PROOF FOR LOCALIZED ENERGY OR INERTIA IN CURVED SPACETIME

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Abstract. General Relativity is known for its local character; unlike the omnipresence, i.e., instancy/immediacy of Einstein's "spooky action" while Quantum Entanglement. Hence, it is expected that a local observer can measure/harvest the energy. This means that the famous problem of energy localization should have a positive solution. I introduce an inertial coordinate system (a local inertial tetrad) and derive conservation laws from the covariant four-dimensional divergence of the energy-momentum tensor. As an introduction to the revealing power of such tetrads, different mathematical methods have coincided in showing that Black Holes can start shrinking and, in the finale, completely vanish the falling test objects; the annual pointing of the rotational axis of Earth on the North Star area is also explained.

PACS2010: 88.05.Gh, 71.15.Nc, 88.60.nh, 88.05.Rt, 88.85.J-, 04.70.Bw, 02.40.Xx.
MSC2020: 83C10, 83C40, 83C55, 83C57, 83C75
Keywords: Energy, Black Hole, Singularity.

1. Importance of this research

Please read my explorations in the field of inertia. Why inertia? Please, recall that Sir Isaac Newton's first law is all about inertia.

Yes, the notion of an "inertial frame of reference in General Relativity" is known [1], but upon a deep examination, I know that I have made substantial progress because I have convincingly discovered (confirmed in many alternative ways) a never known effect: a Black Hole can apply size reduction to the falling objects even at a significant distance from the event horizon, and this is not in contradiction with the observed "Spaghettification/noodle effect" in Ref. [2].

As well as the "energy localization problem", which is troubling the Physics Community, is convincingly solved in my paper.

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2. Introduction and Summary of results

There is the demand for an inertial tetrad in the Galilean Postulate of Relativity because the measurement apparatue is acceleration-dependent, vibration-dependent. Therefore, strictly speaking, Science must be done with an observers from the inertial measurement systems only.

By recalling the basic need to study problems in an inertial coordinate system (ICS tetrad), I found no problem with the local conservation of the fundamental laws of Nature. But others have faced significant problems [3].

As an application of such tetrads, different mathematical methods have coincided in showing that Black Holes can start shrinking and, in the finale, completely vanish the falling test bodies. How? First of all, consider for a moment a distant analogy. It is the collapsing cloud of dust. It is natural to feel the shrinking of yourself together with the cloud, being a free-falling observer in a shrinking dust cloud.

In the case of a Black Hole, there are two opposite effects: EffectN and EffectS. The distant observer never experiences a test body crossing the event horizon (due to gravitational time dilation). Hence, he sees that the falling body starts to fall slower and shrinks in the radial direction. This is EffectS. The opposite EffectN is that the legs of the falling body (the latter should belong not to a doomed astronaut but to a test dummy of scientific endeavor) are closer to the singularity than the head; hence, under Newton's Laws, the Black Hole spaghettified the falling body. And indeed, this happens at the first stage of accretion and fall of matter (as seen in Ref. [2]).

But in the second phase, the factor EffectS grows stronger and overcomes EffectN. The body begins to shrink and to self-collapse. Why? It is being told to public that the final stage of any falling body is the smallest point at the central singularity. Hence, there cannot be an exclusive stretching out between the two evolutionary states of the body: an initial large object and a final tiny dot. There has to be stretching first. But near the "dot stage" (at $r = 0$), there must be shrinking to prepare the object to unite with the dot.

Therefore, shrinking has to happen, even to an indefinite size at $r = r_m > 0$. I have "seen" this happening outside the central singularity (latter one is at $r = 0$), where the geodesic motion stops: no trace of the falling body at $0 < r < r_m$. Notably, the "Riemann curvature tensor" has no "physical" singularity at $r = r_m$, as I have seen in a local tetrad using methods of Ref. [8].

Such effect occurs at the radial straight line falling with $\theta = \text{const}$, $\phi = \text{const}$, where the falling body re-bounce at r_m is not possible. The re-bounce would mean that at $r = r_m$, the body "flies by", having the closest approach to the Black Hole singularity. Such an $r_m \neq 0$ effect is characteristic for the Kerr, Reissner-Nordström, and Kerr–Newman space-times. In the case of Reissner-Nordström space-time, the direct radial falling is studied, so it has no bouncing or "turning" point. Moreover, the Kerr space-time is considered, and falling goes along the axis of rotation $\theta = 0$, so it has no turning point. In the case of the general falling with $\theta \neq 0$, the space-time point with r_m has zero velocity of the falling body: $u^r = u^\theta = u^\phi = 0$, which points not to a turning point but to a complete stop and vanishing by shrinking to an indeterminate size.

