Part 1: A Sketch of Effective Learning Using Technology

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Abstract

A problem set from a calculus text is solved. The goal is to see whether or not modern calculators and a CAS are genuinely helpful.

Introduction

Being a child of the late 60's early 70's, I seem to be enamored of Thomas's 4th Edition *Calculus and Analytic Geometry* [3]. I've checked out later versions of this classic [4] and they just don't seem to quite resonate with my aging brain as well. I've taught from Larson and Edward's [1] equivalent and have a copy of Stewart's even more frequently adopted *Calculus*[1] [2]. I really appreciate both of these a lot. But ...

It is curious, in MHO, that this edition of Thomas is before TI calculators and computer algebra systems (CASs), yet the problems seem to be generally more challenging in Thomas than the contemporary books mentioned. Thomas does provide at the end of chapters sets of problems that seem designed to synthesize knowledge of everything to date in the book. This combined with relatively few and just drill problems after sections seem the perfect combination of *getting to know the material* and then *tweaking one's understanding of it within an evolving general framework* later. That said, there are places in Stewart and Larson where things have evolved for the better: see the three's derivations of the product and quotient rules, for example. Newer is better, but still my aesthetic has it that all

[1] The other author of interest is Thomas Apostol and his two volume calculus texts.

could be helped by rough drafts first that suggest the truth of some math, then some drill, and then, maybe, proofs and rigor last.

I call this scientific math. You make an hypothesis, say, that finding limits of functions might be helped by knowing their derivatives. You fiddle with some easy cases and, if lucky, start to have the *emotion* that you are on the right track. You do a bunch of problems and build up even more confidence and then you start to dream about how you might really prove your hypothesis is true – with what assumptions (weakened or strengthened).

We'll attempt a live demonstration here. The main drift of this article is to solve a end of section problem set from Thomas (18.7 on Indeterminant Forms (L'Hopital)) freely using Maple and a TI-84(CE) as investigative tools; compare with Faraday or Galileo fiddling with instruments in pursuit of a theory about atoms and planets. We'll first give a sketch as to why L'Hopital (not a French hospital, hence forward (LH)[2]) might be, should be true. Then onto a bunch of problems (compliments of Thomas). Finally, we will attempt (with lots of help from more advanced books) proofs. We might even say under what circumstances you can clearly solve 1^∞ forms[3] and others with clear procedures that always work.

Derivatives and Limits

Starting with knowledge of the mean value theorem (MVT), one can do a simple simplification of a complex fraction:

$$\frac{f'(t_f)}{g'(t_g)} = \frac{\frac{f(t)-f(0)}{t-0}}{\frac{g(t)-g(0)}{t-0}} = \frac{f(t)-f(0)}{g(t)-g(0)}.$$

As $t \to 0$ the choices for t_f and t_g get pinched; that is for all practical purposes *made equal* and out pops (hops) LH. Let's see if this is true with some combinations of quotients of functions that are blisteringly obvious easy cases. We note that the antecedent of interest is when both f and g go to zero; you can't just substitute in $t = 0$.

Note to teachers: you might call this idea the *go ahead and show 'em the first draft*, rather than implying the fiction that math just springs out of some genius's mind effortlessly; except in this writer's case, of course.

[2] Actually this is the same guy, the Australian francophile that wanted a small kangaroo like pet for his daughter and so invented the hare. Last weird footnote, I promise.

[3] See Thomas explication of such indeterminate forms, leaves LSA in the dust.

Some Evidence

Back in college algebra [**?**], we studied horizontal and vertical asymptotes. These are of special interest when trying to graph a rational function, a polynomial over a polynomial. Horizontal asymptotes show where a function goes as x goes towards infinity. There was the rule that if the degree of the numerator (N_d) is the same as that of the denominator (D_d), the limit as $x \to \infty$ will be the ratio of the leading coefficients of the two: N/D. Indeed in the linear case, its pretty darn simply true: $\lim_{x\to\infty} ax/bx = a/b$ and, low and behold, the derivatives using LH indeed are just a and b and their ratio, a ghost from a grave in Denmark tells us that's a/b. Duh! If the limit is towards $x = 0$, we do have a division by zero afloat, but I bet we still get a/b.

How about

$$\lim_{x\to 0} \frac{x^3 + 2x^2 + 3x}{3x^3 + 2x^2 + x}?$$

According to our horizontal asymptote rule this should be $1/3$ as $x \to \infty$. Using a calculator's table and graph features we get some evidence that that is true. If we take the derivatives repeatedly of N and D it distills to the obvious (if LH is true):

$$\overset{LH}{=} \frac{3x^2 + 4x + 3}{9x^2 + 4x + 1} \overset{LH}{=} \frac{6x + 4}{18x + 4} \overset{LH}{=} \frac{6}{18} \overset{LH}{=} \frac{1}{3}. \tag{1}$$

Remember that's towards infinity. We could have multiplied by a form of 1, namely $1/x^3$ for a purely algebraic solution. What about towards 0. I claim it is 3. Put these functions into your calculator and look at a table of values. They get (from the left and right) closer and closer to 3; that's $f'(0)/g'(0) = 3/1$! Look at the far left of (1); put in the zeros; it's allowed.

We have some evidence that if both go to infinity or are zero at a constant our LH seems to work. But rational functions! Seems too easy. How about some trig!

By way of a long and laborious geometric argument we determined – that is Thomas, Larson, and Stewart[4] determined that

$$\lim_{t\to 0} \frac{\sin(t)}{t} = 1.$$

[4] Henceforth (actually pastforth too!) TLS, sorry no C; Thomas Apostol, not "the apostol" also I'm sure gives a proof, but I determined HWOOHM. I'm not going to play on TSA, I assure you.

