The Zeta function and the Euler-Maclaurin Formula

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Abstract

The Riemann Zeta function is very famous because hidden within it lies the much-desired prime counting function. In this paper, we will unlock the door using the Euler-Maclaurin formula and present the proof of the Riemann Hypothesis.

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1 Introduction.

1.1 Definition of the Riemann Zeta Function.

The Riemann Zeta function is defined as follows: $\mathbb{C} \to \mathbb{C}$

$$\zeta(s) := \sum_{m=1}^{\infty} \frac{1}{m^s} \tag{1}$$

where: $s \in \mathbb{C}$

This series converges when $Re[s] > 1$

Riemann succeeded in giving it analytic continuation, obtaining the following functional equation [3]:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

This function has a unique pole, which occurs at $\zeta(1)$.

1.2 Formulas for specific cases of the Riemann Zeta Function.

Formulas for specific cases of the Riemann Zeta Function are also known. For instance, there's a formula to calculate the zeta function when 's' is a positive even number, deduced by Euler. Additionally, there's another formula to calculate the zeta function when 's' is a negative integer, obtained by combining the functional equation with Euler's deduced formula:

Let: $s = 2k, k \in \mathbb{N}$

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{3}$$

Let $s = -k, k \in \mathbb{N}$

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \tag{4}$$

Where: B_{2k} and B_{-k} are Bernoulli numbers.

2 Sum of powers and Bernoulli numbers.

The Bernoulli numbers B_j are a sequence of successive rational numbers of significant importance in number theory. They arise in combinatorics, in the expansion of the tangent function and hyperbolic tangent function via Taylor series, and in the Euler-Maclaurin equation

One way to derive these numbers is through recursive addition:

$$B_j = -\frac{1}{1+j} \sum_{m=0}^{j-1} \binom{1+j}{m} B_m \tag{5}$$

Bernoulli numbers arise when seeking a function to compute the sum of powers. Originally derived by Jakob Bernoulli, the formula for these numbers is outlined below, albeit refined and presented with modern nomenclature [2]:

$$\sum_{m=1}^{n} m^k = S_k(n) = \sum_{p=1}^{1+k} A_p n^p \tag{6}$$

Where: $k \in \mathbb{N} \cup \{0\}$, *and* S_k *is a function of* n.

A_p is obtained by:

$$A_p = \frac{(-1)^{1+k-p}}{1+k} \binom{1+k}{p} B_{1+k-p} \tag{7}$$

3 Deduction of a formula to compute the sum of power $\sum m^k$ expressed as a sum of higher-order derivatives of the function m^k.

In this section, we will rearrange equation (6) and compare it to a sum involving higher-order derivatives of the function m^k.

Factoring $\frac{1}{1+k}$ and expanding the summation in equation (6), we can determine $S_k(n)$ as follows:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^k(k+1)!}{1!k!}B_k n + \frac{(-1)^{k-1}(k+1)!}{2!(k-1)!}B_{k-1}n^2 + \frac{(-1)^{k-2}(k+1)!}{3!(k-2)!}B_{k-2}n^3 + \right.$$
$$\left. \frac{(-1)^{k-3}(k+1)!}{4!(k-3)!}B_{k-3}n^4 + \cdots + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^0(k+1)!}{(k+1)!(0)!}B_0 n^{k+1}\right]$$

We will now reverse the order of the terms in the sum and simplify by reducing the factorials in each term.

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^2(k+1)!}{(k-1)!2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1 k!(k+1)}{k!1!}B_1 n^k + \frac{(-1)^2(k-1)!k(k+1)}{(k-1)!2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0}{0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)}{1!}B_1 n^k + \frac{(-1)^2 k(k+1)}{2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

The formula can be rewritten as a summation involving a product:

$$\sum_{m=1}^{n} m^k = S_k(n) = \frac{1}{1+k}n^{k+1} + \sum_{p=2}^{1+k}\frac{(-1)^{p-1}B_{p-1}}{(1+k)(p-1)!}\prod_{m=1}^{p-1}(2+k-m)n^{2+k-p} \tag{8}$$

With careful observation, we can derive an equivalent formula featuring a summation that incorporates higher-order derivatives of the function n^k, as shown below:

$$\sum_{m=1}^{n} m^k = S_k(n) = \int n^k dn + \sum_{p=1}^{k}\frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) \tag{9}$$

Also:

$$\sum_{m=1}^{n}\frac{1}{m^{-k}} = S_{-k}(n) = \int \frac{1}{n^{-k}}dn + \sum_{p=1}^{k}\frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^{-k}}\right) \tag{10}$$

Hence, we can conclude that equations (8), (9), and (10) are equivalent.

