Further investigations on Euler's odd perfect numbers

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Abstract

It is a long standing question whether there exists an odd perfect number. This article establishes a complete theory in order to prove that if an odd perfect number n exists then $n = pm^2$ with p prime and p $\equiv 1 \pmod 4$, and gcd $(p, m) = 1$.

1 Introduction

A perfect number is a positive integer that is equal to the sum of its positive divisors excluding itself. All presently known perfect numbers are even [2].

Whether there exists an odd perfect number is one of the oldest open problems in mathematics.

Perfect numbers [6][3][4][7] have continued to be of interest for thousands of years because they have been studied by some of the brightest mathematicians in history, but have yet to reveal the entirety of their nature. Around 330 B.C, the great mathematician Euclid was instrumental in the advances made in the study of perfect numbers. The result of Euclid's studies of perfect numbers is Euclid's Perfect Number Theorem.
Euclid's Theorem [5] is widely considered to be the first step mankind took to understanding the nature of perfect numbers.

Euclid's Perfect Number Theorem states that
If $2^p - 1$ is a prime number, then $(2^{p-1})(2^p - 1)$ is a perfect number.

Naturally, the question that arises after proving Euclid's Perfect Number Theorem is whether it describes all perfect numbers.

That is to say, are all perfect numbers of the form $(2^{p-1})(2^p - 1)$?
The next significant step in the answering of this question and the understanding of perfect numbers was made two thousand years after Euclid's results by Swiss mathematician Leonhard Euler. Euler proved that Euclid's formula for perfect numbers holds true for all even perfect numbers.

Euler's Perfect Number Theorem [6] states that
if n is an even perfect number, then it is of the form $n = (2^{p-1})(2^p - 1)$ where p is some prime and $2^p - 1$ is a Mersenne prime.

Not only did Euler provide us with a useful theorem for even perfect numbers, he also provided us with an

equally impactful theorem that allowed us to study the form of odd perfect numbers.
Euler's Odd Perfect Number Theorem [5] states that
Any odd perfect number n (if it exists) must be of the form $n = p^\beta m^2$ with p prime and $p \equiv \beta \equiv 1 (\text{mod } 4)$, and gcd $(p, m) = 1$.

One observation can be found in [1] where Touchard determined the pattern of an odd perfect number(assuming it's existence).
Touchard's Theorem states that any odd perfect number(if it exists) must have the form $12m + 1$ or $36m + 9$.

The next section in this article gives a detailed explanation to conclude that if odd perfect number exists then β must be 1.

2 On the existence of Odd perfect Number

By Euler's perfect number theorem we know that if n is an odd perfect number then $n = p^\beta m^2$ where p is prime, m is odd, $p \equiv \beta \equiv 1 (\text{mod } 4)$ and $\gcd(p, m) = 1$.

Our aim is to prove that $\beta = 1$. On the contrary, suppose that an odd perfect number $n = p^\beta m^2$ exists where $\beta > 1$ i.e., $\beta \geq 5$.
With this assumption we begin this section. At first, we note that $\sigma(m^2)$ is

an odd natural for every odd natural m. In fact, if $m_1, m_2, ..., m_i$ are all distinct proper ($\neq 1$) odd factors of m^2 where $m_j < m$, for all $j = 1, 2...., i$ then there are odd factors $k_1, k_2, ..., k_i$, where $k_j > m$ for all $j = 1, 2...i$ and $m_1.k_1 = m_2.k_2 = ... = m^2$. Then the sum $S = (m_1 + k_1) + (m_2 + k_2) + ... + (m_i + k_i)$ is an even natural and consequently, $\sigma(m^2) = S + 1 + m + m^2$ is odd.
At the end of the article we shall conclude that such m and p cannot exist.

Before this conclusion let us establish the following results :

Theorem 1.

(i) $\sigma(m^2) < 2m^2$

(ii) $2\frac{p-1}{p}m^2 < \sigma(m^2)$.

Proof. We have
$\sigma(p^\beta m^2) = 2p^\beta m^2$. Then $\sigma(p^\beta)\sigma(m^2) = 2p^\beta m^2$.
Now $\sigma(p^\beta) = 1 + p + p^2 + ... + p^\beta > p^\beta$ and hence $\sigma(m^2) < 2m^2$. Thus (i) is proved.
Now $\sigma(p^\beta) = \frac{p^{\beta+1}-1}{p-1} < \frac{p^{\beta+1}}{p-1}$.
Again $\sigma(p^\beta)\sigma(m^2) = 2\frac{p^{\beta+1}}{p-1}\frac{p-1}{p}m^2$. Hence it follows that $2\frac{p-1}{p}m^2 < \sigma(m^2)$. Thus (ii) is proved.

