A totally connected superluminal Natario warp drive spacetime with constant velocities

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February 7, 2024

Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However one the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble. The behavior of a photon sent to the front of the warp bubble in the case of a Natario warp drive with constant velocity and a lapse function is the main purpose of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ spacetime with lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline the final result found in our work which is the following one: For the case of the lapse function the Horizon do not exists at all.Due to the extra terms in the lapse function that affects the whole spacetime geometry this solution allows to circumvent the problem of the Horizon.

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1 Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8],[9],[10] and mainly [11] and [23]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7],[8] and mainly [11] and [21]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7],[8] and mainly [20]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8],[9],[10],[11],[20],[21] and [23]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

[1] do not violates Relativity

Alcubierre([16]) used the so-called 3+1 Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([15]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [16], see also eq 1 pg 3 in [1])[2][3] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.

The Natario warp drive equation that obeys the $3+1$ ADM formalism with constant velocities is given below:(see Appendix E in [12])

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2d\theta^2 \quad (1)$$

However some important things must be outlined in both the Alcubierre or Natario warp drive spacetimes:

- 1)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs.They do not accelerate or de-accelerate and travel always with a constant speed.But a real warp drive must "know" how to accelerate for example from 0 to a speed of 200 times faster than light in the beginning of an interstellar journey and in the end of the journey it must de-accelerate again to 0 in the arrival at the destination point which means to say of course a distant star.

- 2)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs raised to the square in their equations for the negative energy density.An accelerating warp drive probably must have the terms of variable velocities or accelerations included in the expression for the negative energy density since this energy is responsible for the generation of the warp drive spacetime.

- 3)-The warp drives as proposed by Alcubierre or Natario always have the so-called lapse function of the ADM formalism always equal to 1

A lapse function with values different than 1 adapted to the Natario warp drive that obeys the $3+1$ ADM formalism with constant velocities must possess the following properties:

- inside the warp bubble(flat spacetime where the spaceship is located)the lapse function is equal to 1

- outside the warp bubble(flat spacetime where an external observer watches the ship passing by)the lapse function is also equal to 1

- in the Natario warped region(warp bubble walls curved spacetime) the lapse function must possesses a large value at least greater than or equal to the modulus of the ship velocity to keep the warp bubble totally connected.

The Natario warp drive equation that obeys the $3+1$ ADM formalism with constant velocities and a lapse function α is given below:(see Appendix J in [23])

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2d\theta^2 \quad (2)$$

[2]see also Appendix E in [12]

[3]see the Remarks section on our system to quote pages in bibliographic references

In this work we discuss the Horizon problem for both the Natario warp drive spacetime equations in the $1+1$ ADM formalism with and without the lapse function at constant velocities and we arrive at the conclusion that while the equation without the lapse function suffers from the problem of the Horizon and cannot control the warp bubble the new equation with the lapse function possessing a value greater than or equal to the bubble velocity in modulus can circumvent the problem of the Horizon because in this case the warp bubble is totally connected due to the presence of the lapse function.

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pgs $[106(a)], [98(b)]$ in [17] and pgs $[394(a)][377(b)]$ in [18]).

What Alcubierre and Natario did was an attempt to reproduce the expansion of the Universe in a local way creating a local spacetime distortion that expands the spacetime behind a ship and contracts spacetime in front reproducing the superluminal expansion of the Universe moving away the departure point in an expansion and bringing together the destination point in a contraction.The expansion-contraction can be seen in the abs of the original Alcubierre paper in [1].Although Natario says in the abs of his paper in [2] that the expansion-contraction does not occurs in its spacetime in pg 5 of the Natario paper we can see the expansion-contraction occurring however the expansion of the normal volume elements or the trace of the extrinsic curvature is zero because the contraction in the radial direction is exactly balanced by the expansion in the perpendicular directions.

An excellent explanation on how a spacetime distortion or a perturbation pushes away a spaceship from the departure point and brings the ship to the destination point at faster than light speed can be seen at pg 34 in [3],pgs $[260(a)260(b)][261(a)261(b)]$ in [4].Note that in these works it can be seen that the perturbation do not obeys the time dilatation of the Lorentz transformations hence the speed limit of Special Relativity cannot be applied here.