4 Deduction of a new equation to compute the Zeta Function from the Euler-Maclaurin formula.

Considering the renowned Euler-Maclaurin Formula [1], which approximates the sum of a function $f(m), \mathbb{R} \to \mathbb{R}$, assuming it is differentiable q times:

$$\sum_{m=a+1}^{b} f(m) = \int_a^b f(x)dx + \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!}\left[f^{(p-1)}(b) - f^{(p-1)}(a)\right] + r_q \tag{11}$$

Where:

$$r_q = (-1)^q \int_0^n \frac{B_{q+1}(m - \lfloor m \rfloor)}{(q+1)!} f^{(q+1)}(m) dm \tag{12}$$

$f^{(p-1)}(b)$ and $f^{(p-1)}(a)$ represent the $(p-1)$-th derivative of the function $f(m)$ evaluated at b and a, respectively, while r_q denotes the residual error of the q-th derivative approximation and $B_j(x)$ is a Bernoulli polynomial calculated using the following formula:

$$B_j(x) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} B_m x^{j-m} \tag{13}$$

Setting the limits of the summation in equation (11) to $a = 0$ and $b = n$, and defining the function $f(m)$ as m^s, where $f(m)$ is differentiable q times, we obtain:

$$\sum_{m=1}^{n} m^s = \int_0^n m^s dm + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dm^{p-1}} (m^s) \Big|_0^n + r_q \tag{14}$$

Incorporating the initial limits of the integral and the derivatives, and adding them to the residual error, we can write the following:

$$\sum_{m=1}^{n} m^s = \int n^s dn + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} (n^s) + R_q \tag{15}$$

Here, R_q represents the cumulative total residual error.

We can similarly derive an expression for the sum of the function $f(m) = \frac{1}{m^s}$.

$$\sum_{m=1}^{n} \frac{1}{m^s} = \int \frac{1}{n^s} dn + \sum_{p=1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^s} \right) + R_q \tag{16}$$

If $n \to \infty$, and we regard R_q as a function of s, then $R_q(s)$ can be expressed by solving equation (16):

$$R_q(s) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^s} - \int \frac{1}{n^s} dn - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^s} \right) \right] \tag{17}$$

Expanding Equation (17), we get:

$$R_q(s) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^s} - \left(\frac{1}{1-s} n^{1-s} + \frac{(-1)^1 B_1}{1!} n^{-s} + \frac{(-1)^2 B_2}{2!} (-s) n^{-1-s} + \frac{(-1)^3 B_3}{3!} (-s)(-1-s) n^{-2-s} + \cdots \right) \right] \tag{18}$$

We will now examine three cases for equation (18), but before that, let's consider that the variable $s \in \mathbb{C}$:

Case I. For $Re[s] > 1$.
Applying the limit yields:

$$R_q(s) = \left[\sum_{m=1}^{\infty} \frac{1}{m^s} - \left(\frac{1}{1-s} 0 + \frac{(-1)^1 B_1}{1!} 0 + \frac{(-1)^2 B_2}{2!} (-s) 0 + \frac{(-1)^3 B_3}{3!} (-s)(-1-s) 0 + \cdots \right) \right]$$

$$R_q(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \iff Re[s] > 1 \tag{19}$$

Hence, we can conclude that when $Re[s] > 1$, the value of $R_q(s)$ equals $\zeta(s)$:

$$If \quad Re[s] > 1 \Rightarrow R_q(s) = \zeta(s) \tag{20}$$

Case II. When $s = 1$.
Applying the limit yields:

$$R_q(1) = \left[\sum_{m=1}^{\infty} \frac{1}{m^1} - \left(\frac{1}{0} + \frac{(-1)^1 B_1}{1!} 0 + \frac{(-1)^2 B_2}{2!}(-1)0 + \frac{(-1)^3 B_3}{3!}(-1)(-1-1)0 + \cdots\right)\right]$$

$$R_q(1) = \infty - (\infty + 0 + 0 + 0 + \cdots)$$

$$R_q(1) = indeterminate$$

Hence, it can be concluded that when $s = 1$, the value of $R_q(1)$ is indeterminate, and it equals $\zeta(1)$:

$$If \quad s = 1 \Rightarrow R_q(s) = \zeta(s) \tag{21}$$

Case III. Let's consider $s = -k$, where $k \in \mathbb{N}$.
Equation (17) can be rewritten by splitting the summation into two parts: the first part with limits from $p = 1$ to $p = k$, and the second summation from $p = k + 1$ to $p = q$, resulting in the expression as follows:

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \int n^k dn - \sum_{p=1}^{q=k} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) - \sum_{p=k+1}^{q} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right] \tag{22}$$

It is evident that the integral with the first summation is equivalent to equation (9), but with a negative sign. Hence, it can be expressed as follows:

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \sum_{m=1}^{n} m^k - \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right]$$

The first two summations cancel each other out, resulting in the following expression:

$$R_q(-k) = \lim_{n\to\infty} \left[- \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right)\right] \tag{23}$$

Next, extending the series:

$$R_q(-k) = \lim_{n\to\infty} \left[-\left(\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(k!)n^0 + \frac{(-1)^{k+2} B_{k+2}}{(k+2)!}(k)(k-1)\cdots n^{-1} + \frac{(-1)^{k+3} B_{k+3}}{(k+3)!}(k)(k-1)\cdots n^{-2} + \cdots\right)\right]$$

Applying the limit, then simplifying the factorials, we get:

$$R_q(-k) = -\frac{(-1)^{k+1} B_{k+1}}{(k+1)!}(k!)$$

$$R_q(-k) = \frac{(-1)^k B_{k+1}}{k+1}$$

As $B_{2j+1} = 0$ for $j \in \mathbb{N}$, we can express:

$$R_q(-k) = -\frac{B_{k+1}}{k+1} \iff k \in \mathbb{N} \tag{24}$$

Upon comparing equation (24) with equation (4), we find that they are equivalent. Hence:

$$If \quad k \in \mathbb{N} \;\Rightarrow\; R_q(-k) = \zeta(-k) \tag{25}$$

We are prepared to present the following theorem:

Theorem 1 *Using equations (20), (21), and (25), according to the principle of analytic continuation, we can express:*

$$\zeta(s) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^s} - \int\frac{1}{n^s}dn - \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^s}\right)\right] \tag{26}$$

Where $s \in \mathbb{C}$, and q represents the number of times the function $\frac{1}{n^s}$ can be differentiated.