Theorem 2. The inequality $p > 2m^2$ cannot be true.

Proof. If possible, let $p > 2m^2$.
By theorem 1, we have
$\sigma(m^2) < 2m^2$ and
$$(\frac{p-1}{p})2m^2 < \sigma(m^2) \quad (1)$$
But $\frac{2m^2}{p} < 1$(by assumption). Therefore, $2m^2 - \frac{2m^2}{p} > 2m^2 - 1 \geq \sigma(m^2)$, (note that both $2m^2$ and $\sigma(m^2)$ are naturals). Hence, $(\frac{p-1}{p})2m^2 > \sigma(m^2)$, a contradiction to (1).

Theorem 3. $m^2 < p < 2m^2$ cannot be true.

Proof. If possible, let $m^2 < p < 2m^2$. We shall first prove

Lemma 1. If $p < 2m^2$ and then $\sigma(m^2) + 1 < 2m^2$.

Proof of lemma 1. By Theorem 1 (i) we have
$\sigma(m^2) < 2m^2$. If possible, let $\sigma(m^2) + 1 = 2m^2$. Now,
$\sigma(p^\beta)\sigma(m^2) = 2p^\beta m^2 = p^\beta(\sigma(m^2) + 1)$
$\Rightarrow \sigma(m^2)(\sigma(p^\beta) - p^\beta) = p^\beta$.
$\Rightarrow \sigma(m^2)\sigma(p^{\beta-1}) = p^\beta$
$\Rightarrow \sigma(m^2) = \frac{p^\beta}{1+p+p^2+....+p^{\beta-1}}$
$\Rightarrow \sigma(m^2) < p$.
$\Rightarrow \sigma(m^2) + 1 < p$.(because, both $\sigma(m^2)$ and p are odd naturals).

But we have $p < 2m^2 = \sigma(m^2) + 1$. There arises a contradiction. Hence the Lemma is proved.
Now we proceed to prove the Theorem. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (2)
Since $m^2 < p$, therefore, $\frac{2m^2}{p} < 2$. By the lemma we have $\sigma(m^2) + 1 < 2m^2$. Then $\sigma(m^2) + 2 < 2m^2$ (since both $\sigma(m^2) + 1$ and $2m^2$ are even). So, $2m^2 - \frac{2m^2}{p} > 2m^2 - 2 > \sigma(m^2)$, which contradicts (2). This completes the proof.

Theorem 4. $p < m^2 < \frac{3p}{2}$ cannot be true.

Proof. If possible, let $p < m^2 < \frac{3p}{2}$. We prove the following

Lemma 2. If $p < m^2$ then $\sigma(m^2) + 2 < 2m^2$.

Proof of lemma 2. Since $p < m^2 < 2m^2$, by Lemma 1, we have
$\sigma(m^2) + 1 < 2m^2$. But $\sigma(m^2) + 1$ is even (since $\sigma(m^2)$ is odd) and hence $\sigma(m^2) + 2 < 2m^2$. Hence the Lemma is proved.
We shall now prove the Theorem. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (3)
Since $2m^2 < 3p$. Therefore, $\frac{2m^2}{p} < 3$. By lemma 2 we have $\sigma(m^2) + 2 < 2m^2$ i.e., $\sigma(m^2) < 2m^2 - 2$. So,
$2m^2 - \frac{2m^2}{p} > 2m^2 - 3 \geq \sigma(m^2)$, which contradicts (3). This completes the proof.

Theorem 5. $\frac{3p}{2} < m^2 < 2p$ cannot be true.

Proof. If possible, let $\frac{3p}{2} < m^2 < 2p$. We first prove

Lemma 3. If $\frac{3p}{2} < m^2$ then $\sigma(m^2) + 3 < 2m^2$.

Proof of lemma 3. Since $p < m^2$, by Lemma 2, we have $\sigma(m^2) + 2 < 2m^2$.

If possible, let $\sigma(m^2) + 3 = 2m^2$.