An accelerated warp drive can only exists if the astronaut in the center of the warp bubble can somehow communicate with the warp bubble walls sending instructions to change its speed.But for signals at light speed the Horizon exists at least for the warp drive with constant velocity and without the lapse function.So light speed cannot be used to send signals to the front of the bubble in this case.(see pg 16 in [7] and pg 21 in [8]).Besides in the Natario warp drive with constant velocity the negative energy density covers the entire bubble.(see Appendices B,C and D in [23]).Since the negative energy density have repulsive gravitational behavior the photon of light if possible to reach the bubble walls would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls(see pg $[116(a)][116(b)]$ in [13]). The solution that allows contact with the bubble walls was presented in pg 28 in [7] and pg 31 in [8]. Although the light cone of the external part of the warp bubble is causally disconnected from the astronaut who lies inside the large bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro bubble so these bubbles can be "engineered" to be sent to the large bubble. This idea seems to be endorsed by pg 34 in [3],pgs $[268(a)268(b)]$ in [4] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

Horizons were deeply covered in the warp drive literature but always for constant velocities and without lapse functions.(see pg 6 in [2],pg 16 in [7],pg 21 in [8],pg 34 in [3],pgs [268(a)268(b)] in [4]).The behavior of a photon sent to the front of the warp bubble in the case of a warp drive with constant velocity and a lapse function is the main purpose of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ spacetime with or without the lapse function at constant velocities using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline(or underline or reinforce) the final results found in our work which are the following ones:

- 1)-In the case of the Natario warp drive with fixed velocity and without lapse functions the Horizon exists as expected and in agreement with the current literature.
- 2)-In the case of the Natario warp drive with constant velocity and a lapse function the Horizon do not exists at all.Due to the extra terms provided by the lapse function that affects the whole spacetime geometry this solution with lapse functions have different results when compared to the solution without lapse functions.Remember that we are presenting our results using step by step mathematics in order to better illustrate our point of view.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes.

This work must be regarded as a companion work to our works in [12] and [23] which are required readings in order to understand some of the mathematics used in this text.

Although this work covers the lapse function as depicted in [23] for the Natario warp drives with constant velocities future works will appear using lapse functions to cover the accelerated Natario warp drives depicted in [12],[19] and [22].

We would recommend for beginners or introductory students a first reading of our work in [23] in order to fully understand the mathematical properties and capabilities of the lapse function.

2 The equation of the Natario warp drive spacetime metric with a constant speed vs in the original $3+1$ ADM formalism and a lapse function $\alpha = 1$

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix E in [12] for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{3}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{4}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2])(see also Appendix A in [12] for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{5}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{6}$$

The covariant shift vector components X_{rs} and X_{θ} are given by:

$$X_{rs} = 2v_s n(rs)\cos\theta \tag{7}$$

$$X_{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{8}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{9}$$

But since $X_{rs} = X^{rs}$ the equation can be written as given below:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{10}$$

Examining the Natario warp drive equation in a $1 + 1$ spacetime:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{11}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s n(rs) \tag{12}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.

3 The equation of the Natario warp drive spacetime metric with a constant speed vs in the original $3+1$ ADM formalism using a lapse function α always equal to 1 in the regions inside and outside the Natario bubble but with large values in the Natario warped region.

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix J in [23] for details)

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{13}$$

This section is almost similar to the previous section with the Natario vector and the Natario shape function being equal to their counterparts of the previous section however with a difference:the lapse function α

In a $1+1$ spacetime the equatorial plane we get:

$$ds^2 = (\alpha^2 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{14}$$

The equation above was written using both contravariant and covariant shift vector components of the Natario vector at the same time.

Since $X_{rs} = X^{rs}$ the equation in the $1+1$ spacetime can be written as given below:

- 1)-contravariant form;all the shift vector components of the Natario vector are contravariant

$$ds^2 = (\alpha^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{15}$$

- 2)-covariant form:all the shift vector components of the Natario vector are covariant

$$ds^2 = (\alpha^2 - (X_{rs})^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{16}$$

The equal contravariant and covariant shift vector component X_{rs} and X^{rs} are then:

$$X^{rs} = X_{rs} = 2v_s n(rs) \tag{17}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.