5 Graphical interpretation of the Riemann Zeta function.

Let's define the function $S^*_{-s}(n)$ as follows:

$$S^*_{-s}(n) := \int\frac{1}{n^s}dn + \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^s}\right) \tag{27}$$

When we vary n while computing the series $\sum_{m=1}^{n}\frac{1}{m^s}$ and plot it on the complex plane, it forms a sequence of points arranged in the shape of a logarithmic spiral, with its center being $\zeta(s)$ (see Figure 1).

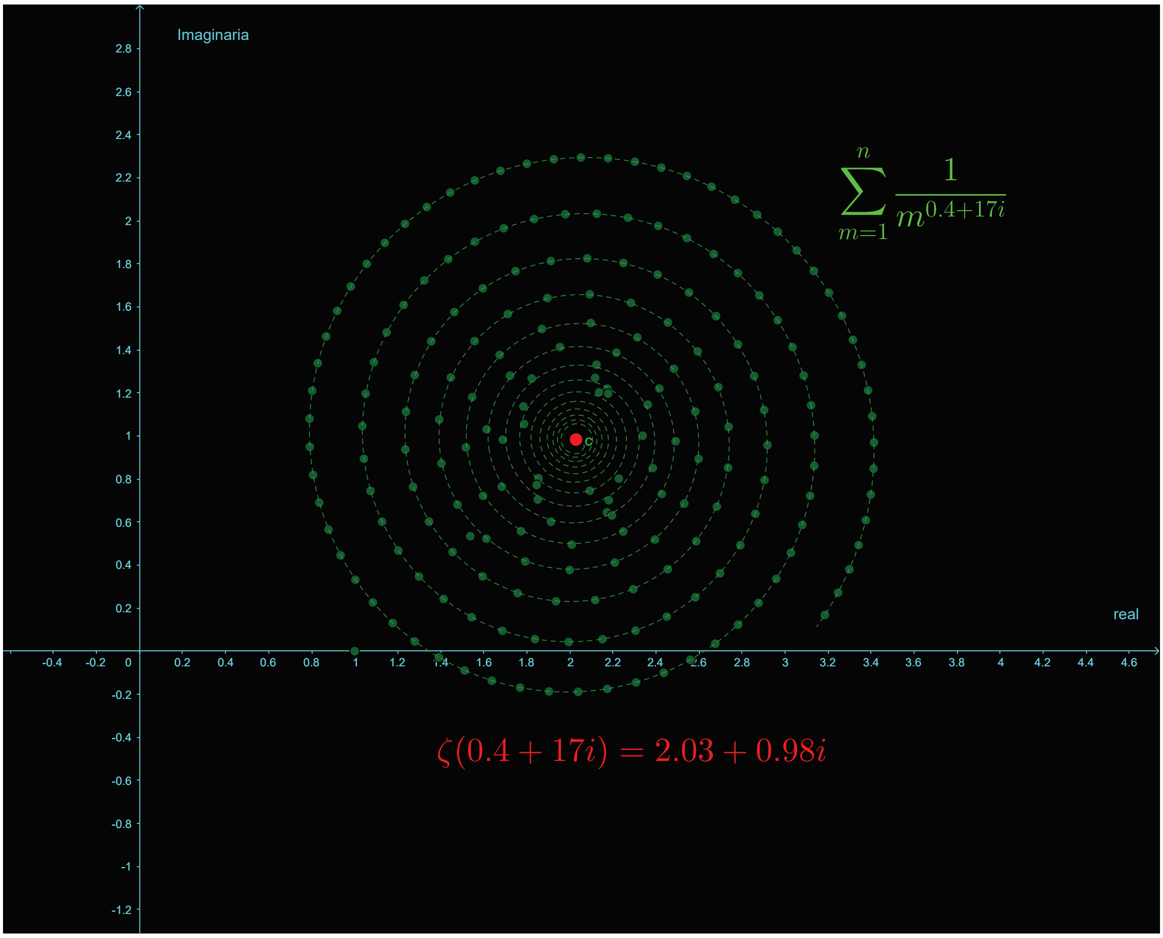

Figure 1: *Example of logarithmic spiral of the series* $\sum_{m=1}^{n}\frac{1}{m^s}$ for $s = 2 + 27i$ where $\zeta(s)$ is the center of the spiral.

When $Re[s] > 1$, the points of the spiral converge towards its center, but if $Re[s] \leq 1$, the series of points diverges from its center. The function $S^*_{-s}(n)$ forms a spiral with the center at the origin of the complex plane, but by adding $\zeta(s)$, the spiral approaches the series of points, and both spirals converge as $n \to \infty$ (see Figure 2).