Now, $\frac{3p}{2} < m^2 \Rightarrow 3p < 2m^2 \Rightarrow 3p + 1 \leq 2m^2$. Applying Theorem 1(ii) we get $(p-1)2m^2 < p\sigma(m^2)$
$\Rightarrow (p-1)(3p+1) < p\sigma(m^2)$
$\Rightarrow (p-1)(3p+1) + 1 \leq p\sigma(m^2)$
$\Rightarrow 3p - 2 \leq \sigma(m^2)$ (4)
Now,
$\sigma(p^\beta)\sigma(m^2) = 2p^\beta m^2 = p^\beta(\sigma(m^2) + 3)$
$\Rightarrow \sigma(m^2)(\sigma(p^\beta) - p^\beta) = 3p^\beta$.
$\Rightarrow \sigma(m^2)\sigma(p^{\beta-1}) = 3p^\beta$
$\Rightarrow (3p-2)\sigma(p^{\beta-1}) \leq 3p^\beta$ (by (4))
$\Rightarrow (3p-2)(1 + p + p^2 + ... + p^{\beta-1}) \leq 3p^\beta$, which is false (since $\beta \geq 5$). Hence the lemma is proved.
Now let us prove the Theorem. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (5)
We have assumed that $2m^2 < 4p$. Therefore, $\frac{2m^2}{p} < 4$. By lemma 3 we have $\sigma(m^2) + 3 < 2m^2$ i.e., $\sigma(m^2) + 4 < 2m^2$(since $\sigma(m^2) + 3$ and $2m^2$ are both even). So,
$2m^2 - \frac{2m^2}{p} > 2m^2 - 4 > \sigma(m^2)$, which contradicts (5). This completes the proof.

Theorem 6. $2p < m^2 < \frac{5p}{2}$ cannot be true.

Proof. If possible, let $2p < m^2 < \frac{5p}{2}$. Let us prove

Lemma 4. If $2p < m^2$ then $\sigma(m^2) + 4 < 2m^2$.

Proof of lemma 4. By Lemma 3, we have $\sigma(m^2) + 3 < 2m^2$. But $\sigma(m^2) + 3$ is even (since $\sigma(m^2)$ is odd) and hence $\sigma(m^2) + 4 < 2m^2$. Hence the Lemma is proved.

Now we move to prove the Theorem. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (6)
Since $2m^2 < 5p$. Therefore, $\frac{2m^2}{p} < 5$. By lemma 4 we have $\sigma(m^2) + 4 < 2m^2$ i.e., $\sigma(m^2) + 5 \leq 2m^2$. So,
$2m^2 - \frac{2m^2}{p} > 2m^2 - 5 \geq \sigma(m^2)$, which contradicts (6). This completes the proof.

Theorem 7. $\frac{5p}{2} < m^2 < 3p$ cannot be true.

Proof. If possible, let $\frac{5p}{2} < m^2 < 3p$. First we prove

Lemma 5. If $\frac{5p}{2} < m^2$ then $\sigma(m^2) + 5 < 2m^2$.

Proof of lemma 5. By Lemma 4, we have
$\sigma(m^2) + 4 < 2m^2$.

If possible, let $\sigma(m^2) + 5 = 2m^2$.

Now, $\frac{5p}{2} < m^2 \Rightarrow 5p < 2m^2 \Rightarrow 5p + 1 \leq 2m^2$. Applying Theorem 1(ii) we get $(p-1)2m^2 < p\sigma(m^2)$
$\Rightarrow (p-1)(5p+1) < p\sigma(m^2)$
$\Rightarrow (p-1)(5p+1) + 1 \leq p\sigma(m^2)$
$\Rightarrow 5p - 4 \leq \sigma(m^2)$ (7)
Now,
$\sigma(p^\beta)\sigma(m^2) = 2p^\beta m^2 = p^\beta(\sigma(m^2) + 5)$
$\Rightarrow \sigma(m^2)(\sigma(p^\beta) - p^\beta) = 5p^\beta$.
$\Rightarrow \sigma(m^2)\sigma(p^{\beta-1}) = 5p^\beta$
$\Rightarrow (5p-4)\sigma(p^{\beta-1}) \leq 5p^\beta$ (by (7))
$\Rightarrow (5p-4)(1 + p + p^2 + ... + p^{\beta-1}) \leq 5p^\beta$, which is false. Hence the lemma is proved.
Now let us prove the Theorem. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (8)
We have assumed that $2m^2 < 6p$. Therefore, $\frac{2m^2}{p} < 6$. By lemma 5, we have $\sigma(m^2) + 5 < 2m^2$ i.e., $\sigma(m^2) + 6 < 2m^2$(since $\sigma(m^2) + 5$ and $2m^2$ are both even). So,
$2m^2 - \frac{2m^2}{p} > 2m^2 - 6 > \sigma(m^2)$, which contradicts (8). This completes the proof.

Theorem 8. $3p < m^2 < \frac{7p}{2}$ cannot be true.

Proof. If possible, let $3p < m^2 < \frac{7p}{2}$. Let us prove

Lemma 6. If $3p < m^2$ then $\sigma(m^2) + 6 < 2m^2$.

