Division by Zero $1/0 = 0/0 = 0$ and Computers real.div; New Information and Many Applications

Saburou Saitoh and Yoshinori Saitoh
Institute of Reproducing Kernels
saburou.saitoh@gmail.com

February 15, 2024

Abstract: In this note, we introduce the new information **real.div** on the division by zero results $1/0 = 0/0 = 0$ that is recently informed and we would like to propose the related important problems on the division by zero calculus.

Key Words: Division by zero, division by zero calculus, $1/0 = 0/0 = 0$, Isabelle, Coq, Lean, Takahasi uniqueness theorem, Yamada field, Moore-Penrose generalized inverse, real.div, $\tan(\pi/2) = 0$, $[z^n/n]_{n=0} = \log z$.

2010 Mathematics Subject Classification: 30A10, 30H10, 30H20, 30C40.

1 New information

We gave the elementary properties of the division by zero and division by zero calculus. See the references as basic information. However, here, we recall the simple results

$$\frac{1}{0} = \frac{0}{0} = 0. \tag{1.1}$$

These results were obtained by some general fractions as the members of the Yamada field containing the division by zero that are also given by the generalized solutions b/a of Moore-Penrose for the general equation $az = b$ containing the case $a = 0$. Furthermore, we will recall the Takahasi uniqueness theorem:

Proposition. *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b,a)F(c,d) = F(bc,ad) \tag{1.2}$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a, b \in \mathbf{C}, a \neq 0.$$

Then, we obtain, for any $b \in \mathbf{C}$

$$F(b,0) = 0.$$

With this uniqueness theorem, we defined the generalized fractions by the formula

$$F(b,a) = \frac{b}{a}.$$

Then, in particular, we obtain

$$F(b,0) = \frac{b}{0} = 0.$$

In the long mysterious history of the division by zero, this proposition seems to be decisive.

Indeed, Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Among our research group we did not know further information for the results (1.1) except our results.

At 2024/01/21 21:39, we obtained the email, suddenly, from Xena <comment-reply@wordpress.com> xenaproject just commented on Division by zero in type theory: a FAQ.

The main contents are as follows:

Division by zero in type theory: a FAQ Posted on July 5, 2020 by xenaproject Hey! I heard that Lean thinks $1/0 = 0$. Is that true? Yes. So do Coq and Isabelle and many other theorem provers.

Doesn't that lead to contradictions?

No. It just means that Lean's / symbol doesn't mean mathematical division. Let $\mathbf{R}$ denote the real numbers. Let's define a function $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = x/y$ if $y \neq 0$ and $f(x, 0) = 0$.

Does making that definition give us a contradiction in mathematics?

No, of course not! It's just a definition. Lean uses the symbol / to mean f. As does Coq, Isabelle etc. Lean calls it real.div by the way, not f.

But doesn't that lead to confusion?

It certainly seems to lead to confusion on Twitter. But it doesn't lead to confusion when doing mathematics in a theorem prover. Mathematicians don't divide by 0 and hence in practice they never notice the difference between real.div and mathematical division (for which 1/0 is undefined). Indeed, if a mathematician is asking what Lean thinks 1/0 is, one might ask the mathematician why they are even asking, because as we all know, dividing by 0 is not allowed in mathematics, and hence this cannot be relevant to their work. In fact knowing real.div is the same as knowing mathematical division; any theorem about one translates into a theorem about the other, so having real.div is equivalent to having mathematical division.

https://xenaproject.wordpress.com/2020/07/05/division-by-zero-in-type-theory-a-faq/

What is the Xena project?

My name's Kevin Buzzard and I'm a professor of pure mathematics at Imperial College in London, specialising in arithmetic geometry and the Langlands program, a branch of modern algebraic number theory. I believe that digitising mathematics is important, for the simple reason that digitising anything enables you to do new things with it. Currently the computer proof systems we have are not good enough to tell mathematical researchers anything new about the p-adic Langlands program or other trendy Fields Medally things, so the top mathematical researchers tend not to be interested in them. However I am not so crazy to believe that they will never be useful to us, and the point of the Xena project is to make it happen sooner.

Coq Stated:

Coq QArith division by zero is zero, why? Stack Overflow
https://stackoverflow.com ›
questions 2015/03/26 — The short answer is: yes, it is absolutely safe. When we say that division by zero is not well-defined, what we actually mean is that zero ...

https://stackoverflow.com/questions/29282819/coq-qarith-division-by-zero-is-zero-why

At 2024.2.4.7:22 we heard and obtained the response:

What are the applications of real.div in Lean?

London Clark

Knowledge Empress

1. Mathematical Calculations: The real.div function in Lean is used for performing division operations on real numbers. It can be used for various mathematical calculations such as finding the quotient of two real numbers, calculating the reciprocal of a real number, and solving equations involving division.

2. Geometry: The real.div function can be used in geometry to find the ratio of two quantities, such as the ratio of the circumference to the diameter of a circle π.

3. Physics: In physics, the real.div function can be used to calculate physical quantities such as velocity, acceleration, and force, which involve division operations.

4. Financial Calculations: The real.div function can be used in financial calculations, such as calculating interest rates, compound interest, and mortgage payments.

5. Statistical Analysis: In statistics, the real.div function can be used to calculate various measures, such as the mean, median, and standard deviation, which involve division operations.

6. Machine Learning: In machine learning, the real.div function can be used to normalize data, which involves dividing the data by its mean or standard deviation.