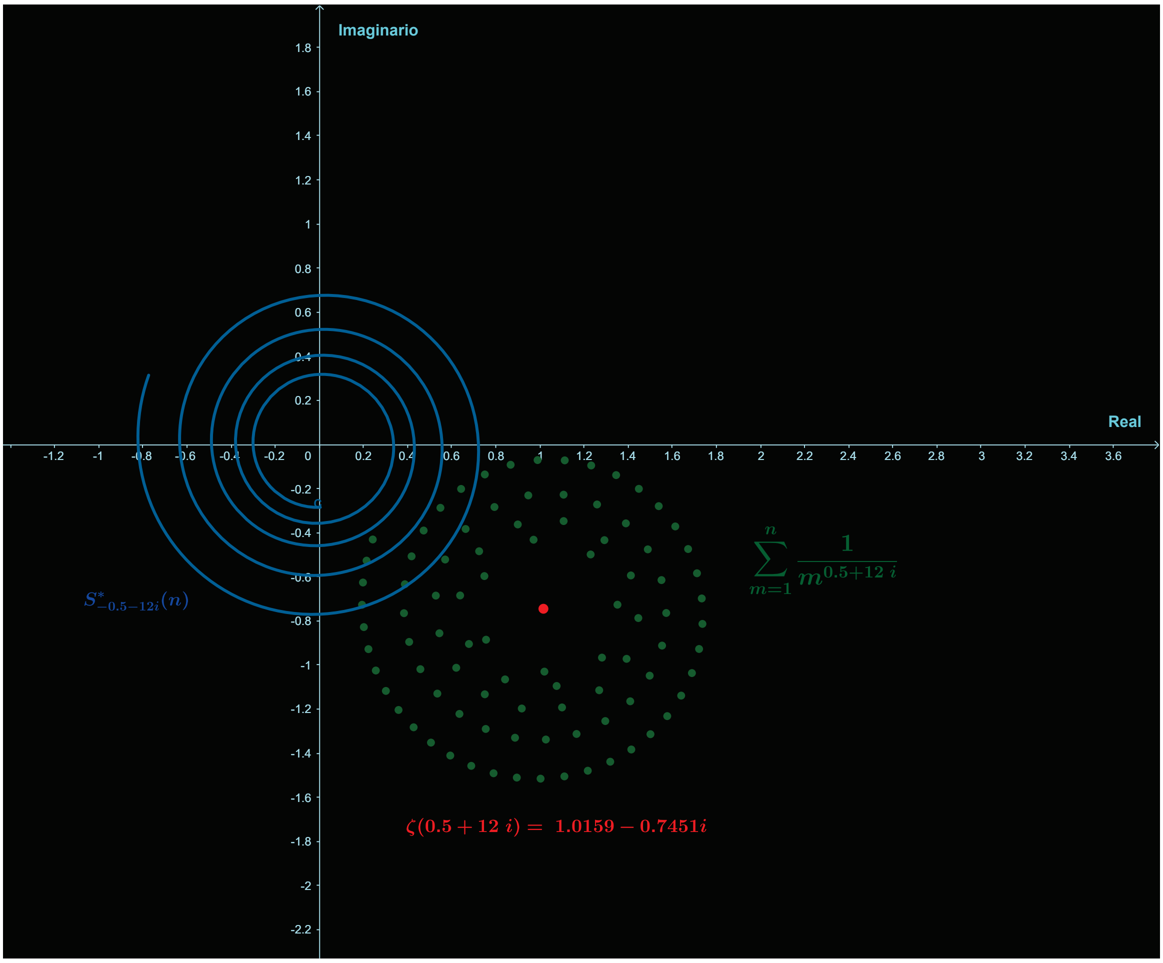

Figure 2: *Complex plane plot showing the spirals represented by the function $S^*_{-s}(n)$ (in blue), the series $\sum_{m=1}^{n} \frac{1}{m^s}$ (in green), and the point $\zeta(s)$ (in red).*

When separately plotting the real and imaginary parts of the series of points and the function $S_{-s}(n)$, we observe periodic functions with variable amplitude and period. The amplitude increases when $Re[s] < 1$ and decreases when $Re[s] > 1$, while the frequency depends on the term $Im[s]\ln(n)$. The axis or mean value of the periodic functions will be:

