glass Ising model), reflecting natural phenomena. Therefore, it is equivalent to validating machine learning and neural network models. The Ising model addresses the problem of determining the values of nodes, while machine learning focuses on determining the values of weights. Because the above equation is mathematically sound or represents natural phenomena, it suggests the possibility of analyzing and understanding machine learning. Consequently, there is potential for creating more accurate models or machine learning models through numerical analysis without the need for learning.

Next, differentiating both sides of equation (17) yields the following.

$$\frac{\partial \langle n_i \rangle}{\partial t} = \langle n_i \rangle (1 - \langle n_i \rangle) \left(\xi_i + \frac{\partial \xi_i}{\partial t} t \right) \quad (18)$$

Since equation (18) is only valid at t=∞, expressing it in terms of the variable xi, which holds for all t, results in the following.

$$\frac{\partial x_i}{\partial t} = x_i (1 - x_i) \left(\xi_i + \frac{\partial \xi_i}{\partial t} t \right) \quad (19)$$

$$\xi_i = \varepsilon_{ii} + \sum_{j \neq i} \varepsilon_{ij} x_j$$

Express the derivative as follows,

$$\dot{x}_i$$

Transforming it results in,

$$\dot{x}_i - x_i (1 - x_i) t \dot{\xi}_i = x_i (1 - x_i) \xi_i \quad (20)$$

By multiplying by the inverse matrix of A, the following is obtained.

$$\dot{x}_i = A^{-1} \cdot \vec{b} = f(x) \quad (21)$$

Matrix A and vector b are as follows.

$$A_{i,j} = \begin{cases} 1 & (for\ j = i) \\ -x_i (1 - x_i) t \varepsilon_{i,j} & (for\ j \neq i) \end{cases}, \quad \vec{b}_i = x_i (1 - x_i) \xi_i$$

Therefore, this is subject to Euler's method [2].

$$\dot{x}_i = f_i(x_j)$$

$$\tilde{x}_i^{(t+dt)} = x_i^{(t)} + f_i\left(x_j^{(t)}\right) dt$$

$$x_i^{(t+dt)} = x_i^{(t)} + \frac{1}{2}\left(f_i\left(x_j^{(t)}\right) + f_i\left(\tilde{x}_i^{(t+dt)}\right) \right) dt \quad (22)$$

Starting from an initial value of 0.5 for x, it's possible to determine the ground state at t=∞. Furthermore, if the number of steps required for practical accuracy is on the order of polynomial time, it can be said that P=NP.

Discussion

In mathematics, there exists an unsolved problem known as the P vs NP problem. This problem, a major question in computer science, raises the issue of whether all problems for which solutions can be efficiently verified can also be efficiently found. Here, "efficiently" means that the solution can be computed within polynomial time relative to the size of the problem. "Polynomial time" refers to an algorithm's efficiency, indicating that the algorithm takes time proportional to n raised to a constant power (where n is the size of the problem). In essence, the P vs NP problem asks whether all problems that can be verified quickly (NP) are also solvable quickly (P).

Now, in response to this problem, it has been possible to determine the ground state of the spin glass Ising model using Euler's method [2]. Since solving the problem of finding the ground state of the spin glass Ising model is equivalent to solving all NP-complete problems [1], if the number of steps required for the Euler method's variations to be added up is on the order of polynomial time, it implies that P=NP.

In summary, if P=NP to a practical degree, it holds significant potential for solving various business and societal issues that were previously challenging. If you are interested, please feel free to contact the email address provided below.

Inquiry

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References

[1] Hidetoshi Nishimori, Iwanami Shoten, New Physics Selection, "Spin Glass Theory and Information Statistical Mechanics", February 10, 2016.

[2] Steven H. Strogatz, "Nonlinear Dynamics And Chaos", January 30, 2015.