Remember that the lapse function must possess the following properties:

- inside the warp bubble(flat spacetime where the spaceship is located)the lapse function is equal to 1
- outside the warp bubble(flat spacetime where an external observer watches the ship passing by)the lapse function is also equal to 1
- in the Natario warped region(warp bubble walls curved spacetime) the lapse function must possesses a large value at least greater than or equal to the modulus of the ship velocity to keep the warp bubble totally connected.

4 Horizons(causally disconnected portions of spacetime geometry in the equation of the Natario warp drive spacetime metric with a constant speed vs in the original $1+1$ ADM formalism and with a lapse function $\alpha = 1$)

The mathematical discussions of this section uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for constant speed vs in a $1+1$ spacetime:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{18}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s n(rs) \tag{19}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for constant speed vs in a $1+1$ spacetime we have:

$$0 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{20}$$

Dividing both sides by dt^2 we have:

$$0 = (1 - [X^{rs}]^2) + 2(X^{rs}\frac{drs}{dt}) - (\frac{drs}{dt})^2 \tag{21}$$

Making the following algebraic substitution:

$$U = \frac{drs}{dt} \tag{22}$$

We have:

$$0 = (1 - [X^{rs}]^2) + 2(X^{rs})U - U^2 \tag{23}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$U^2 - 2(X^{rs})U - (1 - [X^{rs}]^2) = 0 \tag{24}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X^{rs}) \pm \sqrt{4((X^{rs})^2) + 4(1 - [X^{rs}]^2)}}{2} \tag{25}$$

$$U = \frac{2(X^{rs}) \pm \sqrt{4((X^{rs})^2) + 4 - 4([X^{rs}]^2)}}{2} \tag{26}$$

The simplified algebraic expression becomes:

$$U = \frac{2(X^{rs}) \pm \sqrt{4}}{2} \tag{27}$$

Which leads to:

$$U = \frac{2(X^{rs}) \pm 2}{2} \tag{28}$$

And the final result is then given by:

$$U = X^{rs} \pm 1 \tag{29}$$

The above equation have two possible solutions U respectively $U = X^{rs} + 1$ and $U = X^{rs} - 1$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X^{rs} - 1 \tag{30}$$

$$U_{rear} = X^{rs} + 1 \tag{31}$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X^{rs} - 1 = 2v_s n(rs) - 1 \tag{32}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) and assuming a continuous behavior for $n(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2v_s n(rs)$ from 0 to vs we can clearly see that inside the bubble $2v_s n(rs) = 0$ because $n(rs) = 0$ and outside the bubble $2v_s n(rs) = vs$ because $n(rs) = \frac{1}{2}$ and assuming also continuous values from 0 to vs with $vs > 1$[4] then somewhere in the Natario warped region where $0 < n(rs) < \frac{1}{2}$ we have the situation in which $2v_s n(rs) = 1$ because 1 lies in the continuous interval from 0 to vs and in consequence :

$$U_{front} = X^{rs} - 1 = 2v_s n(rs) - 1 = 1 - 1 = 0!!! \tag{33}$$

The result is zero !!!.The photon sent to the front of the bubble stops!!!.A Horizon is established!!!.The front part of the bubble is causally disconnected from the observer in the center of the bubble.So photons at light speed cannot be used to send signals to the front of the bubble.(see pg 6 in [2],pg 16 in [7],pg 21 in [8],pg 34 in [3],pgs [268(a)268(b)] in [4]).The place where the photon stops is the place where $n(rs) = \frac{1}{2vs}$ and with $vs > 1$ this place lies well within the Natario warped region.$0 < \frac{1}{2vs} < \frac{1}{2}$

[4]Remember that we are working with Geometrized Units in which $G = c = 1$

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for constant speed vs in a $1+1$ spacetime.But we know that in the Natario warp drive the negative energy density covers the entire bubble.(see Appendices B,C and D in [23]).Since the negative energy density have repulsive gravitational behavior(see pg [116(a)][116(b)] in [13]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 28 in [7] and pg 31 in [8]. Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg [96(a)][83(b)] in [14] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy. This idea seems to be endorsed by pg 34 in [3],pgs [268(a)268(b)] in [4] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