One of my methods is the study of a falling drop of "perfect fluid". To avoid any misconception, I declare that my "density equation" [Eq. (33)] is applied to the falling of a small drop, not a star's evolution/collapse. Here I have given my results and a verbal explanation for them. But below are the methods and mathematics of General Relativity.

3. The idea of Energy localization and inertial coordinate system

The Christoffel Symbols ($\Gamma^{\hat{\alpha}}_{\hat{u}\,\hat{\mu}}$) are necessary in curved coordinate systems; it means their metric is not Minkowski. [8] Professor Lev Davidovich Landau has proven on a single page of his famous book "Theory of Fields" that there is a coordinate transformation at any point of space-time, in that all Christoffel Symbols turn to zero. The proof might be in Ref. [4], but I used the original, 1960 AD edition, printed in his native languadge. In the following, under $x^\nu = x^\nu(\{x^{\hat{\mu}}\})$ is meant just that Landau's transformation.

The definition of covariant derivatives [8] imply that if $T^{\hat{u}\,\hat{\mu}}_{;\hat{u}} = 0$ holds together with $\Gamma^{\hat{\alpha}}_{\hat{u}\,\hat{\mu}} = 0$ holding, then $T^{\hat{u}\,\hat{\mu}}_{,\hat{u}} = 0$ holds too. The latter formula means that the classical conservation laws do hold in the vicinity of the space-time point $P(\{x^\nu\})$.

Here and in the following, the index with a comma means an ordinary derivative with respect to the space-time coordinate $x^{\hat{u}}$ while the index with semicolon means the covariant derivative using Christoffel symbols.

Here and in the following, I am using the "Einstein summation convention." For example, $a^\nu\, b_\nu \;\equiv\; a^t\, b_t + a^r\, b_r + a^\theta\, b_\theta + a^\phi\, b_\phi$, where $x^\nu = t,\, r,\, \theta,\, \phi$ are so called "curvature coordinates," or "coordinates of background spacetime." To cite an Encyclopedia of 2023 AD, *Einstein summation is a notational convention that implies summation over a set of indexed terms in a formula. It is used in mathematics and physics to simplify expressions involving vectors, matrices, and tensors. The main rule of Einstein summation is that repeated indices are implicitly summed over. Each index can appear at most twice in any term. It was introduced by Albert Einstein in 1916 AD.*

And if you calculate such coordinate systems along the world-line of the observer Dmitri Martila, then Dmitri will see the energy-momentum conservation in his vicinity. Such co-moving coordinate systems will have the denotation $x^{\hat{u}}$. Therefore,

$$T^{\hat{u}\,\hat{\mu}}_{,\hat{u}} \equiv \frac{\partial T^{\hat{u}\hat{\mu}}}{\partial x^{\hat{u}}} = 0\,. \tag{1}$$

My contribution is that the inertial orthonormal tetrad (in the following, it is ICS – inertial coordinate system) is such a coordinate system at event $P(\{x^\nu\})$. The local matrix of the coordinate transformation $x^\nu = x^\nu(\{x^{\hat{\mu}}\})$ is

$$M^\nu_{\hat{\mu}} = \frac{\partial x^\nu}{\partial x^{\hat{\mu}}}\bigg|_{\{x^{\hat{\mu}}\}=0}\,. \tag{2}$$

The localized (i.e., the entire set $\{x^{\hat{\mu}}\} \to 0$) tetrad is a local ICS system of four vectors given at "space-time event" $P(\{x^\nu\}) \equiv P(\{x^{\hat{\mu}}\} = 0)$. The vectors are: $e^\nu_{\hat{\mu}} \equiv M^\nu_{\hat{\mu}}$. They are numerated by $\{\hat{\mu}\} = \hat{0}, \hat{1}, \hat{2}, \hat{3}$. The index $\{\nu\} = 0, 1, 2, 3$ numerates components of a vector in a background space-time.

The rate of a vector in ICS has

$$\frac{d\,A^{\hat{u}}}{d\tau} = e^{\hat{u}}_\alpha \frac{D\,A^\alpha}{d\tau}\,, \tag{3}$$

where the covariant τ-derivative $D\,A^\alpha/d\tau$ is a tensor [4, 8]. All this means that $d\,A^{\hat{u}}/d\tau$ is a tensor too.