$$y_{Re} = Re[\zeta(s)]$$
$$y_{Im} = Im[\zeta(s)]$$

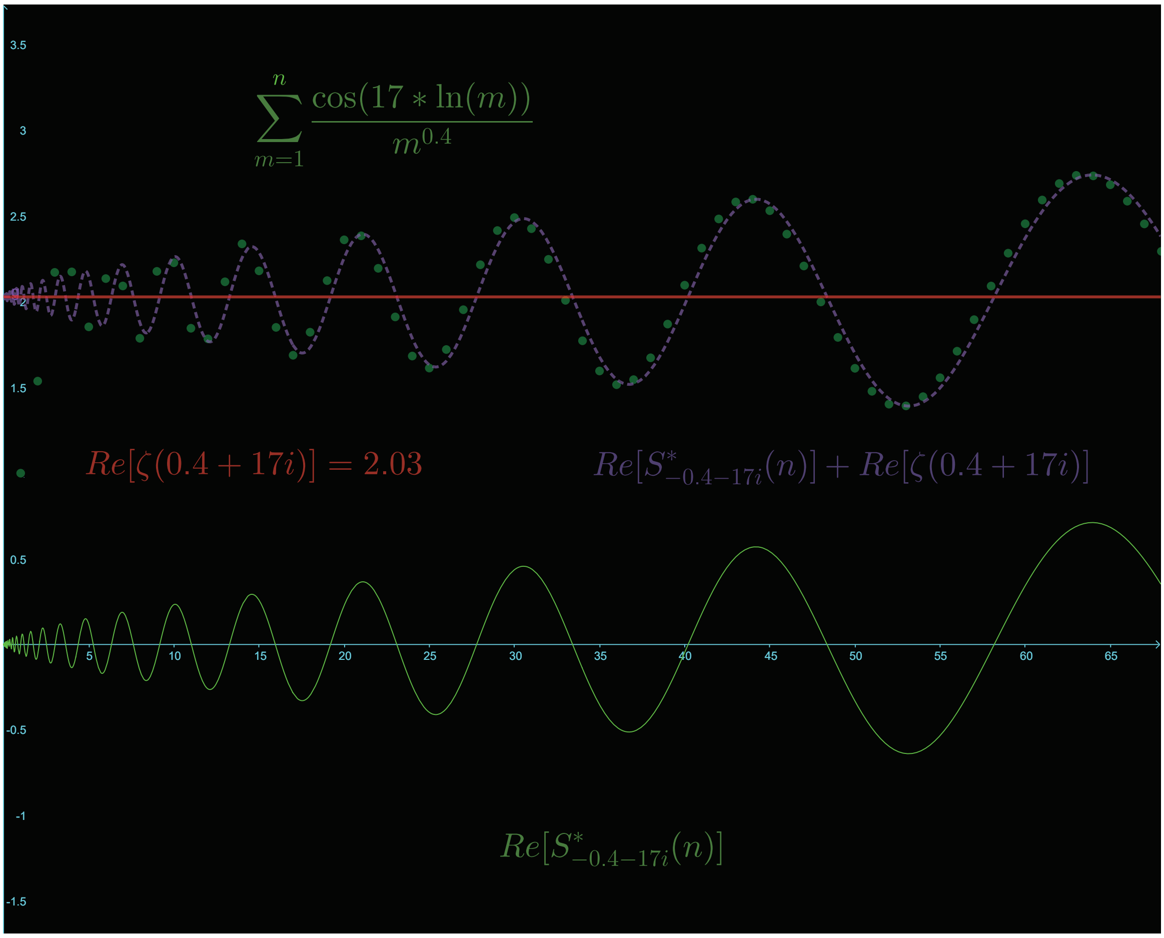

Figure 3: *Graphical representation of the functions* $Re\left[S^{(}_{-s}n)\right]$ *and the series* $\sum_{m=1}^{n} \frac{1}{m^{Re[s]}} \cos(Im[s] \ln m)$, *oscillating around its mean value* $Re\left[\zeta(s)\right]$.

By adding $Re[\zeta(s)]$ to the function $Re[S^{(}_{-s}n)]$, we align $Re[S^{(}_{-s}n)]$ with the series $\sum_{m=1}^{n} \frac{1}{m^{Re[s]}} \cos(Im[s] * \ln m)$ as $s \to \infty$ (see Figure 3).

Similarly, by adding $Im[\zeta(s)]$ to the function $Im[S^{(}_{-s}n)]$, we align $Im[S^{(}_{-s}n)]$ with the series $-\sum_{m=1}^{n} \frac{1}{m^{Re[s]}} \sin(Im[s] * \ln m)$ as $s \to \infty$ (see Figure 4).