Proof of lemma 6. By Lemma 5, we have $\sigma(m^2) + 5 < 2m^2$. But $\sigma(m^2) + 5$ is even (since $\sigma(m^2)$ is odd) and hence $\sigma(m^2) + 6 < 2m^2$. Hence the Lemma is proved.

Now we move to prove the Theorem. By Theorem 1 (ii), we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (9)

Since $2m^2 < 7p$. Therefore, $\frac{2m^2}{p} < 7$. By lemma 6 we have $\sigma(m^2) + 6 < 2m^2$ i.e., $\sigma(m^2) + 7 \leq 2m^2$. So, $2m^2 - \frac{2m^2}{p} > 2m^2 - 7 \geq \sigma(m^2)$, which contradicts (9). This completes the proof.

Theorem 9. $\frac{7p}{2} < m^2 < 4p$ cannot be true.

Proof. If possible, let $\frac{7p}{2} < m^2 < 4p$. We now prove

Lemma 7. If $\frac{7p}{2} < m^2$ then $\sigma(m^2) + 7 < 2m^2$.

Proof of lemma 7. By Lemma 6, we have $\sigma(m^2) + 6 < 2m^2$.

If possible, let $\sigma(m^2) + 7 = 2m^2$.

Now, $\frac{7p}{2} < m^2 \Rightarrow 7p < 2m^2 \Rightarrow 7p + 1 \leq 2m^2$. Applying Theorem 1(ii) we get $(p-1)2m^2 < p\sigma(m^2)$

$\Rightarrow (p-1)(7p+1) < p\sigma(m^2)$

$\Rightarrow (p-1)(7p+1) + 1 \leq p\sigma(m^2)$

$\Rightarrow 7p - 6 \leq \sigma(m^2)$ (10)

Now,

$\sigma(p^\beta)\sigma(m^2) = 2p^\beta m^2 = p^\beta(\sigma(m^2) + 7)$

$\Rightarrow \sigma(m^2)(\sigma(p^\beta) - p^\beta) = 7p^\beta$.

$\Rightarrow \sigma(m^2)\sigma(p^{\beta-1}) = 7p^\beta$

$\Rightarrow (7p-6)\sigma(p^{\beta-1}) \leq 7p^\beta$ (by (10))

$\Rightarrow (7p-6)(1 + p + p^2 + ... + p^{\beta-1}) \leq 7p^\beta$, which is false. Hence the lemma is proved.

Now let us prove the Theorem. By Theorem 1 (ii), we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (11)

We have assumed that $2m^2 < 8p$. Therefore, $\frac{2m^2}{p} < 8$. By lemma 7, we have $\sigma(m^2) + 7 < 2m^2$ i.e., $\sigma(m^2) + 8 < 2m^2$(since $\sigma(m^2) + 7$ and $2m^2$ are both even). So,
$2m^2 - \frac{2m^2}{p} > 2m^2 - 8 > \sigma(m^2)$, which contradicts (11). This completes the proof.

Now after proving the following two results we shall use Mathematical Induction:

Result A If $\frac{(2k-1)p}{2} < m^2 \Rightarrow \sigma(m^2) + (2k-1) < 2m^2$ for some k, then $kp < m^2 \Rightarrow \sigma(m^2) + 2k < 2m^2$.

Result B If $kp < m^2 \Rightarrow \sigma(m^2) + 2k < 2m^2$ for some k, then $\frac{(2k+1)p}{2} < m^2 \Rightarrow \sigma(m^2) + (2k+1) < 2m^2$.

Proof of **Result A**. Suppose that $\frac{(2k-1)p}{2} < m^2 \Rightarrow \sigma(m^2)+(2k-1) < 2m^2$ for some k. Now let $kp < m^2$. Then $\frac{(2k-1)p}{2} < m^2$ and hence by assumption $\sigma(m^2) + (2k-1) < 2m^2$. But $\sigma(m^2) + (2k-1)$ is even. So, $\sigma(m^2)+(2k-1)+1 < 2m^2$ i.e., $\sigma(m^2)+2k < 2m^2$. Thus Result A is proved.