Therefore, if $A^{\hat{u}}$ is meant to be conserving, i.e., $A^{\hat{u}} = \overrightarrow{\mathrm{const}}$, then

$$\frac{d\,A^{\hat{u}}}{d\tau} = 0\,, \quad \frac{D\,A^\alpha}{d\tau} = 0\,. \tag{4}$$

Then,

$$A^\alpha = e^\alpha_{\hat{u}}\,A^{\hat{u}}\,, \quad A^{\hat{u}}\,\frac{D\,e^\alpha_{\hat{u}}}{d\tau} = 0\,, \tag{5}$$

where both of Eqs. (4) were used. Now, because $A^{\hat{u}}$ can be an arbitrary conserving vector, the necessary condition for Eq. (5) to hold is

$$\frac{D\,e^{\alpha}_{\hat{u}}}{d\tau} = \frac{d\,e^{\alpha}_{\hat{u}}}{d\tau} + \Gamma^{\alpha}_{\beta\,\gamma}\,e^{\beta}_{\hat{u}}\,u^{\gamma} = 0\,. \tag{6}$$

Latter is my definition of an inertial tetrad; these are such crucially important tetrads, in which the conservation of vectors and tensors is possible. Hence, the energy-momentum conservation happens there, making energy harvesting a local process.

This formula solves the Energy Localization problem in General Relativity in the following way. The famous formula is [4, 8]

$$T^{\nu\mu}_{;\nu} = 0\,. \tag{7}$$

Energy-momentum conservation

$$T^{\hat{u}\,\hat{\mu}}_{,\hat{u}} = 0 \tag{8}$$

is going on in the ICS. Left-hand side of the Eq. (3) is a tensor; therefore, $T^{\hat{u}\,\hat{\mu}}_{,\hat{u}}$ is also tensor, but in ICS. It must hold because, using the coordinate transformation, the original tensor components $T^{\nu\mu}_{;\gamma}$ become tetrad components $T^{\hat{u}\,\hat{\mu}}_{,\hat{\gamma}}$ And then, I sum over $\hat{u} = \hat{\gamma}$ using Einstein's summation rule.

All this means that

$$\Gamma^{\hat{\alpha}}_{\hat{u}\,\hat{\mu}} = 0\,. \tag{9}$$

Please, recall that due to the Strong Equivalence Principle, the free-moving laboratory's physical laws are independent of gravity. [9] Latter means that the Christoffel symbols are not necessary; so, they can vanish.

It turned that Eq. (6) is the definition of a geodesic vector. [8] Therefore, tetrad vectors $e^{\nu}_{\hat{\mu}}$ are all geodesic vectors in ICS,

$$\frac{D\,e^{\nu}_{\hat{0}}}{d\tau} = \frac{D\,e^{\nu}_{\hat{1}}}{d\tau} = \frac{D\,e^{\nu}_{\hat{2}}}{d\tau} = \frac{D\,e^{\nu}_{\hat{3}}}{d\tau} = 0\,. \tag{10}$$

For instance, one has [8]

$$\frac{D\,e^{\nu}_{\hat{3}}}{d\tau} = \frac{d\,e^{\nu}_{\hat{3}}}{d\tau} + \Gamma^{\nu}_{\mu\,\alpha}\,e^{\mu}_{\hat{3}}\,u^{\alpha}\,, \tag{11}$$

where u^{α} is the four-dimensional velocity of ICS, to which the tetrad vectors are "attached".

Working in the Schwarzschild metric, I have managed to find the following ICS,

$$e^{\hat{t}}_{\hat{\mu}} = \left(\frac{4\sqrt{70}}{35},\, 0,\, 0,\, -\frac{10}{\sqrt{7}} \right) , \tag{12}$$

$$e^{\hat{r}}_{\hat{\mu}} = \left(\frac{2}{\sqrt{35}}\cos(w\,\tau),\, -\frac{\sqrt{5}}{2}\sin(w\,\tau),\, 0,\, -\frac{20\sqrt{14}}{7}\cos(w\,\tau) \right) , \tag{13}$$