5 Horizons(causally disconnected portions of spacetime geometry in the equation of the Natario warp drive spacetime metric with a constant speed vs in the original $1 + 1$ ADM formalism using a lapse function α always equal to 1 in the regions inside and outside the Natario bubble but with large values in the Natario warped region)

The mathematical discussions of this section uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for constant speed vs in a $1+1$ spacetime:

$$ds^2 = (\alpha^2 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{34}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s n(rs) \tag{35}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for constant speed vs in a $1+1$ spacetime we have:

$$0 = (\alpha^2 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{36}$$

Dividing both sides by dt^2 we have:

$$0 = (\alpha^2 - [X^{rs}]^2) + 2(X^{rs}\frac{drs}{dt}) - (\frac{drs}{dt})^2 \tag{37}$$

Making the following algebraic substitution:

$$U = \frac{drs}{dt} \tag{38}$$

We have:

$$0 = (\alpha^2 - [X^{rs}]^2) + 2(X^{rs})U - U^2 \tag{39}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$U^2 - 2(X^{rs})U - (\alpha^2 - [X^{rs}]^2) = 0 \tag{40}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X^{rs}) \pm \sqrt{4((X^{rs})^2) + 4(\alpha^2 - [X^{rs}]^2)}}{2} \tag{41}$$

$$U = \frac{2(X^{rs}) \pm \sqrt{4((X^{rs})^2) + 4\alpha^2 - 4([X^{rs}]^2)}}{2} \quad (42)$$

The simplified algebraic expression becomes:

$$U = \frac{2(X^{rs}) \pm \sqrt{\alpha^2}}{2} \quad (43)$$

Which leads to:

$$U = \frac{2(X^{rs}) \pm \alpha}{2} \quad (44)$$

And the final result is then given by:

$$U = X^{rs} \pm \alpha \quad (45)$$

The above equation have two possible solutions U respectively $U = X^{rs} + \alpha$ and $U = X^{rs} - \alpha$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X^{rs} - \alpha \quad (46)$$

$$U_{rear} = X^{rs} + \alpha \quad (47)$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X^{rs} - \alpha = 2v_s n(rs) - \alpha \quad (48)$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) and assuming a continuous behavior for $n(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2v_s n(rs)$ from 0 to vs we can clearly see that inside the bubble $2v_s n(rs) = 0$ because $n(rs) = 0$ and outside the bubble $2v_s n(rs) = vs$ because $n(rs) = \frac{1}{2}$ and assuming also continuous values from 0 to vs with $vs > 1$[5] then in the Natario warped region where $0 < n(rs) < \frac{1}{2}$ we never have the situation in which $2v_s n(rs) = \alpha$ because α is greater than $2v_s n(rs)$ in modulus and α never lies in the continuous interval from 0 to vs and in consequence :

$$U_{front} = X^{rs} - \alpha = 2v_s n(rs) - \alpha \neq 0!!! \quad (49)$$

The result is not zero !!!.The photon sent to the front of the bubble never stops!!!.A Horizon is not established!!!.

[5]Remember that we are working with Geometrized Units in which $G = c = 1$

This is a situation different than the previous case in which:

$$U_{front} = X^{rs} - 1 = 2v_s n(rs) - 1 = 1 - 1 = 0!!! \quad (50)$$

A large lapse function may perhaps keeps the Natario warped region causally connected.Like in the previous case this is the geometrical point of view of the Natario warp drive equation for constant speed vs in a $1+1$ spacetime with a lapse function and we know that in the Natario warp drive the negative energy density covers the entire bubble.(see Appendices B,C and D in [23]).Since the negative energy density have repulsive gravitational behavior(see pg $[116(a)][116(b)]$ in [13]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.[6]