Proof of **Result B**. Suppose that $kp < m^2 \Rightarrow \sigma(m^2) + 2k < 2m^2$ for some k. Now let $\frac{(2k+1)p}{2} < m^2$. Then $kp < m^2$ and hence by assumption $\sigma(m^2)+2k < 2m^2$. We shall prove that $\sigma(m^2)+(2k+1) < 2m^2$. If possible, let $\sigma(m^2) + (2k+1) = 2m^2$. Now $\frac{(2k+1)p}{2} < m^2 \Rightarrow (2k+1)p < 2m^2 \Rightarrow (2k+1)p+1 \le 2m^2$. Applying Theorem 1(ii) we get $(p-1)2m^2 < p\sigma(m^2)$
$\Rightarrow (p-1)((2k+1)p+1) < p\sigma(m^2)$
$\Rightarrow (p-1)((2k+1)p+1)+1 \le p\sigma(m^2)$
$$\Rightarrow (2k+1)p - 2k \le \sigma(m^2) \tag{12}$$
Now,
$\sigma(p^\beta)\sigma(m^2) = 2p^\beta m^2 = p^\beta(\sigma(m^2) + (2k+1))$
$\Rightarrow \sigma(m^2)(\sigma(p^\beta) - p^\beta) = (2k+1)p^\beta$.
$\Rightarrow \sigma(m^2)\sigma(p^{\beta-1}) = (2k+1)p^\beta$
$\Rightarrow ((2k+1)p - 2k)\sigma(p^{\beta-1}) \le (2k+1)p^\beta$ (by (12))
$\Rightarrow ((2k+1)p-2k)(1 + p + p^2 + ... + p^{\beta-1}) \le (2k+1)p^\beta$, which is false. Thus Result B is proved.

Hence by Mathematical Induction we now have proved the following Lemma :

Lemma for Odd Perfect Number

(i) For every natural k, $kp < m^2 \Rightarrow \sigma(m^2) + 2k < 2m^2$.

(ii) For every natural k, $\frac{(2k+1)p}{2} < m^2 \Rightarrow \sigma(m^2) + (2k+1) < 2m^2$.

Now We can prove

Theorem 10.

(i) $kp < m^2 < \frac{(2k+1)p}{2}$ cannot be true for any natural k.

(ii) $\frac{(2k+1)p}{2} < m^2 < (k+1)p$ cannot be true for any natural k.

Proof of (i). if possible, let $kp < m^2 < \frac{(2k+1)p}{2}$ hold for some natural k. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (13)
Since $2m^2 < (2k+1)p$. Therefore, $\frac{2m^2}{p} < 2k+1$. By lemma (i) for Odd Perfect Number we have $\sigma(m^2) + 2k < 2m^2$ i.e., $\sigma(m^2) + (2k+1) \leq 2m^2$. So,
$2m^2 - \frac{2m^2}{p} > 2m^2 - (2k+1) \geq \sigma(m^2)$, which contradicts (13). This completes the proof of (i).

Proof of (ii). If possible, let $\frac{(2k+1)p}{2} < m^2 < (k+1)p$ hold for some natural k. By Theorem 1 (ii),
we have $2m^2 - \frac{2m^2}{p} < \sigma(m^2)$ (14)
We have assumed that $2m^2 < (2k+2)p$. Therefore, $\frac{2m^2}{p} < 2k+2$. By lemma (ii) for Odd Perfect Number we have $\sigma(m^2) + (2k+1) < 2m^2$ i.e., $\sigma(m^2) + (2k+2) < 2m^2$ (since $\sigma(m^2) + (2k+1)$ and $2m^2$ are both even). So, $2m^2 - \frac{2m^2}{p} > 2m^2 - (2k+2) > \sigma(m^2)$, which contradicts (14). This completes the proof.

Thus from all the above theorems we have the general result as follows:

Theorem 11.

If an Euler's odd perfect number $n = p^\beta m^2$ exists (where $\beta > 1$) then the following results hold.

(i) $p > 2m^2$ cannot be true.

(ii) $m^2 < p < 2m^2$ cannot be true.

(iii) For any natural n, $np < m^2 < \frac{(2n+1)p}{2}$ cannot be true.

(iv) For any natural n, $\frac{(2n+1)p}{2} < m^2 < (n+1)p$ cannot be true.

But m and p are naturals. So, at least one of the four inequalities as mentioned in Theorem 11 must hold. This can be justified as follows.
If $p < m^2$ then we must have a greatest natural, say, n^* such that $n^*p < m^2$ (note that $\gcd(p, m) = 1$, hence, equality cannot occur,i.e., $m^2 \neq n^*p$). Then it follows that
either $n^*p < m^2 < \frac{(2n^*+1)p}{2}$ or $\frac{(2n^*+1)p}{2} < m^2 < (n^* + 1)p$ must hold (since $\gcd(p, m) = 1$, equality never occurs).

Thus it is clearly justified that if m and p exist then at least one inequality must hold. This observation contradicts Theorem 11. Hence m and p cannot exist.

Now we definitely have the following

CONCLUSION : If an odd perfect number exists then β must be 1 .

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