$$e^{\hat{\theta}}_{\hat{\mu}} = (0,\, 0,\, r,\, 0)\,, \tag{14}$$

$$e^{\hat{\phi}}_{\hat{\mu}} = \left(\frac{2}{\sqrt{35}}\sin(w\,\tau),\, -\frac{\sqrt{5}}{2}\cos(w\,\tau),\, 0,\, -\frac{20\sqrt{14}}{7}\sin(w\,\tau) \right) . \tag{15}$$

where $w = \sqrt{10}/100$, $M = 1$, $r = 10 = \text{const}$. Let this tetrad be co-moving with Earth, namely, all four vectors are in the center of the Earth. These tetrad vectors enable a conserving vector ($A^{\hat{u}}$), which is constantly directed into North Star. Why? Because this tetrad vectors contain periodic functions $\cos(w\,\tau)$ and $\sin(w\,\tau)$ only.

The real-life situation was simplified because it has not acknowledged the size of the planet; the latter was taken as zero for the sake of argument. But a more detailed application of ICS systems should reveal the slow precession of the planet axis. A recent paper about precession is in Ref. [5].

4. Four mutually consistent methods

A fluid drop falls along the geodesic line because the drop size is negligible. There is water in heaven [6]. As a background example, I consider the Schwarzschild metric of the space-time, $g_{\nu\mu} = \text{diag}(-(1 - 2M/r), 1/(1-2M/r), r^2, r^2\sin^2\theta)$. One finds velocity vector using the "integral of motion" $u_t = -E = \text{const}$, and the norm given by $u_\nu\, u^\nu = -1$. The non-zero components are [8]

$$u_t = -E\,, \quad u_r = -\frac{\sqrt{E^2 - 1 + (2M/r)}}{1-(2M/r)}\,, \tag{16}$$

where $E = \sqrt{1-(2M/r_0)}$. The M, Q, $S^{\hat{u}}$, τ, and r are being measured in meters: they are "geometrized." The initial velocity (at $r = r_0$) is zero, $u_r = 0$. The free-falling ICS has a time-like geodesic vector $e^{\hat{0}}_\nu = u_\nu$ and space-like vectors $e^{\hat{1}}_\nu = (A, H, 0, 0)$ (which is radially directed), $e^{\hat{2}}_\nu = (0, 0, r, 0)$, and $e^{\hat{3}}_\nu = (0, 0, 0, r\sin\theta)$, with the inner product $e^{\hat{q}}_\alpha\, e^{\hat{u}\,\alpha} = \eta^{\hat{q}\,\hat{u}} = \text{diag}(-1, 1, 1, 1)$.

5. First method: Alternative to the known deviation Equation

A free, small particle falls following a "geodesic" trajectory in the four-dimensional space-time, i.e., a geodesic world-line. There can be several small particles freely moving in space-time along geodesics. While following the propagation of neighboring geodesics, people see that they start to deviate from each other more and more, going each own way.

The derivation of the Geodesics Deviation Equation is in Ref. [8], pages 58 and 291. The bundle of geodesic world-lines is $x^\alpha = x^\alpha(\lambda, \eta)$ and the tangent vector to a geodesic trajectory (one from the bundle) is $u^\alpha = \partial x^\alpha / \partial \lambda$.

People are writing very complicated papers (unlike my simple paper) because they are using the famous second-order Deviation Equation [7], latter is in Eq. (30). However, I present an easily accessible way to study problems through the lens of the first-order Geodesics Deviation Equation in Eq. (20). Please note that, unlike the known Geodesics Deviation Equation, Eq. (20) includes the property of the bundle of geodesic world-lines: a starting area with $E = E(r_0) \equiv \eta$, whereas proper time runs along each geodesic $\tau \equiv \lambda$.

One can write open:

$$U^\alpha(\{x^\nu\};\, \lambda,\, \eta) = U^\alpha(\{x^\nu(\lambda,\, \eta)\};\, \lambda,\, \eta) = u^\alpha(\lambda,\, \eta) \tag{17}$$

with

$$U^\alpha_{,\nu} \equiv \frac{\partial U^\alpha(x^0, x^1, x^2, x^3;\, \lambda,\, \eta)}{\partial x^\nu} \neq 0\,. \tag{18}$$