But now, if only LH is true, that horror becomes

$$\lim_{t\to 0}\frac{\sin(t)}{t} \overset{LH}{=} \frac{\cos(t)}{1} \overset{L}{=} 1^{5}$$

Okay. Its your turn. Make up two functions that are 0 at 0 or both go to infinity. What else do we need to be true? Right: they have to be differentiable. So how about something with $\ln(x)$ and e^x, trigs (T), hyperbolics (H), their (and general) inverses (I). What other function types have we seen in Thomas by the 18th chapter? How about

$$\frac{e^x}{\ln x} \text{ or } \frac{\ln x}{e^x}$$

as $x \to \infty$, $x \to -\infty$, $x \to 1$ the last from the left, the right? Go easy on yourself: take TH over polynomials where you know the T or H is going to 0. Inverses of TH going to zero over polynomials without a constant. Regular over inverses of THs? Parametric? Polar? Integer functions (aka sequences)?

Better go with a systematic approach: find all functions that go to infinity in Thomas (George, not the apostol) and all those that go to zero. Next make a table with the top row one function, the left column another, form the division inside your table. But division is just one operation: how about if we do subtraction, addition, exponentiation, taking of a function as in making a composite function. Don't no matter what I say attempt to come to a systematic understanding of functions. No matter what you do don't attempt to make a top row with derivative, integral, and series and a right column of all the major function types: LQCPTHIPPV stands for what? Do not attempt to answer that! You are to have a scattered understanding that math is not amenable to systematic treatments: obey your book, your teacher, do not question their authority, make up your own problems and come to a unique understanding of the subject. That is an order. Don't think, obey, obey, obey.

We mention one function divided by one function, but technically two function that are headed to 0 as x or t goes to zero has a product that does the same. So, imagine, use you imagination to comprehend two over two such functions. Press your brain to imagine all the possible sums, differences, products, and powers of any number of functions. If you sense your body temperature (brain thermostat going bongers), have a cool aid with lots (lust, slots, lost) of ice. Hopefully you

[5] I had a teacher who would take off you just plopped a equal sign in for limit. Hopefully, the notation is obvious. I might try to get Latex to put in the two types of limits using `\underset`. Can you guess how I got the LH over the equal sign? $\underset{0}{-}\underset{\mathit{0}}{-}$!

could model such before and after the aid with a linear function. You'd have to manually try all the possible slope and y-intercepts to find the closest fit. We better review.

Pop Quiz 1

This is a timed test. Determine your life expectancy via a statistical analysis of your biological capacity as given by your genome (see me). Then determine key personal behaviors that shorten or lengthen this number (milliseconds preferred). Finally, calculate time to senility, dementia, severe disease leading to inability to focus. Make sure to factor in your stupid ass personal behaviors and their shortening effects. The difference between now and then is how long you have. And it isn't long.

1. You must not use a calculator to determine the best fit curve for a phenomenon if it involves your personal behaviors (like what you eat). (T/F)

2. There is a special favorable dispensation towards humans that allows them to engage in horrible behaviors the aggregate of which (absent this dispensation) would certainly render the planet uninhabitable. (T/F)

3. Mathematical inclined people armed with the purport of this article should be held in detention in Gitmo for their own, Facebook, and Exxon executives own good. (T/F)

4. The observation that this car we're in is going 100 miles per hour towards a solid two mile thick brick wall of environmental catastrophe should never be mentioned to anyone. (T/F)

5. Assess the total tonnage of your energy expenditure, the weight of your carbon footprint to date (round to 10 decimals, make it in grams). (Long essay)

6. Determine a list of top ten reasons why some diety shouldn't strike you dead to preserve human life on Earth. (Short answer) (Hint: You may try things like your race, sex (how much you've had or would like to have and haven't had yet), maybe your donations to PETA, other: you don't wear fur. Good luck!)

7. Speculate on why books exist with titles like *God is a Mathematician*. (Short answer. Be specific. Be sure to reference at least three inspired texts, like the Torah, Koran, and something by Confucius. Statements about the author will not, with certainty, get you a clear A for this quiz or the course.)

8. Speculate as to how it is that academics at your institution, despite knowing the problem of all students and faculty driving separately in personal cars to campus and the pointless (relative to mass transportation) expenditure of fossil fuels, nevertheless did so, are doing so, and never will even imagine to think to stop doing so. (Hint: you can say things like *they are just a version of wild, stupid animals not able to behave commensurate with science. Please don't attack administrators at your school by name or something mean like that. Don't mention, above all where I, the author works.*

9. Your goal upon graduation should clearly be to fly around in a fancy personal jet, drive some kind of Italian made car, and live in a 4-10 thousand square foot house, import fine cigars, and generally assert your status as a function of carbon footprint like you were trained form birth to do. (T/F)

10. Some such statements like *humans you must stop all hostilities, you must allow your scientist to determine where fertile ground remains, there plant, and distribute, as best you can, fairly to all humans* is absolutely ludicrous, falling on deaf ears, pointless. (T/F) Extra credit: speculate on whether or not the statement is the only solution to modern global problems, the only way to stave off certain disaster wherein no one will thrive – not the rich, not the poor: all will be dead or wishing they were.

References

[1] Larson, R, Edwards, B.H. (2010). *Calculus*, 9th ed., Boston, Mass.: Cengage Brooks/Cole.

[2] Stewart, J. (2012). *Calculus*, 7th ed. Boston, Mass: Cengage Brooks/Cole.

[3] Thomas, G. (1968). *Calculus and Analytic Geometry*, 4th ed., Reading, Mass: Addison-Wesley.

[4] Thomas, G. (2014). *Thomas' Calculus*, 13th ed., Boston: Pearson.