Note that α must be much greater than $2v_s n(rs)$ in modulus otherwise if α is less than or equal to $2v_s n(rs)$ then the equation $U_{front} = X^{rs} - \alpha = 2v_s n(rs) - \alpha$ would have a Horizon when $2v_s n(rs) = \alpha$ or when $n(rs) = \frac{\alpha}{2v_s}$ assuming a continuous growth from 0 to v_s.Also note that if $\alpha = v_s$ in modulus then $n(rs) = \frac{\alpha}{2v_s} = \frac{v_s}{2v_s} = \frac{1}{2}$ and when $n(rs) = \frac{1}{2}$ this means the region outside the bubble according to pg 5 in [2].So if the Horizon is established outside the bubble at the end of the Natario warped region this means that the entire Natario warped region in which $0 < n(rs) < \frac{1}{2}$ is then totally connected and can be signalized or controlled by an astronaut.

A value of α lesser than v_s in modulus for example $\alpha = \frac{v_s}{2}$ would mean an $n(rs) = \frac{\alpha}{2v_s} = \frac{\frac{v_s}{2}}{2v_s} = \frac{1}{4}$ and the Horizon would appear inside the Natario warped region because $0 < \frac{1}{4} < \frac{1}{2}$.

In order to avoid the Horizon and keeps the Natario warped region totally connected the lapse function must have values greater than or equal to the bubble velocity in modulus .

[6] This can perhaps explain the negative value of $2v_s n(rs) - \alpha \neq 0$ when $\alpha >> 2v_s n(rs)$

6 Conclusion:

In this work we presented two equations for the warp drive spacetime according to Natario with constant velocity in the $3 + 1$ ADM formalism:

One of the equations possesses a lapse function α which have the values of 1 inside the bubble where the spaceship resides(flat spacetime) and outside the bubble where an external observer resides watching the ship passing by(also flat spacetime) but possessing a large value in the Natario warped region(curved spacetime) superior in modulus to the speed of the bubble while the other equation do not have a lapse function at all(or have a lapse function that is always equal to 1).

The original equation of the Natario warp drive spacetime with constant velocity vs that obeys the $3 + 1$ ADM formalism without a lapse function is this one:

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2d\theta^2 \quad (51)$$

The original equation of the Natario warp drive spacetime with constant velocity vs that obeys the $3 + 1$ ADM formalism with a lapse function is this one:

$$ds^2 = (\alpha^2 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2d\theta^2 \quad (52)$$

Some very important things both equations have in common and also have a fundamental difference:

- 1)- Both equations satisfies the Natario definition and condition for a warp drive spacetime using the same Natario shape function $n(rs)$ which gives 0 inside the bubble $\frac{1}{2}$ outside the bubble and $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.
- 2)- The same Natario shape function $n(rs)$ appears in the contravariant and covariant components of both Natario vectors.
- 3)-The difference between both equations is precisely the lapse function:in the first case its value is always equal to 1 and this no not affect the mathematical structure
- 4)-In the second case the lapse function have the following values:
  - inside the warp bubble(flat spacetime where the spaceship is located)the lapse function is equal to 1
  - outside the warp bubble(flat spacetime where an external observer watches the ship passing by)the lapse function is also equal to 1
  - in the Natario warped region(warp bubble walls curved spacetime) the lapse function must possesses a large value at least greater than or equal to the modulus of the ship velocity to keep the warp bubble totally connected.

A real and fully functional warp drive must encompasses accelerations or de-accelerations in order to go from 0 to 200 times light speed or even faster in the beginning of an interstellar journey and to slow down to 0 again in the end of the interstellar journey.

An accelerated warp drive can only exists if the astronaut in the center of the warp bubble can somehow communicate with the warp bubble walls sending instructions to change its speed.But for signals at light speed the Horizon exists at least for the Natario warp drive with constant velocity and no lapse function at all(or lapse function always equal to 1)so light speed cannot be used to send signals to the front of the bubble in this case of fixed velocities and no large lapse function.

In this work we analyzed the behavior of photons being sent to the front of the warp bubble at a constant velocity by an observer in the center of the bubble using the null-like geodesics of General Relativity $ds^2 = 0$ in both the Natario warp drive metrics with or without large lapse functions in the simplified case of the $1 + 1$ ADM formalism.Accelerated warp drives appears in [12],[19] and [22] but without lapse functions.These will appear in future works using also lapse functions.