Because mathematically speaking

$$\frac{\partial^2 x^\alpha}{\partial \eta\, \partial \lambda} = \frac{\partial^2 x^\alpha}{\partial \lambda\, \partial \eta}\,, \tag{19}$$

one has

$$\frac{\partial n^\alpha}{\partial \lambda} = \frac{\partial u^\alpha}{\partial \eta}\,, \tag{20}$$

where $n^\alpha = \partial x^\alpha / \partial \eta$. With $n^\alpha = n^{\hat{u}}\, e^\alpha_{\hat{u}}$, where $n^{\hat{u}}$ is the projection of the vector n^α onto the ICS. This turns into

$$\frac{d\, n^{\hat{u}}}{d\lambda}\, e^\alpha_{\hat{u}} = \frac{\partial\, u^\alpha}{\partial \eta} - n^{\hat{u}}\, \frac{\partial\, e^\alpha_{\hat{u}}}{\partial \lambda}\,. \tag{21}$$

Now, because of Eq. (17), one has

$$\frac{\partial u^\alpha}{\partial \eta} \equiv U^\alpha_{,\nu}\, \frac{\partial x^\nu}{\partial \eta} + \frac{\partial U^\alpha}{\partial \eta}\,, \tag{22}$$

where

$$\frac{\partial\, x^\nu}{\partial \eta} = n^\nu = n^{\hat{u}} e^\nu_{\hat{u}}\,. \tag{23}$$

To make my point, I will consider a small pressure-free ($p = 0$) dust cloud in this section of the paper.

If the body falls in the Schwarzschild metric with proper time $\tau \equiv \lambda$, I come to

$$M\, n^{\hat{1}} + \frac{d\, n^{\hat{1}}}{d\tau}\, r\, \sqrt{r^2\,(E^2 - 1) + 2\, M\, r} - r^2 = 0\,, \tag{24}$$

and the τ-derivative (note that $r = r(\tau)$) of both sides of Eq. (24) results in

$$\frac{d^2\, n^{\hat{1}}}{d\tau^2} = \frac{2M}{r^3}\, n^{\hat{1}}\,. \tag{25}$$

The proper (i.e., directly measurable) distance is given by $S^{\hat{u}} = \Delta\eta\, n^{\hat{u}}$, if the constant $\Delta\eta$ is small.

The Strong Equivalence principle implies [9], what the same time shall be in the locality/vicinity of the observer, namely $S^{\hat{0}} = 0$. By taking τ-derivative of both sides of $S^{\hat{0}} = 0$, I come to $d\, S^{\hat{0}}/d\tau = 0$. So, solution given in Eqs. (24) and (25) was derived using fixed $S^{\hat{0}} = d\, S^{\hat{0}}/d\tau = 0$. Therefore, $S^{\hat{1}}$ can be recognized as the proper distance between the dust particles. Vector $S^{\hat{1}}$ is radially directed, i.e., it is placed along the line which connects a falling body and the Schwarzschild Black Hole.

Amazingly, despite the positive acceleration of deviation, the radial size of the body can shrink,

$$f = \frac{d^2\, S^{\hat{1}}}{d\tau^2} > 0\,, \quad \frac{d\, S^{\hat{1}}}{d\tau} < 0\,, \tag{26}$$

if $M\, n^{\hat{1}} > r^2$. I am giving the following explanation for it. The deviation forces (f) are not forces at all. Why? The Strong Equivalence Principle stays clear: the Physics of the small laboratory is not affected by the outside curvature of space-time. [9] So, introducing an alien force f into such an "oasis" is conceptually wrong.

6. Second Method: Known Deviation Equation agrees

The pressure is still absent in this section: $p = 0$. In the inertial tetrad, one has

$$\frac{d^n\, h^{\hat{u}}}{d\tau^n} = e^{\hat{u}}_\alpha\, \frac{D^n\, h^\alpha}{d\tau^n}\,, \tag{27}$$

where

$$h^{\hat{u}} = e^{\hat{u}}_{\mu}\, h^{\mu}\,, \qquad h^{\alpha} = e^{\alpha}_{\hat{u}}\, h^{\hat{u}}\,. \tag{28}$$

The inertial tetrad is defined by

$$\frac{D\, e^{\alpha}_{\hat{u}}}{d\tau} = \frac{d\, e^{\alpha}_{\hat{u}}}{d\tau} + \Gamma^{\alpha}_{\beta\,\gamma}\, e^{\beta}_{\hat{u}}\, u^{\gamma} = 0\,. \tag{29}$$

