Calculating the Gaussian Curvature of Spacetime

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Abstracts

We are working on applications of the Schwarzschild metric to the cosmos. Here we have calculated the curvature of space-time in an isotropic and homogeneous universe with a constant energy density, relating this universe to the Schwarzschild metric using the Birkhoff-Jebsen theorem. Already in this metric it is possible to calculate gaussian curvatures and thus we have found a solution to this problem. The result $0{,}1.10^{-52}$ m^{-2}, is very close to zero and this agrees with the opinion of many scientists. To reach this result, we have found an equation that relates the Gaussian curvature of the space-time of the Schwarzschild model with several cosmic parameters and through it and the Birkhoff-Jebsen theorem we have established a relationship between the Schwarzschild metric and the universe of the standard cosmological model. This has allowed us to calculate the proportionality factor between curvature and mass density. With this, calculating the curvature of space-time is immediate.

Keywords: Curvature of space-time, General Relativity.

1.-Search for an equation that relates the gauss curvature of space-time to cosmological parameters into Schwarzschild metric.

First, in the Schwarzschild metric we are going to obtain an equation that relates the Gaussian curvature of space-time k, with some cosmic parameters, such as the gravitational mass that is causing the curvature M, the universal gravitation constant G, and the speed of light "c".

Schwarzschild solves the equations of the general theory of relativity [1] for a model of a point gravitational mass and a surrounding empty space, establishing a metric and an equation for space-time that turns out to be stationary in time and with spherical symmetry, resulting in a 2D surface, (the Flamm paraboloid), which is represented in Fig. 1.

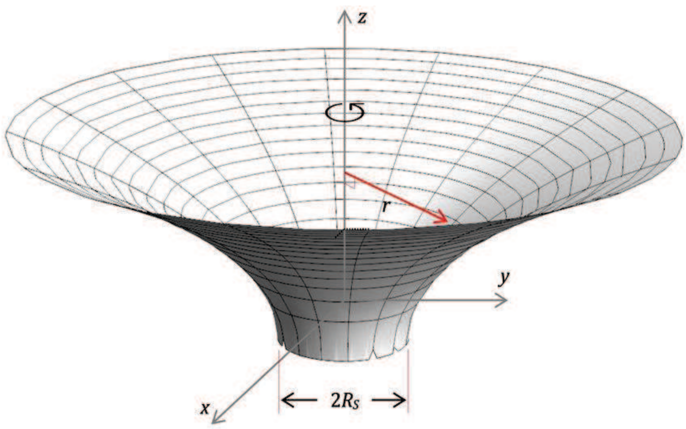

Fig. 1 Space-time in the Schwarzschild metric. Flamm paraboloid

Flamm's paraboloid, mathematical solution to the Schwarzschild metric, is a 2D surface inserted in a space R^3. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the point mass "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [2], and with it we are going to calculate the Gaussian Curvature. (Rs = Schwarzschild radius)

Surface parameters (r, φ)

$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r\cos\varphi$

$y = r\,\mathrm{sen}\varphi$

$z = 2(Rs(r - Rs))^{1/2}$

Vector Equation of the Surface

$\mathbf{f\,(x,y,z) = (r\cos\varphi,\ r\,sen\varphi,\ 2(Rs(r-Rs))^{1/2})}$

Determination of velocity, acceleration, and normal vectors to the surface

$\eth f/\eth\varphi = (-r\,\mathrm{sen}\varphi, r\cos\varphi, 0)$

$\eth^2 f/\eth\varphi^2 = (-r\cos\varphi,\ -r\,\mathrm{sen}\varphi,\ 0)$

$\eth f/\eth r = (\cos\varphi, \mathrm{sen}\varphi, (r/Rs - 1)^{-1/2})$

$\eth^2 f/\eth r^2 = (0,\ 0,\ (-1/(2R_s)).\,(r/Rs - 1)^{-3/2})$

$\eth f/\eth\varphi\eth r = (-\mathrm{sen}\varphi,\ \cos\varphi,\ 0)$

$$\mathbf{n} = \frac{(\eth f/\eth\varphi \times \eth f/\eth r)}{[\frac{\eth f}{\eth\varphi} x \frac{\eth f}{\eth r}]}$$

$(\eth f/\eth\varphi \times \eth f/\eth r) = (r\cos\varphi/(r/Rs - 1)^{1/2},\ r\,\mathrm{sen}\varphi/(r/Rs - 1)^{1/2},\ -r)$

$[\frac{\eth f}{\eth\varphi} X \frac{\eth f}{\eth r}] = r\,((1/(r/Rs - 1)) + 1)^{1/2}$

Curvature and curvature parameters

$$\textbf{Gauss curvature} \quad \mathbf{k = LN - M^2/EG - F^2}$$

$L = \eth^2 f/\eth\varphi^2.\ \mathbf{n} = \mathbf{-r(r/Rs)^{-1/2}}$

$N = \eth^2 f/\eth r^2.\ \mathbf{n} = \mathbf{(1/2Rs)\,(r/Rs)^{-1/2}\,(r/Rs - 1)^{-1}}$

$M = (\eth f/\eth\varphi\eth r).\ \mathbf{n} = \mathbf{0}$

$G = \eth f/\eth r.\ \eth f/\eth r = \mathbf{1 + (1/(r/Rs - 1))}$

$E = \eth f/\eth\varphi.\ \eth f/\eth\varphi = \mathbf{r^2}$

$F = \eth f/\eth\varphi.\ \eth f/\eth\varphi = \mathbf{0}$

$$\mathbf{k = -Rs/2r^3 = -GM/c^2r^3} \quad (1)$$

for Schwarzschild radius, $Rs = 2GM/c^2$

2- Calculating the curvature of space-time in a homogeneous and isotropic universe

We are going to calculate the Gaussian curvature for a homogeneous isotropic universe with a constant energy density, and we will do it at a generic inner point. Let's calculate the Gaussian curvature for a time "t"

To do this, we will use the results obtained in this work with respect to our curvature equation (1), and the Birkhoff-Jebsen theorem.