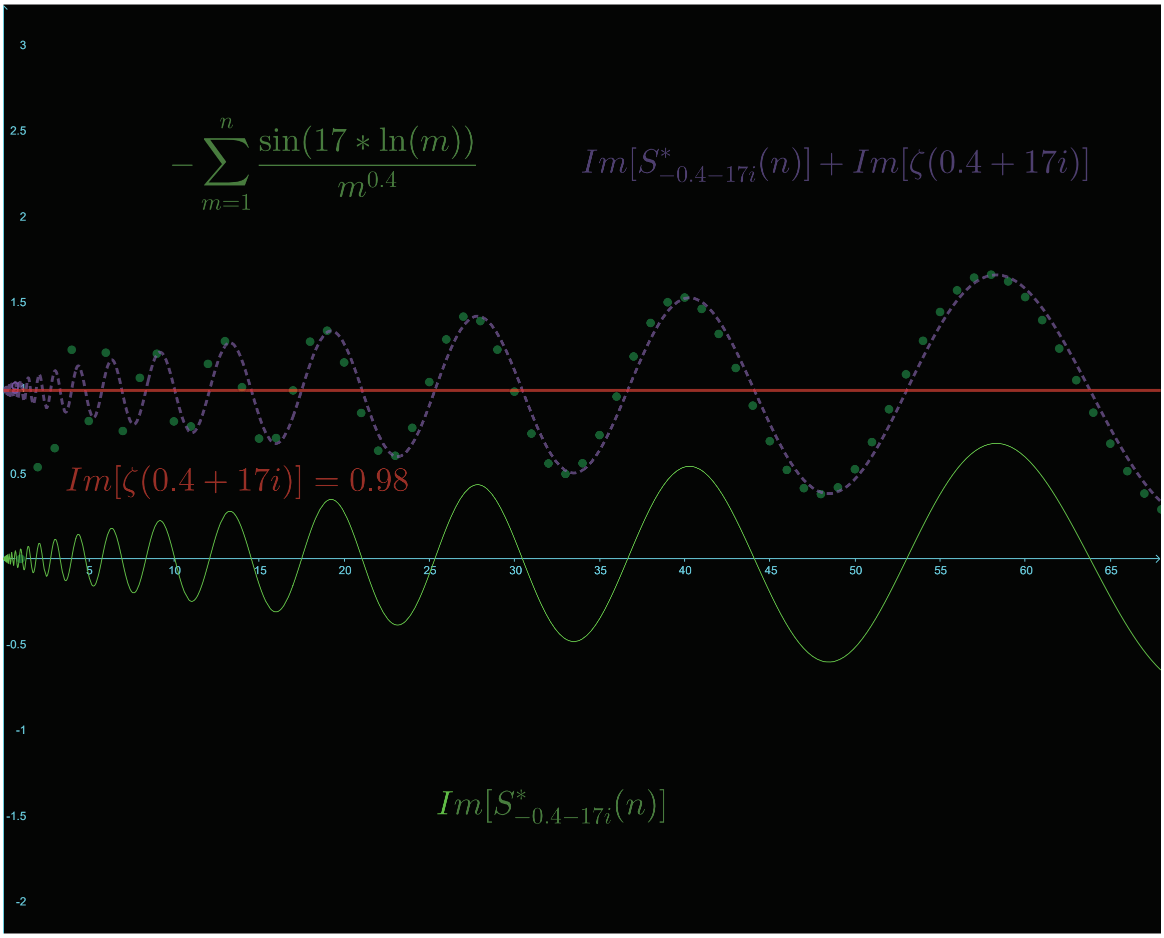

Figure 4: *Graphical depiction of the functions* $Im\left[S^(_{-s}n)\right]$ *and the series* $\sum_{m=1}^{n}\frac{1}{m^a}\sin(-Im[s]\ln n)$, *oscillating around its mean value* $Im\left[\zeta(s)\right]$.

6 The Riemann Hypothesis.

6.1 The critical strip of the Zeta Function.

Utilizing the Riemann functional equation (2), it becomes feasible to derive $\zeta(s)$ for values of $Re[s]<0$, as it relies on $\zeta(1-s)$, which converges. Nonetheless, obtaining $\zeta(s)$ becomes infeasible when $0\leq Re[s]\leq 1$, as it would necessitate knowledge of $\zeta(1-s)$, which also fails to converge (refer to Figure 5).

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{s\pi}{2}\right)\Gamma(1-s)\zeta(1-s)$$

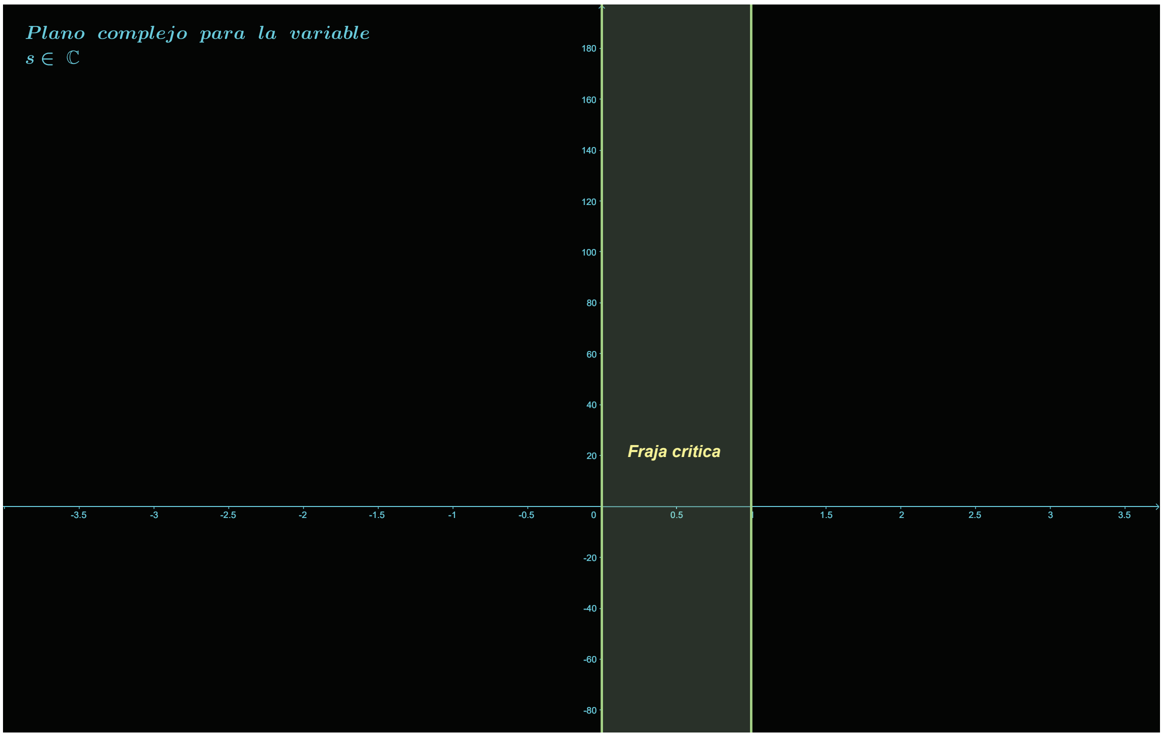

Figure 5: *Critical region in the complex plane, where for the variable $s \in \mathbb{C}$, it becomes impossible to ascertain $\zeta(s)$ using the Riemann Functional Equation.*

6.1.1 Trivial zeros of the Zeta function.

By examining the Riemann functional equation, it becomes evident that for negative even values of s, $\zeta(s) = 0$. This occurs because the term $\sin(\frac{s\pi}{2})$ equals zero. These points where the function equals zero are referred to as *trivial zeros.*

$$\zeta(-2k) = 2^{-2k}\pi^{-2k-1}\sin\left(\frac{-2k\pi}{2}\right)\Gamma(1+2k)\zeta(1+2k) = 0$$

Where $k \in \mathbb{N}$

When s is a positive even number, the function $\zeta(s)$ is nonzero, since the term $\sin(\frac{s\pi}{2})$ cancels with the poles of the gamma function $\Gamma(1-s)$.