We used quadratic equations to analyze the behavior of photons being sent to the front of the warp bubble.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons and we arrived at the following results:

- 1)-Natario warp drive metric with lapse function always equal to 1 : The dominant term in the solution of the quadratic form is the contravariant spatial component $X^{rs} = 2v_s n(rs)$ and the solution or the root of the quadratic form is given by:

$$U_{front} = X^{rs} - 1 = 2v_s n(rs) - 1 \tag{53}$$

  Note that when $X^{rs} = 2v_s n(rs) = 1$ the final result is zero the photon stops and an Horizon will appear:

$$U_{front} = X^{rs} - 1 = 2v_s n(rs) - 1 = 1 - 1 = 0!!! \tag{54}$$

- 2)-Natario warp drive metric with a large lapse function with a value greater than the bubble speed in modulus in the Natario warped region:

Unlike the previous case where the solution of the quadratic form reduces to zero in this case provided a large lapse function α greater than $2v_s n(rs)$ in modulus the root of the quadratic form never reduces to zero and is given by:

$$U_{front} = X^{rs} - \alpha = 2v_s n(rs) - \alpha \neq 0!!! \tag{55}$$

Note that α must be much greater than $2v_s n(rs)$ in modulus otherwise if α is less than or equal to $2v_s n(rs)$ then the equation $U_{front} = X^{rs} - \alpha = 2v_s n(rs) - \alpha$ would have a Horizon when $2v_s n(rs) = \alpha$ or when $n(rs) = \frac{\alpha}{2v_s}$ assuming a continuous growth from 0 to v_s.Also note that $\alpha = v_s$ in modulus then $n(rs) = \frac{\alpha}{2v_s} = \frac{v_s}{2v_s} = \frac{1}{2}$ and when $n(rs) = \frac{1}{2}$ this means the region outside the bubble according to pg 5 in [2].So if the Horizon is established outside the bubble in the end of the Natario warped region this means that the entire Natario warped region in which $0 < n(rs) < \frac{1}{2}$ is then totally connected and an astronaut can send signals to control the bubble.

In order to avoid the Horizon and keeps the Natario warped region totally connected the lapse function must have values greater than or equal to the bubble velocity.

The Natario warp drive spacetime is a very rich environment to study the superluminal features of General Relativity because now we have two spacetime metrics and not only one and the second metric can be totally controlled by an astronaut.

Because collisions between the walls of the warp bubble and the hazardous particles of the Interstellar Medium(IM) would certainly occurs in a real superluminal interstellar spaceflight we borrowed the idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection of the spaceship and the crew members in the Natario warp drive equation for constant speed vs(see pg 46 in [11]).

Our idea was to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the dangerous IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

A submicroscopic outer radius of the bubble being the only part in contact with our Universe would mean a submicroscopic surface exposed to the collisions against the hazardous IM particles thereby reducing the probabilities of dangerous impacts against large objects (comets asteroids etc) enhancing the protection level of the spaceship and hence the survivability of the crew members.

Any future development for the Natario warp drive must encompass the more than welcome idea of Chris Van Den Broeck and this idea can also be easily implemented in the Natario warp drive with lapse functions.Since the Broeck idea is independent of the Natario geometry wether we consider lapse functions or not we did not covered the Broeck idea here because it was already covered in [11] and in order to discuss the geometry of a Natario warp drive with lapse functions the Broeck idea is not needed here however the Broeck idea must appear in a real Natario warp drive with variable velocity vs and lapse functions concerning realistic superluminal interstellar spaceflights.

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.We also dont know how to generate the lapse function either.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario-Broeck warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario-Broeck warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

7 Remarks

References [4],[13],[14],[15],[16], [17] and [18] are standard textbooks used to study General Relativity and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [15] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader

The number of pages mentioned in the bibliographic references stored as e-prints in arXiv or HAL is the number of the page displayed in the bottom line of the Adobe PDF Acrobat Reader

8 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[7]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[8][9]

[7]special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[8]"Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[9]appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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