It is known that [8]

$$\frac{D^2\, n^{\alpha}}{d\tau^2} = -R^{\alpha}_{\mu\,\rho\,\nu}\, u^{\mu}\, u^{\nu}\, n^{\rho}\,, \tag{30}$$

where $R^{\alpha}_{\mu\,\rho\,\nu}$ is the famous Riemann curvature tensor. If $S^{\hat{0}} = 0$ is fixed, then

$$\frac{d^2\, S^{\hat{1}}}{d\tau^2} = -e^{\hat{1}}_{\alpha}\, R^{\alpha}_{\mu\,\rho\,\nu}\, u^{\mu}\, u^{\nu}\, (e^{\rho}_{\hat{1}}\, S^{\hat{1}})\,, \tag{31}$$

which, in the case of the Schwarzschild metric, gives

$$\frac{d^2\, S^{\hat{1}}}{d\tau^2} = \frac{2M}{r^3}\, S^{\hat{1}}\,, \tag{32}$$

exactly matching Eq. (25).

My way of getting Eq. (31) was the following. I multiplied both sides of Eq. (30) by $e^{\hat{u}}_{\alpha}$ and used Einstein's rule of summation (over index α). After that, I used Eq. (27) with $n = 2$. The formula from Section 5, which is $S^{\hat{u}} = \Delta\eta\, n^{\hat{u}}$, where $\Delta\eta = \text{const}$, was used too.

7. Third Method: density from energy-momentum

This section allows a non-zero pressure $p \neq 0$. It is known (from Ref. [8], pages 226–227, see Appendix), that the rate of compression of a perfect fluid behaves as

$$\frac{d\,\rho}{d\tau} = -(\rho + p)\, u^{\nu}_{;\nu}\,. \tag{33}$$

If one inserts the velocity u^{ν} into the divergence, one gets to know that $u^{\nu}_{;\nu}$ behaves like $1/u^r$, latter shows $-\infty$ in the limit $r \to r_m$. For the Schwarzschild Black Hole, one has

$$H = u^{\nu}_{;\nu} = M\, \frac{4\, r - 3\, r_0}{\sqrt{2\, M\, r_0\, r^3\, (r_0 - r)}}\,, \tag{34}$$

with the zero at $r = 3\, r_0/4$ as the start of the compression. At the initial moment (i.e., $r = r_0$), $H > 0$, and it is infinite. It behaves like $1/\sqrt{r_0 - r}$. It means: the drop's density goes down but in a finite proportion, $\int (d\rho/d\tau) d\tau < \infty$. Why? It is the very start of "Spaghettification" [2], predicted by Newton's Theory of Gravity.

Then the $H < 0$ stage starts at $r = 3\,r_0/4$: the drop shrinks. Notably, this happens at an infinite distance from the Black Hole if r_0 is infinite. This effect does not fit the intuition, where the gravity deviation forces are trying to rip apart the "falling astronaut body" in a tragic scenario. This would be an unexpected result for Sir Newton's age, even though I have a weak gravity field at $r = (3/4)\,r_0 \gg 2\,M$. The deadly ripping with extremely large $H > 0$ never begins; however, $H < 0$ holds at $r = r_m = 0$ and is infinite. At this moment, H behaves like $-1/r^{3/2}$, the integral of which is diverging at the curvature singularity $r = 0$.

At a more complicated Black Hole than idealistic Schwarzschild black hole, $r_m \neq 0$ holds. The drop's density while r approaches r_m diverges because of

$$\frac{d\,\rho}{\rho} = \left(-H - H\,\frac{p}{\rho}\right)\,d\tau\,, \tag{35}$$

which is the rewritten Eq. (33). Integration of both sides of Eq. (35) produces

$$\ln(C\,\rho) = \int\left(-H - H\,\frac{p}{\rho}\right)\,d\tau = \int\left(\frac{H}{u^r} + \frac{H}{u^r}\,\frac{p}{\rho}\right)\,dr = \infty\,, \tag{36}$$

where C is a constant of integration; and the definition of radial velocity component $u^r = dr/d\tau$ was used. Why? The H behaves like $1/u^r$ at r_m, but $u^r = 0$ at $r \to r_m$.