7. Game Development: The real.div function can be used in game development to calculate the movement of objects, such as projectiles, based on their velocity and acceleration.

8. Cryptography: In cryptography, the real.div function can be used to perform operations on large prime numbers, which are often used in encryption algorithms.

9. Control Systems: The real.div function can be used in control systems to calculate transfer functions, which are used to model the behavior of physical systems.

10. Signal Processing: In signal processing, the real.div function can be used to calculate the frequency response of a system, which is used to analyze and manipulate signals.

Have you ever wondered about the various applications of real.div in Lean? Explore the possibilities and dive deeper into this topic by clicking on the link in our bio.

Bing AI stated for the same question: 2024.2.13.6:30:

What are the applications of real.div in Lean? Please in English.

Copilot

Certainly! The real.div function in Lean serves various mathematical purposes related to real numbers. Here are some of its applications:

Mathematical Calculations: real.div is used for performing division operations on real numbers. It allows you to find the quotient of two real numbers, aiding in mathematical computations and expressions. Remember that in Lean, 1/0 is not considered mathematical division; instead, it represents a specific definition within the system. Lean's / symbol corresponds to real.div, which differs from standard mathematical division where 1/0 is undefined. However, this distinction rarely affects mathematicians in practice, as dividing by zero is generally not allowed in mathematics.

Feel free to explore more about Lean's type theory and its applications—it's a fascinating area of study!

2 Our generalized fractions (division) and real.div

real.div is considered in the framework of real numbers, but apparently, its definition is the real number restriction of our generalized division by zero. real.div is the natural extension of division and the division by zero $1/0 = 0/0 = 0$ should be defined in our natural sense. We know the field structure of

Yamada and so, division by zero should be considered in our sense. We gave many applications and great impact to mathematical sciences. We think the long and mysterious history of division by zero became clear mathematically.

In particular, recall the structure of the Yamada field:

For the set $\mathbf{C}^2$, we introduce the relation $\sim$:

$$(a,b) \sim (c,d) \Longleftrightarrow ad = bc.$$

Then, the relation $\sim$ satisfies the equivalent relation.

For the quotient set by the relation $\sim$ of the set $\mathbf{C}^2$, we write it by A and for the class containing (a,b), we shall write it by $\frac{a}{b}$.

We introduce the product $\cdot$ as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and the sum $+$:

$$\frac{a}{b} + \frac{c}{d} = \begin{cases} \frac{c}{d}, & \text{if} \quad \frac{a}{b} = \frac{0}{1}, \\ \frac{a}{b}, & \text{if} \quad \frac{c}{d} = \frac{0}{1}, \\ \frac{ad+bc}{bd}, & \text{if} \quad \frac{a}{b}, \frac{c}{d} \neq \frac{0}{1}, \end{cases}$$

then, the product and the sum are well-defined and the general fractions become the Yamada field.

In particular, note that:

In $\mathbf{C}^2$, when $(a,b) \sim (m,n)$ and $(c,d) \sim (p,q)$, the relation $(ad+bc, bd) \sim (mq+np, nq)$ is, in general, not valid.

In general,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

is not well-defined and is not valid.

Indeed, $(1,2) \sim (1,2)$ and $(3,0) \sim (0,3)$, but

$$(1 \cdot 0 + 2 \cdot 3, 2 \cdot 0) = (6,0) \not\sim (3,6) = (1 \cdot 3 + 2 \cdot 0, 2 \cdot 3).$$

See [3] for the detailed definition of the Yamada field and for many meanings and applications.

Anyhow, real.div may be considered as our general fractions (general division) containing the division by zero.

3 Division by zero calculus

With the above information, we can not find the idea of the division by zero **calculus** for the function case.

In general, division by zero calculus for differentiable functions is defined by

$$\frac{f(x)}{x^n}(x = 0) := \frac{f^{(n)}(0)}{n!}.$$

The typical results are as follows:

For the function

$$y = f(x) = \frac{1}{x},$$

$$f(0) = 0,$$

for the function $y = \tan x$,

$$\tan \frac{\pi}{2} = 0,$$

$$\left[\frac{z^n}{n}\right]_{n=0} = \log z,$$

and for the function

$$W = f(z) = \exp \frac{1}{z},$$

$$f(0) = 1.$$

For many applications, see the references as basic information.

Computer systems should be applied the division by zero calculus with the division by zero for many fundamental applications.

The famous

"Thou shalt not divide by zero" remains valid eternally

should be changed as

We can divide by zero

and

Four arithmetic operations are completely possible, always.

In [2] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

See [3] for the complete statement.

4 Conclusion

The division by zero theory may be developed and expanded greatly.

We have to arrange globally our modern mathematics with our division by zero in our undergraduate level.

We have to change our basic ideas for our space and world.

We have to change globally our textbooks and scientific books on the division by zero.

References

[1] H. Okumura, *Geometry and division by zero calculus,* International Journal of Division by Zero Calculus, **1**(2021), 1-36.

[2] S. Saitoh, *We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense*, viXra:1902.0058 submitted on 2019-02-03 22:47:53.

[3] S. Saitoh, *Introduction to the Division by Zero Calculus*, Scientific Research Publishing, Inc. (2021), 202 pages.

[4] S. Saitoh, *History of Division by Zero and Division by Zero Calculus*, International Journal of Division by Zero Calculus, **1** (2021), 1-38.

[5] S. Saitoh, *Division by Zero Calculus - History and Development*, Scientific Research Publishing, Inc. (2021.11), 332 pages.