6.1.2 Nontrivial zeros of the Zeta function.

There exist additional zeros of $\zeta(s)$; nevertheless, it has been demonstrated that these zeros reside within the critical strip. Referred to as nontrivial zeros, they can only be determined through numerical methods as they cannot be calculated using the Riemann functional equation (see Figure 6).

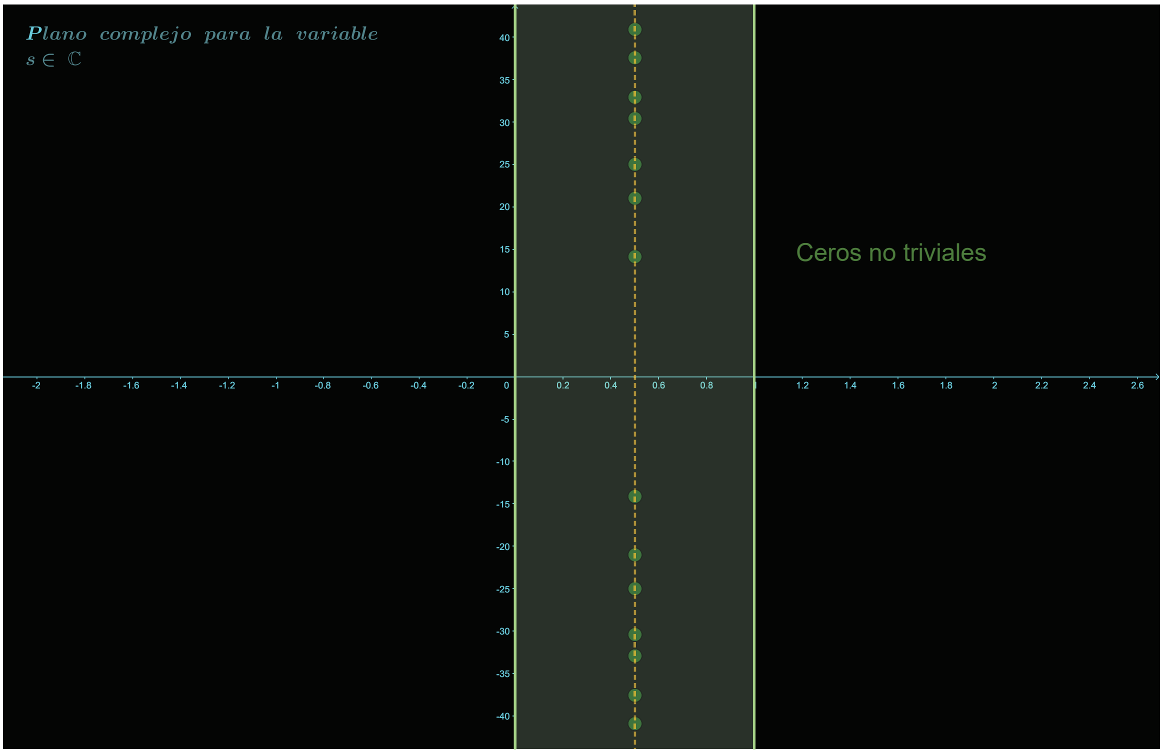

Figure 6: *Critical region in the complex plane, where for the variable $s \in \mathbb{C}$, it is impossible to ascertain $\zeta(s)$ using the Riemann Functional Equation.*

In 1859, the German mathematician Georg Friedrich Bernhard Riemann, in his doctoral thesis 'On prime numbers less than a given magnitude' [4], while developing an explicit formula for calculating the number of prime numbers less than x, conjectured that: ***'The real part of every nontrivial zero of the Zeta Function $\zeta(s)$ is $\frac{1}{2}$'***. Riemann was confident in this assertion, yet unable to prove it, leaving it as one of the most significant unproven hypotheses for over 160 years.

7 The proof of the hypothesis.

Consider $s \in \mathbb{C} : s = a + bi$, where $a, b \in \mathbb{R}$ and is a nontrivial zero such that $\zeta(s) = 0$.