8. Fourth Method: geometric density change

In this section, a small layer of pressure-free and turbulence-free dust is falling in a space-time with the Schwarzschild Black Hole. The entire cloud remains to evolve within the solid angle $\Omega = \text{const}$ throughout the entire process of the fall. The proper thickness of the dust layer is $S^{\hat{1}}$. Then, for pure geometrical reason, $\rho = K/(S^{\hat{1}}\,r^2)$, where the $K = \text{const}$. Then, by taking the τ-derivative of both sides, i.e.,

$$\frac{d\rho}{d\tau} = \frac{d}{d\tau}\left(\frac{K}{S^{\hat{1}}(\tau)\,r^2(\tau)}\right)\,, \tag{37}$$

one has

$$\frac{d\rho}{d\tau} = \rho\,\frac{3\,r_0 - 4\,r}{2\,\sqrt{r_0\,r^3\,(r_0 - r)}} + O(\Delta\eta)\,, \tag{38}$$

which coincided with Eqs. (33) and (34) for a small falling object (dust cloud) with initial radial size Δr_0. If the Δr_0 is small, $\Delta\eta$ is small as well because they are connected via $E(r_0) \equiv \eta$. The latter formula is from Section 5.

I have used $p = 0$ and Eqs. (24) and (37) with $M = 1/2$.

9. Abrupt-end-geodesics

Now, I consider a Kerr Black Hole with mass $M = 1/2$ and rotation $a = 1/4$. A test body starts falling from $\theta_0 = \pi/4$ at a large distance $r_0 = 20$ with zero initial velocity. The radial component of the velocity vector is given by [8]

$$u^r \equiv \frac{d\,r}{d\tau} = -\frac{\sqrt{B}}{r^2 + (1/16)\,\cos^2\theta}\,, \tag{39}$$

where $B = -(640/12801)\,r^4 + r^3 - (742460/155672961)\,r^2 + (12481/194576)\,r - (62405/622691844)$. Presence of $\sqrt{B}$ demands positivity of B; but for $r < r_m = 1/640$ one has $B < 0$. Therefore, there is no falling body in $0 \le r < r_m$ because the trajectory is impossible in $r < r_m$. Note that the Black Hole tidal forces do not stretch the body apart but compress it to a point size. At the $r = r_m$, $u^r = u^\theta = u^\phi = 0$ holds (to demonstrate, I have used [8]), which points not to a turning/bouncing point but to a complete stop and vanishing by shrinking to an indeterminate size.

Now, I consider velocity of free fall in the Reissner-Nordström metric [8],

$$u^r \equiv \frac{dr}{d\tau} = -\frac{\sqrt{B}}{r^2}\,, \quad u^\phi = u^\theta = 0\,, \tag{40}$$

where $B = E^2\,r^4 - (r^2 - 2\,M\,r + Q^2)\,r^2$. Let me choose $Q = 1/5$ and $M = 1/2$. Zero initial velocity ($B = 0$ at $r = r_0 = 20$) determines the trajectory with

$$E = \frac{\sqrt{9501}}{100}\,. \tag{41}$$

Therefore

$$B = -\frac{499}{10000}\,r^4 + r^3 - \frac{r^2}{25}\,, \tag{42}$$

which is negative in $r < r_m = 20/499$. Thus, at r_m one has $u^r = 0$.

10. Appendix: density rate for perfect fluid

Consider a drop of "perfect fluid" falling into a Black Hole. Because the drop is small, every part of it has about velocity of the fall. The equation of matter is $T^{\mu\,\nu}_{;\nu} = 0$; thus, $u_\mu\,T^{\mu\,\nu}_{;\nu} = 0$, where

$$T^{\mu\,\nu} = (\rho + p)\,u^\mu\,u^\nu + p\,g^{\mu\,\nu}\,. \tag{43}$$

To write open,

$$-(\rho+p)_{,\nu}\, u^{\nu} - (\rho+p) u^{\nu}_{;\nu} + (\rho+p)\, u^{\nu}\, u^{\mu}_{;\nu}\, u_{\mu} + p_{,\nu}\, u^{\nu} = 0\,, \tag{44}$$

where $u^{\mu}_{;\nu}\, u_{\mu} = 0$ was used because $(u^{\mu}\, u_{\mu})_{;\nu} = (-1)_{;\nu} = 0$. Then

$$-\frac{d(\rho+p)}{d\tau} - (\rho+p)\, u^{\nu}_{;\nu} + \frac{d\,p}{d\tau} = 0 \tag{45}$$

because I have velocity definition: $u^{\nu} = dx^{\nu}/d\tau$.

Finally,

$$\frac{d\rho}{d\tau} = -(\rho+p)\, H\,, \tag{46}$$

where $H = u^{\nu}_{;\nu}$.

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