2.1- Birkhoff–Jebsen theorem

We make a brief comment on this theorem of mathematics applied to the theory of generalized relativity. First, we summarize Professor Fulvio Melia in reference [3] to explain it.

“If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere”.

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

In general relativity, Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be given by the Schwarzschild metric.

2.2- Calculating the curvature

Let us consider applying our equation (1) to our universe, for this we consider a sphere of radius r inside, the Birkhoff-Jebsen theorem assures us that if we want to calculate the space-time curvature on the surface of the sphere, we could consider only the interaction with the gravitational mass found inside. Furthermore, as the Birkhoff-Jebsen theorem assures us that the solution is given by the Schwarzschild metric. The curvature equation (1) that we have obtained can be applicable in this case, taking into account that the interaction with the interior points of the sphere is, that is, the gravitational field on the surface of the sphere is reduced to an interaction with a point mass of equal magnitude in the center of the sphere and in this case the equation to calculate curvatures of the Schwarzschild model is applicable. that we have found.

Since the mass density “ρ_m” in this universe is constant, it will be constant in every sphere that we are considering and thus we can write:

$$M = \rho_m(4\pi r^3/3)$$

According to equation (1), we have:

$$k = -GM/c^2r^3$$

substituting result:

$$k = -4\pi G\, \rho_m/3c^2$$

Equation found, which relates the curvature of space-time to mass density, valid at any inner point in our model of the universe.

$$\mathbf{k/\rho_m = -4\pi G/3c^2 = -0{,}3104.10^{-26}\ \ m/Kg}$$

k is the Gaussian curvature (m^{-2}) and ρ_m is the mass density (Kg/m^{-3})

This equation is independent of the radius of the sphere that we are considering for the calculation of the curvature and assigns the same value of curvature at each point of the universe, which is characterized by a constant energy density and presents the properties of isotropy and uniformity. That value of curvature, which depends only on the energy density and is the same at all points in our model of our universe, we will call "curvature of space-time" and now we will calculate it:

As mass density I take the density of matter [4], (baryonic and cool dark matter)

$$\rho_m = 0{,}28.10^{-26}\ Kg/m^3$$

$$k = -4\pi G\rho_m/3c^2 = (-0{,}3104.10^{-26}).\rho_m$$

the curvature of space-time result:

Result of the calculation of the curvature of space-time
$k = 0{,}1.10^{-52}\ m^{-2}$

It should be noted that our result does not resolve the sign of the curvature, it only gives an absolute value for it.

3.- Calculation of curvature in the FLRW metric

Although the gaussian curvature of space-time has never been measured, there are recent experimental data for the measurement of a curvature parameter Ω_k [5]. Starting from Friedmann's equation [6] and our equation that relates curvature and mass density, we can establish a relationship between the two magnitudes.

Let's make this development.

Friedmann´s equation:

$$H^2 = (a'/a)^2 = 8\pi G\rho/3 - kc^2/a^2$$

being H the Hubble constant, “a” the scale factor and “ρ” the energy density.

In a universe dominated by matter, such as ours now, energy density can be described as:

$$\rho = \rho_m + \rho_\Lambda$$

ρ_m is the mass energy density,

ρ_Λ is the vacuum energy density.

Friedmann's equation can be written like this:

$$1 = \Omega_m + \Omega_\Delta + \Omega_k \quad (2)$$

Ω_k is the measured curvature parameter

$\Omega_m = 8\pi G\rho_m/3H^2$

$\Omega_\Delta = 8\pi G\rho_\Lambda/3H^2$

In addition, according to the reference [5]:

$\Omega_k = 0{,}001 \pm 0{,}002$

In addition, according to the reference [7]

$\rho_\Lambda = (0.603 \pm 0.013) \times 10^{-26}$ kg/m^3

In addition, according to our equation that relates mass density to curvature of space-time,

$$\Omega_m = 8\pi G\rho_m/3H^2$$

$$k/\rho = -4\pi G/3c^2$$

$$\Omega_m = - 2Kc^2/H^2$$

In addition, according to the reference [5]

H = 67,37 ± 0,64 km/s per Megaparsec = $2{,}3.10^{-18}$ m/s per meter

Substituting into equation (2), result:

$$1 = \Omega_m + \Omega_\Delta + \Omega_k$$

$$\Omega_m = 1 - 0{,}001 - 0{,}608 = 0{,}391$$

Curvature of space-time according to the FLRW metric
$k = 0{,}1.10^{-52}$ m^{-2}

4.- Conclusions

We have calculated the curvature of space-time according to a model of an isotropic, homogeneous universe with constant energy density at all its points. We have obtained that the curvature only depends on the value of the mass density and a constant that in turn is a function of the speed of light and the universal gravitation constant. This leads us to the same curvature value, $0{,}1.10^{-52}$ m^{-2}, at each of its points if these are inner points in the topological concept. To obtain this result we have used the Schwarzschild metric, applied in a very special way to our model of the universe, through the Birkhoff-Jebsen theorem. Finally, we have performed a curvature calculation using the FLRW metric and the equation, found by us, that relates mass density to curvature. The result for the curvature is the same.

From all this we deduce that the curvature of space-time can be calculated from Einstein's equations through appropriate models and that our result, using the Schwarzschild metric and the Universe Model ΔCDM, it is approaching a flat universe now.

5.- References

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