According to the Riemann Functional equation, if s is a nontrivial zero, then the following holds:

$$\zeta(a+bi) = \zeta(1-a-bi) = 0 \tag{28}$$

If we substitute $a + bi$ into equation (26) and expand the expression:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{a+bi}} - \int\frac{1}{n^{a+bi}}dn - \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{n^{a+bi}}\right)\right] \tag{29}$$

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{a+bi}} - \frac{1}{1-a-bi}n^{1-a-bi} - \frac{(-1)^1 B_1}{1!}n^{-a-bi} - \frac{(-1)^2 B_2}{2!}(-a-bi)n^{-1-a-bi}\cdots\right] \tag{30}$$

Since $0 < a < 1$, upon applying the limit, the equation simplifies to:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{a+bi}} - \frac{1}{1-a-bi}n^{1-a-bi}\right] \tag{31}$$

Given that $\zeta(a+bi) = 0$, expressing it in polar form yields:

$$0 = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{e^{-bi\ln(m)}}{m^a} - \frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\left[\arctan\left(\frac{b}{1-a}\right)-b\ln(n)\right]}\right] \tag{32}$$

Then:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{e^{-bi\ln(m)}}{m^a}\right]=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\left[\arctan\left(\frac{b}{1-a}\right)-b\ln(n)\right]}\right] \tag{33}$$

Utilizing Euler's properties:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-i\sum_{m=1}^{n}\frac{\sin(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\left\{\cos\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right]\right.\right.$$
$$\left.\left.+i\sin\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right]\right\}\right] \tag{34}$$

Similarly, for the nontrivial zero $s=1-a-bi$, we obtain:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}+i\sum_{m=1}^{n}\frac{\sin(b\ln m)}{m^{1-a}}\right]=\lim_{n\to\infty}\left[\frac{n^{a}}{\sqrt{a^2+b^2}}\left\{\cos\left[arctan\left(\frac{b}{a}\right)-b\ln n\right]\right.\right.$$
$$\left.\left.-i\sin\left[arctan\left(\frac{b}{a}\right)-b\ln n\right]\right\}\right] \tag{35}$$

By considering the real part of equations (34) and (35), we get:

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\cos\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right]\right] \tag{36}$$

$$\lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}\right]=\lim_{n\to\infty}\left[\frac{n^{a}}{\sqrt{a^2+b^2}}\cos\left[arctan\left(\frac{b}{a}\right)-b\ln n\right]\right] \tag{37}$$

Setting the amplitudes equal to each other to compare equations (36) and (37), given that the axis of both functions is $Re[\zeta(a+bi)]=Re[\zeta(1-a-bi)]=0$:

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\cos\left[arctan\left(\frac{b}{1-a}\right)-b\ln n\right] \tag{38}$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{a^2+b^2}}{n^{a}}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}\right]=\lim_{n\to\infty}\cos\left[arctan\left(\frac{b}{a}\right)-b\ln n\right] \tag{39}$$

Performing operations on equation (38) to simplify the equality to a basic periodic function:

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\cos\left[arctan\left(\frac{b}{1-a}\right)\right]\cos(b\ln n)+\sin\left[arctan\left(\frac{b}{1-a}\right)\right]\sin(b\ln n)\right]$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\cos(b\ln n)+\tan\left[arctan\left(\frac{b}{1-a}\right)\right]\sin(b\ln n)\right]$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}\right]=\lim_{n\to\infty}\left[\cos(b\ln n)+\frac{b}{1-a}\sin(b\ln n)\right]$$

$$\lim_{n\to\infty}\left[\frac{\sqrt{(1-a)^2+b^2}}{n^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-\cos(b\ln n)\right]=\lim_{n\to\infty}\frac{b}{1-a}\sin(b\ln n)$$

$$\lim_{n\to\infty}\left[\frac{(1-a)\sqrt{(1-a)^2+b^2}}{bn^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}-\frac{1-a}{b}\cos(b\ln n)\right]=\lim_{n\to\infty}\sin(b\ln n) \tag{40}$$

Similarly, we manipulate equation (39) and obtain:

$$\lim_{n\to\infty}\left[\frac{a\sqrt{a^2+b^2}}{bn^a\cos\left[arctan\left(\frac{b}{a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}-\frac{a}{b}\cos(b\ln n)\right]=\lim_{n\to\infty}\sin(b\ln n) \tag{41}$$

Now we can equate equations (40) and (41):

$$\lim_{n\to\infty}\left[\frac{(1-a)\sqrt{(1-a)^2+b^2}}{bn^{1-a}\cos\left[arctan\left(\frac{b}{1-a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{a}}-\frac{1-a}{b}\cos(b\ln n)\right]=$$

$$\lim_{n\to\infty}\left[\frac{a\sqrt{a^2+b^2}}{bn^a\cos\left[arctan\left(\frac{b}{a}\right)\right]}\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}-\frac{a}{b}\cos(b\ln n)\right] \tag{42}$$

Improving the wording: "Since $a+bi$ is a zero of the Zeta Function, series $\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^a}$ and $\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}$ do not converge; however, their average value is zero. Therefore, we can write:

$$\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{a}}=\sum_{m=1}^{n}\frac{\cos(b\ln m)}{m^{1-a}}=0 \tag{43}$$

Substituting equation (43) into equation (42), we get:"

$$0-\frac{1-a}{b}\cos(b\ln n)=0-\frac{a}{b}\cos(b\ln n) \tag{44}$$

Then, by simplifying terms, factoring, and setting equal to zero, we obtain:

$$(2a-1)\cos(b\ln n)=0 \tag{45}$$

Then, the only solution for the equation is:

$$a=\frac{1}{2} \tag{46}$$

Finally, we can conclude that the Riemann hypothesis is true.

References

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Dedication.

"Call to me and I will answer you and tell you great and unsearchable things you do not know"
(Jeremiah 33:3)

I thank God for hearing my prayers and showing me the way to solve this problem. To Him be the glory. This work is dedicated to my entire family who supported me at all times and during the most difficult moments of my life, especially to my beloved wife Araceli, my two beautiful children Ocram and Arelys, and my parents who never lost faith in me.